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Routing properties of supercubes ☆

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Abstract

Usually each vertex of the $(s+1)$-dimensional hypercube is labeled with a unique integer k with $0 \leqslant k \leqslant 2^{s+1}-1$. The supercube S_N of N nodes with $2^s < N \leqslant 2^{s+1}$ is constructed by merging nodes u and $u-2^s$, with $N \leqslant u \leqslant 2^{s+1}-1$, in the $(s+1)$-dimensional hypercube into a single node labeled as $u-2^s$ and leaving other nodes in the $(s+1)$-dimensional hypercube unchanged. In this paper, we give the exact distance between any two nodes of supercube and present a new shortest path routing algorithm on S_N. Then we show how to construct $\kappa(S_N)$ disjoint paths between any two nodes of the supercube, where $\kappa(S_N)$ is the connectivity of S_N. Finally, we compute the wide diameter and the fault diameter of S_N. We show that both the wide diameter and the fault diameter are equal to $s+2$ if $N \in \{2^{s+1}-2^i+1 \mid 0 \leqslant i \leqslant s-1\}$ and $s+1$ otherwise. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

The rapidly growing need for large scale computation and an ever increasing density of low cost VLSI circuit have resulted in an increasing demand for

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multiprocessing systems consisting of large numbers of interconnected processors. The topology of an interconnection network is a crucial factor for the performance of the network. Many types of interconnection network topologies have been studied. Network topology is usually represented by a graph, where vertices represent processors and edges represent links between processors. Among these topologies, hypercube has been studied extensively as an interconnection network for parallel machines because of advantages such as low message latency and high bandwidth [8]. However, one major constraint of the hypercube topology is that the number of nodes in the network must be 2^s for some positive integer s and cannot be defined for arbitrary number of nodes. Although the incomplete hypercube topology introduced in [5] has removed such a restriction, the incomplete hypercube has serious limitations from the fault-tolerance perspective. A single node failure may disconnect the network. In [9], Sen proposed a family of networks called supercubes which are denoted by S_N and can be realized for any number of nodes N. The supercube contains the hypercube with dimension $\lfloor \log_2 N \rfloor$ as a subgraph. It is isomorphic to a hypercube when the number of nodes is a power of 2. In addition, the supercube has the same excellent characteristics of small diameter and high connectivity as the hypercube. In [14], Yuan studied some topological properties of supercube and extended some results known for the hypercube to the supercube. Some fault-tolerant characteristics are explored in [2,10,11]. In [1], the computational capabilities of the supercube are studied by graph-embedding techniques.

It is important that an interconnection network routes data efficiently among nodes. Therefore, a shortest path routing algorithm that finds the shortest path joining any two nodes is preferred. In [13], Lien and Yuan proposed a distributed shortest path routing algorithm for the supercube. However, the algorithm does not estimate the exact length of shortest path between any pair of nodes of supercube. In this paper, we will present another shortest path routing algorithm which can compute the shortest distance between any pair of nodes for supercubes. Efficient routing can be achieved by using node-disjoint paths. Routing by node-disjoint paths among nodes can not only avoid communication bottlenecks, but also provide alternative path in case of node failures. In this paper, we will discuss the disjoint routing paths among nodes in supercube.

Fault diameter and wide diameter are important measures for interconnection networks. The fault diameter, proposed by Krishnamoorthy and Krishnamurthy [6], estimates the impact of diameter when fault occurs. The wide diameter, proposed by Hsu [3,4], measures the performance of multipath communication. Due to the non-symmetric property and the variation in the number of vertices, the value of fault diameter of supercube is difficult to obtain. Assume $2^s < N \leqslant 2^{s+1}$ for some positive integer s. In [10], it is proved that the fault diameter of S_N is at most $s + 3$. Later, it is claimed in [2] that the fault diameter of S_N is exactly $s+1$ if $N \notin \{2^{s+1}-1, 2^{s+1}-2, 2^s+2^{s-1}+1\}$, and $s+2$ otherwise. However, this is not correct. Recently, Sheu and Hsu [11] proved that the fault diameter of S_N is $s+2$ if $N \in \{2^{s+1}-2^i+1 \mid 0 \leqslant i \leqslant s-1\}$. Thus, the exact value of the fault diameter of the remaining N values of S_N is questionable. In this paper, we compute the exact values of the fault diameter and the wide diameter for S_N.

The outline of this paper is as follows. Definitions, notations, and general graph properties used throughout this paper are introduced in Section 2. In Section 3, a shortest path routing algorithm is proposed. In Section 4, we discuss the disjoint routing paths between any two nodes of the supercube. In Section 5, we compute the wide diameter and fault diameter of the supercube. The last section gives some concluding remarks.

2. Definitions and notations

Now, we formally introduce the definition of the supercubes and give some graph terminologies used in this paper. Most of the graph and interconnection network definitions used in this paper are standard (see, e.g., [7,12]). Let $G=(V,E)$ be a finite, undirected graph. Throughout this paper, node and vertex are used interchangeably to represent the element of V. Edge and link are used interchangeably to represent the element of E. For a vertex u, $N(u)$ denotes the *neighborhood* of u which is the set $\{v \mid (u,v) \in E\}$. Let u, v be any two nodes of G, the *distance* $d_G(u,v)$ between u and v is the length of the shortest path between them. The *diameter* of G, $D(G)$, is the maximum distance between any two nodes in G. The *connectivity* of G, $\kappa(G)$, is the minimum number of nodes whose removal leaves the remaining graph disconnected or trivial. Let $G=(V,E)$ be a graph with $\kappa(G)=\kappa$. It follows from Menger's theorem [12] that there are k *internally node-disjoint* (abbreviated as *disjoint*) *paths* joining any two vertices u and v when $k \leqslant \kappa$.

A *container* $C(u,v)$ between two distinct nodes u and v in G is a set of disjoint paths between u and v. The *width* of a $C(u,v)$, written as $w(C(u,v))$, is its cardinality. The length of a $C(u,v)$, written as $l(C(u,v))$, is the length of the longest path in $C(u,v)$. The w-*wide distance* between u and v is $l(C(u,v))$, where $C(u,v)$ is the minimum length container between u and v with width w. Let κ be the connectivity of G. The *wide diameter* of G, denoted by $D_\kappa(G)$, is the maximum of κ-wide distances among all pairs of nodes u, v in G, $u \neq v$. The *fault diameter* $D^f_{\kappa-1}(G)$ of a connected graph G is the maximum diameter of any subgraph of G obtained by removing at most $\kappa-1$ nodes. Obviously, $D^f_{\kappa-1}(G) \leqslant D_\kappa(G)$ [3].

In this paper, we assume that N and s are positive integers with $2^s < N \leqslant 2^{s+1}$. Let $u = u_{(s)}u_{(s-1)}\cdots u_{(1)}u_{(0)}$ and $v = v_{(s)}v_{(s-1)}\cdots v_{(1)}v_{(0)}$ be two $(s+1)$-bit strings. Let $\vee$ denote the *string or* operator. For example, $11001 \vee 01011 =$

11011. Let $H(u,v)$ denote the indices set $\{i \mid u_{(i)} \neq v_{(i)}\}$. The *Hamming distance* $h(u,v)$ between u and v is defined to be $|H(u,v)|$. The $(s+1)$-*dimensional hypercube* consists of all $(s+1)$-bit strings as its vertices and two vertices u and v are adjacent if and only if $h(u,v)=1$. By convention each vertex of the $(s+1)$-dimensional hypercube is labeled with a unique integer k corresponding to its binary $(s+1)$-bit string with $0 \leqslant k \leqslant 2^{s+1}-1$. The N-node *supercube* is constructed from the $(s+1)$-dimensional hypercube by merging nodes u and $u-2^s$, with $N \leqslant u \leqslant 2^{s+1}-1$, in the $(s+1)$-dimensional hypercube into a single node labeled as $u-2^s$ and leaving other nodes in the $(s+1)$-dimensional hypercube unchanged.

To be precise, let supercube S_N be the graph (V,E). The vertex set V consists of N vertices which are labeled from 0 to $N-1$. Each vertex u $(0 \leqslant u \leqslant N-1)$ can be expressed as an $(s+1)$-bit string $u_{(s)}u_{(s-1)}\cdots u_{(1)}u_{(0)}$ such that $u=\sum_{i=0}^{s} u_{(i)}2^i$. We use $\bar{u}$ to denote the string $\bar{u}_{(s)}\bar{u}_{(s-1)}\cdots\bar{u}_{(1)}\bar{u}_{(0)}$ and use $u^{(k)}$ to denote the string $u_{(s)}u_{(s-1)}\cdots u_{(k+1)}\bar{u}_{(k)}u_{(k-1)}\cdots u_{(0)}$. Observe that two nodes u and v are adjacent in a hypercube if and only if $v=u^{(k)}$ for some k, $0 \leqslant k \leqslant s$. The vertex set V is partitioned into three subsets V_1, V_2, and V_3, where $V_3=\{u \mid u \in V, u_{(s)}=1\}$, $V_2=\{u \mid u \in V, u_{(s)}=0, \text{ and } u^{(s)} \notin V\}$, and $V_1=\{u \mid u \in V, u_{(s)}=0, \text{ and } u^{(s)} \in V\}$. The edge set E is the union of E_1, E_2, E_3, and E_4, where $E_1=\{(u,v) \mid u,v \in V_1 \cup V_2 \text{ and } h(u,v)=1\}$, $E_2=\{(u,v) \mid u,v \in V_3 \text{ and } h(u,v)=1\}$, $E_3=\{(u,v) \mid u \in V_3, v \in V_2 \text{ and } h(u,v)=2\}$, and $E_4=\{(u,v) \mid u \in V_3, v \in V_1, \text{ and } h(u,v)=1\}$. As an example, a supercube with 13 nodes is shown in Fig. 1. In this figure, edges in E_3 are indicated by bold lines. Let $Z^0=V_1 \cup V_2$ and $Z^1=V_3$. Obviously, Z^0 induces an s-dimensional hypercube.

Let $P: x_0, x_1, \ldots, x_k$ be a path of length k from x_0 to x_k such that nodes x_i and x_{i+1} are adjacent to each other, and $x_{i+1}=x_i^{(\alpha_i)}$ with some index α_i, where $0 \leqslant i \leqslant k-1$ and $0 \leqslant \alpha_i \leqslant s$. We also write P as $\langle x_0 \mid \alpha_0, \alpha_1, \ldots, \alpha_{k-1}\rangle$. For example, let P be $u, u^{(3)}, (u^{(3)})^{(1)}, ((u^{(3)})^{(1)})^{(2)}$, we denote P by $\langle u \mid 3,1,2\rangle$.

3. Shortest path routing

In [13], Lien and Yuan proposed a distributed shortest path routing algorithm between any two nodes of supercube. However, their algorithm does not compute the shortest distance between each pair of nodes. In this section, we present a new shortest path routing algorithm which computes the distance between each pair of nodes of supercube. In the following lemmas, we will discuss the distance between any two nodes of supercube.

Lemma 1. *Assume that $u=u_{(s)}u_{(s-1)}\cdots u_{(0)}$ and $v=v_{(s)}v_{(s-1)}\cdots v_{(0)}$ are two vertices of S_N with $u>v$. Then there exists a path P of length $h(u,v)$ joining u to v with all internal nodes less than u. Moreover, P is in Z^1 if both u and v are in Z^1.*

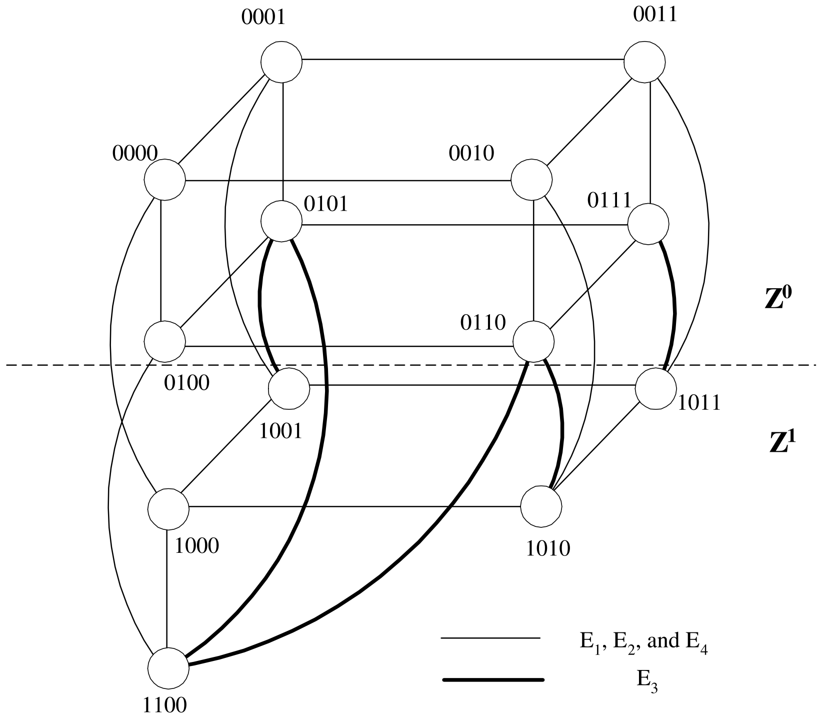


Fig. 1. An example of supercube S_{13}.

Proof. Let $\{\alpha_i\}_{i=0}^{h(u,v)-1}$ be the decreasing sequence of indices in $H(u,v)$. Since $u > v$, we have $u_{(\alpha_0)} = 1$, $v_{(\alpha_0)} = 0$, and $u_{(s)}u_{(s-1)}\cdots u_{(\alpha_0+1)} = v_{(s)}v_{(s-1)}\cdots v_{(\alpha_0+1)}$. Let $P : u, u^{(\alpha_0)}, (u^{(\alpha_0)})^{(\alpha_1)}, \ldots, ((u^{(\alpha_0)})^{(\alpha_1)}\cdots)^{(\alpha_{h(u,v)-1})}$ be a sequence of $(s+1)$-bit strings. Let x be any interior $(s+1)$-bit string in P. By our construction, we have $x_{(\alpha_0)} = 0 < u_{(\alpha_0)}$ and $x_{(s)}x_{(s-1)}\cdots x_{(\alpha_0+1)} = u_{(s)}u_{(s-1)}\cdots u_{(\alpha_0+1)}$. Thus $x < u \leqslant N-1$, that is, x is a vertex of S_N.

Now, assume that both u and v are in Z^1. Then, $u_{(s)} = v_{(s)} = x_{(s)} = 1$, i.e., each internal node x is in Z^1. Thus we have $P \in Z^1$. □

Lemma 2. *Assume that both u and v are nodes in either Z^0 or Z^1. Then $d_{S_N}(u,v) = h(u,v)$.*

Proof. It is observed that any edge (x,y) in $E_1 \cup E_2 \cup E_4$ satisfies $h(x,y) = 1$, and any edge (x,y) in E_3 satisfies $h(x,y) = 2$ and $x_{(s)} \neq y_{(s)}$. Since both u and v are in either Z^0 or Z^1, $u_{(s)} = v_{(s)}$. There are $h(u,v)$ bits, not including the sth bit, that are different from u to v. Hence the length of any path joining u and v is at least $h(u,v)$. Without loss of generality, we assume that $u > v$. By Lemma 1, we have $d_{S_N}(u,v) = h(u,v)$. □

Lemma 3. *Suppose that u is a node in* Z^1 *and v is a node in* Z^0. *Then,* $d_{S_N}(u,v)$ *is at least* $h(u,v)$ *if* $u \vee v < N$, *and at least* $h(u,v)-1$ *otherwise.*

Proof. It is observed that any edge (x,y) in $E_1 \cup E_2 \cup E_4$ satisfies $h(x,y)=1$, and any edge (x,y) in E_3 satisfies: (1) $h(x,y)=2$; (2) $x_{(s)} \neq y_{(s)}$; and (3) $x \vee y \geqslant N$. At least one edge is necessary to change the *i*th bit with $i \in H(u,v)-\{s\}$ in any path joining *u* to *v*. Thus, $d_{S_N}(u,v) \geqslant h(u,v)-1$ for any $u \in Z^1$ and $v \in Z^0$.

Assume that $d_{S_N}(u,v) < h(u,v)$ with $u \in Z^1$, $v \in Z^0$, and $u \vee v < N$. Then, there exists a path *P* in S_N of length $h(u,v)-1$ that joins *u* to *v*. Obviously, *P* contains an E_3 edge, say (x,y), with $x \in Z^1$, $y \in Z^0$, and $y^{(s)} \notin S_N$. Thus, we may write *P* as $u,\ldots,x,y,\ldots,v$. Obviously, we have $x \vee y \geqslant N$. Since $u \vee v < N$, there exists some index $j \notin H(u,v)$ such that $x_{(j)} = \bar{u}_{(j)}$ or $y_{(j)} = \bar{u}_{(j)}$. Thus, the length of *P* is at least $h(u,v)-1$ plus 1 to restore the *j*th bit. We get a contradiction. Thus, $d_{S_N}(u,v)$ is at least $h(u,v)$ if $u \vee v < N$. Hence, the lemma is proved. □

Now we propose the following shortest path routing algorithm in S_N:

Routing algorithm: Let $u = u_{(s)}u_{(s-1)}\cdots u_{(0)}$ and $v = v_{(s)}v_{(s-1)}\cdots v_{(0)}$ be any two nodes in S_N with $u > v$. Construct a shortest path *P* from *u* to *v* as follows:

Case 1. $u,v \in Z^0$ or $u,v \in Z^1$: Let $\{\alpha_i\}_{i=0}^{h(u,v)-1}$ be the decreasing sequence of indices in $H(u,v)$. Construct the path *P* as: $\langle u \mid \alpha_0,\alpha_1,\ldots,\alpha_{h(u,v)-1}\rangle$.

Case 2. $u \in Z^1$ and $v \in Z^0$: Let $A = \{\alpha_0,\alpha_1,\ldots,\alpha_{t-1}\}$ be the indices such that $u_{(\alpha_i)} = 0$ and $v_{(\alpha_i)} = 1$ for $0 \leqslant i \leqslant t-1$, and let $\{\beta_i\}_{i=0}^{h(u,v)-t-2}$ be the sequence of indices in $H(u,v)-\{s\}-A$. Construct the path *P* joining *u* to *v* as:

Case 2.1. $u \vee v < N$: Set $P = \langle u \mid \alpha_0,\alpha_1,\ldots,\alpha_{t-1},s,\beta_0,\beta_1,\ldots,\beta_{h(u,v)-t-2}\rangle$.

Case 2.2. $u \vee v \geqslant N$: Set *P* to be $x_0,x_1,\ldots,x_{h(u,v)-1}$ with $x_0 = u$ and $x_{h(u,v)-1} = v$ as follows: For $0 \leqslant i \leqslant t-1$, let $x_{i+1} = x_i^{(\alpha_i)}$ if $x_i^{(\alpha_i)} \in S_N$, and $x_{i+1} = (x_i^{(\alpha_i)})^{(s)}$ otherwise; and we set the remaining path from x_t to *v* as $\langle x_t \mid \beta_0,\beta_1,\ldots,\beta_{h(u,v)-t-2}\rangle$.

Obviously, if $u \vee v \geqslant N$, then by our algorithm we will go from *u* to a node x_{k-1}, $1 \leqslant k \leqslant t$, in Z^1 such that all of x_i for $1 \leqslant i \leqslant k-1$ are vertices in Z^1 and $x_k = x_{k-1}^{(\alpha_{k-1})}$ is not in S_N. Thus, one E_3 edge $(x_{k-1}, x_k^{(s)})$ can be taken. And the remaining path from $x_k^{(s)}$ to *v* can be constructed as in hypercube. This path is of length $h(u,v)-1$. In the other cases, the length of each path is the Hamming distance between *u* and *v*. For example, let $u = 11000$, $v = 00110$, and $N = 11110$. Since $u \vee v = 11110 \geqslant N$, we construct a shortest path from *u* to *v* as: $u = 11000$, 11100, 01110, $00110 = v$ whose length is $h(u,v)-1 = 3$. Note that $(11100, 01110) \in E_3$.

Applying the routing algorithm, we have the following theorem.

Theorem 1. $d_{S_N}(u,v) = h(u,v)-1$ *if* $u_{(s)} \neq v_{(s)}$ *and* $u \vee v \geq N$, *and* $d_{S_N}(u,v) = h(u,v)$ *if otherwise.*

4. Containers of supercube

Disjoint paths are useful in transferring large amount of data between nodes and offering alternative routes in node failure situations. According to Menger's theorem [12], there exist $\kappa(S_N)$ disjoint paths between any two nodes of S_N. In this section, we construct a container $C(u,v)$ with $w(C(u,v)) = \kappa(S_N)$ for any two nodes u and v in S_N, which will be used in the next section to compute the wide diameter and the fault diameter of supercube.

4.1. Container of S_N with $2^s < N < 2^s + 2^{s-1}$

It is proved in [9] that $\kappa(S_N)$ is s if $2^s < N < 2^s + 2^{s-1}$. The following known result about s disjoint paths of length at most $s+1$ between each pair of nodes of supercube is proved in [13].

Theorem 2. *There exists a container $C(u,v)$ between nodes u and v in S_N with $w(C(u,v)) = s$ and $l(C(u,v)) \leqslant s+1$.*

4.2. Container of S_N with $2^s + 2^{s-1} \leqslant N \leqslant 2^{s+1}$

It is proved in [9] that $\kappa(S_N)$ is $s+1$ if $2^s + 2^{s-1} \leqslant N \leqslant 2^{s+1}$. It follows from Menger's theorem [12] that there are $s+1$ disjoint paths between any two nodes of S_N for $2^s + 2^{s-1} \leqslant N \leq 2^{s+1}$. Therefore, in this section we will discuss the container problem, i.e., disjoint paths problem, of S_N for the case $2^s + 2^{s-1} \leqslant N \leqslant 2^{s+1}$. By the definition of S_N, we have the following lemma.

Lemma 4. *Let $2^s + 2^{s-1} \leqslant N \leqslant 2^{s+1}$. If $u \notin S_N$, then $u_{(s-1)} = 1$. Moreover, if $v_{(s-1)} = 1$, then $v^{(s-1)} \in S_N$.*

Before discussing the disjoint paths problem of the supercube, we need to know the maximum number of disjoint paths between any two nodes of s-dimensional hypercube. The following lemma is constructed in [8].

Lemma 5. *Assume that u and v are two different nodes in the s-dimensional hypercube and $u > v$. There exist s disjoint paths joining u to v, among those, $h(u,v)$ of them are of length $h(u,v)$, and the others are of length $h(u,v)+2$. Moreover, there exists at least one path of length $h(u,v)$ whose internal nodes are all less than u.*

Proof. Let $\{\alpha_i\}_{i=0}^{h(u,v)-1}$ be the decreasing sequence of indices such that $u_{(\alpha_i)} \neq v_{(\alpha_i)}$, and $\{\beta_j\}_{j=0}^{s-h(u,v)-1}$ be the decreasing sequence of indices such that $u_{(\beta_j)} = v_{(\beta_j)}$. For $0 \leqslant i \leqslant h(u,v)-1$ we set $P_i = \langle u \mid \alpha_{0+i}, \alpha_{1+i}, \ldots, \alpha_{h(u,v)-1+i}\rangle$ with the addition of subscripts being performed under modulo $h(u,v)$. Suppose that P_0 is $u, x_1, x_2, \ldots, x_{h(u,v)-1}, v$, we set P_j, for $h(u,v) \leqslant j \leqslant s-1$, as $u, u^{(\beta_{j-h(u,v)})}, x_1^{(\beta_{j-h(u,v)})}, x_2^{(\beta_{j-h(u,v)})}, \ldots, x_{h(u,v)-1}^{(\beta_{j-h(u,v)})}, v^{(\beta_{j-h(u,v)})}, v$. Then $P_0, P_1, \ldots, P_{s-1}$ satisfy our requirement. Moreover, by the similar argument of Lemma 1, it can be checked that all internal nodes of P_0 are less than u. □

Now we construct $s+1$ disjoint paths between any two nodes of the supercube. According to the locations of the source node and the destination node, we construct $s+1$ disjoint paths in the following three lemmas.

Lemma 6. *Let* $1u = 1u_{(s-1)}u_{(s-2)} \cdots u_{(0)}$ and $1v = 1v_{(s-1)}v_{(s-2)} \cdots v_{(0)}$ *be two nodes of* Z^1. *Then there exist* $s+1$ *disjoint paths* $Q_0, Q_1, \ldots, Q_s$ *joining u to v in* S_N *such that the length of* Q_i *is* $h(u,v)$ *for* $0 \leqslant i \leqslant h(u,v)-1$ *and the length of* Q_j *is* $h(u,v)+2$ *for* $h(u,v) \leqslant j \leqslant s$.

Proof. Suppose that $1u > 1v$ and Z^1 is a s-dimensional hypercube. Let $Q_0, Q_1, \ldots, Q_{s-1}$ be the s disjoint paths from $1u$ to $1v$ constructed by Lemma 5 where all internal nodes of Q_0 are less than $1u$. Because some internal nodes of these paths might not be in S_N, we have to modify these paths to be feasible in S_N.

Let $1q$ be any internal node which is not in supercube where q is a s-bit string. We replace $1q$ by $0q$. Obviously, $0q$ is a node of Z^0. By definition of S_N, there exist two edges joining $0q$ to its two neighboring nodes in this path. Thus, in this manner, we can get s disjoint paths from $1u$ to $1v$ such that all nodes are in supercube. Note that such modification does not change the length of any path. Moreover, since all internal nodes of Q_0 are less than $1u$, we conclude that Q_0, written as $1u, 1x_1, 1x_2, \ldots, 1x_{h(u,v)-1}, 1v$, is in Z^1. Finally, we set Q_s as $1u, 0u, 0x_1, 0x_2, \ldots, 0v, 1v$. Then $Q_0, Q_1, \ldots, Q_s$ satisfies our requirement. □

Lemma 7. *Let* $0u = 0u_{(s-1)}u_{(s-2)} \cdots u_0$ *and* $0v = 0v_{(s-1)}v_{(s-2)} \cdots v_{(0)}$ *be two nodes of* Z^0. *There exist* $s+1$ *disjoint paths* $P_0, P_1, \ldots, P_s$ *joining* $0u$ *to* $0v$ *in* S_N *with the following properties*:

1. *The length of* P_i *for* $0 \leqslant i \leqslant h(u,v)-1$ *is* $h(u,v)$.
2. *The length of* P_j *for* $h(u,v) \leqslant j \leqslant s-1$ *is* $h(u,v)+2$.
3. *The length of* P_s *is at most* $h(u,v)+3$ *if* $h(u,v) \leqslant s-2$. *Moreover, this path* P_s *satisfies the following two statements*:
   If $h(u,v)=s$: *The length of* P_s *is* $s+1$ *if exactly one of* $1u$ *and* $1v$ is in S_N, *and is* $s+2$ *if both* $1u$ *and* $1v$ *are in* S_N.
   If $h(u,v)=s-1$: *The length of* P_s *is at most* $s+2$ *if either* $1u$ *or* $1v$ *is in* S_N *and* $u_{(s-1)} = v_{(s-1)} = 1$, *and is at most* $s+1$ *otherwise.*

Proof. Note that Z^0 is isomorphic to an s-dimensional hypercube. Let $P_0, P_1, \ldots, P_{s-1}$ be the paths from $0u$ to $0v$ constructed in Lemma 5. We are going to construct the $(s+1)$th disjoint path P_s such that all of its internal nodes are in Z^1 in the following three cases.

Case 1. Both $1u$ and $1v$ are vertices of Z^1: By Lemma 1, there exists a path Q of length $h(u,v)$ in Z^1 joining $1u$ to $1v$. We set P_s as $0u, 1u, Q, 1v, 0v$ whose length is $h(u,v)+2$.

Case 2. Either $1u$ or $1v$ is in Z^1: Without loss of generality, we assume that $1u$ is a node in Z^1 and $1v$ is not contained in S_N. By Lemma 4, $v_{(s-1)} = 1$ and $1v^{(s-1)}$ is a node of Z^1.

Assume that $1v^{(k)}$ is a node in Z^1 with some $k \in H(u,v)$. Then $(1v^{(k)}, 0v)$ is an E_3 edge. By Lemma 1, there exists a path R of length $h(u,v)-1$ from $1u$ to $1v^{(k)}$ in Z^1. We set P_s as $0u, 1u, R, 1v^{(k)}, 0v$ whose length is $h(u,v)$ $+1$.

Now we assume none of $1v^{(i)}$, for all $i \in H(u,v)$, is a node in supercube. Since $1v^{(s-1)}$ is a node in Z^1, we have $s-1 \notin H(u,v)$. Thus, $u_{(s-1)} = v_{(s-1)} = 1$ and $h(u,v) \leqslant s-1$ in this case. By Lemma 1, there exists a path Q of length $h(1u, 1v^{(s-1)}) = h(u,v)+1$ in Z^1 from $1u$ to $1v^{(s-1)}$. We set P_s as $0u, 1u, Q, 1v^{(s-1)}, 0v$ whose length is $h(u,v)+3$.

Case 3. Neither $1u$ nor $1v$ is in Z^1: By Lemma 4, we have $u_{(s-1)} = v_{(s-1)} = 1$, i.e., in this case $h(u,v) \leqslant s-1$ and both $1u^{(s-1)}$ and $1v^{(s-1)}$ are vertices of Z^1. By definition of supercube, both $(0u, 1u^{(s-1)})$ and $(1v^{(s-1)}, 0v)$ are E_3 edges. By Lemma 1, there exists a path Q of length $h(u,v)$ in Z^1 joining $1u^{(s-1)}$ to $1v^{(s-1)}$. We set P_s as $0u, 1u^{(s-1)}, Q, 1v^{(s-1)}, 0v$ whose length is $h(u,v)+2$.

Obviously, P_s forms the $(s+1)$th disjoint path joining $0u$ to $0v$ whose length is at most $h(u,v)+3$. Suppose that $h(u,v) = s$, the length of P_s is $s+1$ if either $1u$ or $1v$ is in S_N; and is $s+2$ if both $1u$ and $1v$ are in S_N. Suppose that $h(u,v) = s-1$. The length of P_s is at most $s+2$ if either $1u$ or $1v$ is in S_N and $u_{(s-1)} = v_{(s-1)} = 1$, and at most $s+1$ otherwise. Hence, this lemma is proved. □

Lemma 8. *Let* $0u = 0u_{(s-1)}u_{(s-2)}\cdots u_{(0)}$ *and* $1v = 1v_{(s-1)}v_{(s-2)}\cdots v_{(0)}$ *be two nodes of* S_N *with* $0u \in Z^0$ *and* $1v \in Z^1$. *Then there are* $s+1$ *disjoint paths* $Q_0, Q_1, \ldots, Q_s$ *joining* $0u$ *to* $1v$ *in* S_N *with the following properties*:

1. *The length of* Q_i *for* $0 \leqslant i \leqslant h(0u,1v)-2$ *is at most* $h(0u,1v)$; *and every internal node* q *satisfies* $q_{(j)} = u_{(j)}$ *for* $j \notin H(0u,1v)$.
2. *The length of* Q_j *for* $h(0u,1v)$ $-1 \leqslant j \leqslant s-1$ *is at most* $h(0u,1v)+2$.
3. *The length of* Q_s *is at most* $h(0u,1v)+1$. *And every internal node* q *of* Q_s *is in* Z^1. *Let* $h(0u,1v) = s$ *and* t *be the unique index such that* $u_{(t)} = v_{(t)}$. *Then,* $q_{(t)} = \bar{u}_{(t)}$ *only if* $u_{(t)} = 1$ *and* $t = s-1$.
4. *If* $h(0u,1v) = s+1$, *all the* $s+1$ *paths are of length at most* $s+1$.

Proof. Assume that $h(0u, 1v) = 1$. Obviously $u = v$. For $1 \leqslant i \leqslant s$, we set Q_i as $0u, 0u^{(i-1)}, 1v^{(i-1)}, 1v$ if $1v^{(i-1)} \in Z^1$, and $0u, 0u^{(i-1)}, 1v$ otherwise. And we set Q_0 as $0u, 1v$. Obviously, $Q_0, Q_1, \ldots, Q_s$ satisfy our requirement. Hence, in the rest of the proof we assume that $h(0u, 1v) \geqslant 2$.

Obviously, $0u$ and $0v$ are nodes of Z^0, and Z^0 is isomorphic to an s-dimensional hypercube. Let $P_0, P_1, \ldots, P_{s-1}$ be the s disjoint paths constructed by Lemma 5 joining $0u$ to $0v$. We denote the last internal node of P_i be $0l_i$ for every i.

We are going to choose a neighboring node $1z$ of $0u$ in Z^1 and construct a path Q'_s of length at most $h(u, v) + 1$ from $1z$ to $1v$ in Z^1 by Lemma 1. And then we can set Q_s as $0u, 1z, Q'_s, 1v$. Thus, all internal nodes of Q_s are in Z^1 and Q_s satisfies the requirement of statement 3. For simplicity, we discuss only the choice of $1z$ and the length of Q'_s joining $1z$ to $1u$ in the following three cases:

Case 1. $1u$ is in Z^1: We set $1z$ to be $1u$. Thus Q'_s is of length $h(1u, 1v) = h(0u, 1v) - 1$.

Case 2. $1u$ is not in S_N and $1u^{(i)}$ is in Z^1 for some index $i \in H(u, v)$: We set $1z$ to be $1u^{(i)}$ and thus Q'_s is of length $h(1u^{(i)}, 1v) = h(0u, 1v) - 2$. (In case 1 and 2, if $H(0u, 1v) = s$ and t is the unique index such that $u_{(t)} = v_{(t)}$, then it can be checked that each node q of Q'_s satisfies $q_{(t)} = u_{(t)} = v_{(t)}$.

Case 3. $1u$ is not in S_N and $1u^{(i)}$ are not in S_N for all $i \in H(u, v)$: Since $1u$ is not a node of supercube, by Lemma 5, we have $u_{(s-1)} = 1$ and $1u^{(s-1)} \in Z^1$. Thus, $s - 1$ is not contained in $H(0u, 1v)$, and the Hamming distance $h(0u, 1v)$ between $0u$ and $1v$ is at most s in this case. We set $1z = 1u^{(s-1)}$. Thus Q'_s is of length $h(1u^{(s-1)}, 1v) = h(0u, 1v)$.

Obviously, the last internal node of Q'_s is $1l_k$ for some k with $0 \leqslant k \leqslant s - 1$. We extend P_k by appending an edge $(0v, 1v)$ to it and set Q_k to be this new path joining $0u$ to $1v$. For $0 \leqslant i \neq k \leqslant s - 1$, we replace the last edge $(0l_i, 0v)$ of P_i by two consecutive edges $(0l_i, 1l_i)$ and $(1l_i, 1v)$ if $1l_i$ is in Z^1 and by an E_3 edge $(0l_i, 1v)$ otherwise. And we set Q_i to be this new path joining $0u$ to $1v$.

Obviously, $Q_0, Q_1, \ldots, Q_s$ form $s + 1$ disjoint paths. Moreover, the length of Q_i is at most $h(0u, 1v)$ for $0 \leqslant i \leqslant h(0u, 1v) - 2$, and the length of Q_i is at most $h(0u, 1v) + 2$ for $h(0u, 1v) - 1 \leqslant i \leqslant s - 1$. The length of Q_s is at most $s + 1$ if $h(0u, 1v) = s + 1$, and is at most $h(0u, 1v) + 1$ otherwise. Let q be any node in Q_s, $h(0u, 1v) = s$, and t is the unique index such that $u_{(t)} = v_{(t)}$. It can be checked that $q_{(t)} = \bar{u}_{(t)}$ only in case 3. More precisely, $q_{(t)} = \bar{u}_{(t)}$ only if $u_{(t)} = 1$ and $t = s - 1$. Thus, $Q_0, Q_1, \ldots, Q_s$ satisfy our requirement (see Fig. 2). Hence, this lemma is proved. □

According to Lemmas 6–8, we have the following theorem.

Theorem 3. *There exists a container $C(u, v)$ between nodes u and v in S_N with $w(C(u, v)) = s + 1$ and $l(C(u, v)) \leqslant s + 2$.*

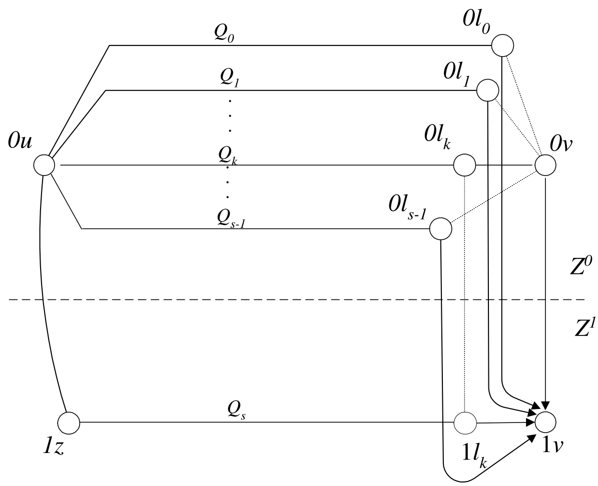


Fig. 2. Illustration for Lemma 8.

4.3. Container of S_N with $2^s + 2^{s-1} \leqslant N \leqslant 2^{s+1}$ and $N \neq 2^{s+1} - 2^i + 1$ for $0 \leqslant i \leqslant s-1$

In Theorem 3, we have proposed a container $C(u,v)$ between nodes u and v in S_N with $w(C(u,v)) = s+1$ and $l(C(u,v)) \leqslant s+2$. However, if $N \notin \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$, it is possible to construct a better container $C(u,v)$ between nodes u and v in S_N with $w(C(u,v)) = s+1$ and $l(C(u,v)) \leqslant s+1$. Therefore, in this section we consider the case $2^s + 2^{s-1} \leqslant N \leqslant 2^{s+1}$ and $N \notin \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$.

Note that here N is at most $11\cdots10$,

$$N - 1 \neq 11\cdots1\overbrace{00\cdots0}^{i} \quad \text{for } 0 \leqslant i \leqslant s-1,$$

s is at least 4, and the connectivity of S_N is $s+1$. Moreover, if $N < 2^{s+1} - 3$, then $N - 1 < 11\cdots100$. And if $N = 2^{s+1} - 2$, then $N - 1 = 11\cdots101$.

The following lemma describes some constructions which are useful for our purpose.

Lemma 9. *Let $u = u_{(s-1)}u_{(s-2)}\cdots u_{(0)}$ and $v = v_{(s-1)}v_{(s-2)}\cdots v_{(0)}$ be two s-bit strings with*: (1) $0 \in H(u,v)$ *and $u_{(t)} = v_{(t)} = 1$ for any index $t \notin H(u,v)$ and* (2) $1u$ *is a node in Z^1. Let $\{\alpha_i\}_0^{m-1}$ be the decreasing sequence of indices in $H(u,v)$ such that $u_{(\alpha_i)} = 0$ and $\{\alpha_j\}_m^{h(u,v)-1}$ be the increasing sequence of indices in $H(u,v)$ such that $u_{(\alpha_j)} = 1$. Set $x_0 = u, x_1 = x_0^{(\alpha_0)}, x_2 = x_1^{(\alpha_1)}, \ldots, x_{m+1} = x_m^{(\alpha_m)}$. Then, the following statements hold*:

1. *There exists an integer k with $1 \leqslant k \leqslant m$ such that neither $1x_k$ nor $1x_{k+1}$ is a node of supercube, but $1x_0, 1x_1, \ldots, 1x_{k-1}$ are all nodes in Z^1.*

2. *If* $\{0, 1, \ldots, s-2\} \subseteq H(u,v)$, *then there exists an index* $\gamma \in H(u,v) - \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$ *such that* $1x_{k-1}^{(\gamma)}$ *is a node in* Z^1 but $1x_k^{(\gamma)}$ *is not a node of supercube.*
3. *Let* $y_1 = x_2^{(\alpha_0)}, y_2 = x_3^{(\alpha_0)}, y_3 = x_4^{(\alpha_0)}, \ldots, y_{k-2} = x_{k-1}^{(\alpha_0)}$, *and* $y_{k-1} = y_{k-2}^{(\alpha_{h(u,v)-1})}$, then $1y_1, 1y_2, \ldots, 1y_{k-1}$ *are all nodes of* Z^1 *and disjoint from* $1x_1, 1x_2, \ldots, 1x_{k-1}$ *where* $1y_1 = 1u^{(\alpha_1)}$.

Proof. The proof is through the following steps.

(1) We first prove that statement 1 holds. Obviously, $1x_{i+1}$ is greater than $1x_i$ since $1x_{i+1} = 1x_i + 2^{\alpha_i}$ for $0 \leqslant i \leqslant m-1$. Because all 0 bits have been complemented, obviously, $1x_m = 11\cdots1$ and $1x_m$ is not a vertex of supercube. Thus, there exists a positive integer k, k is no more than m, such that all $1x_1, 1x_2, \ldots, 1x_{k-1}$ are vertices in Z^1 and $1x_k$ is not a node of supercube.

Before proving that $1x_{k+1}$ is not a node of supercube, we claim that α_{k-1} is at least 1: Assume that $N < 2^{s+1} - 3$. Because $1x_{k-1} = 11\cdots10u_{(\alpha_{k-1}-1)}u_{(\alpha_{k-1}-2)}\cdots u_{(0)} < N \leqslant 11\cdots100$, we have $\alpha_{k-1} \geqslant 2$. Assume that $N = 2^{s+1} - 2$. Because $1x_{k-1} \leqslant N - 1 = 11\cdots101$, $\alpha_{k-1} \geqslant 1$. Therefore, we have $\alpha_{k-1} \geqslant 1$.

Now we show that $1x_{k+1}$ is not a node of supercube, i.e., we will show that $1x_{k+1} \geqslant N$. Suppose that $k < m$. Because $1x_{k+1} > 1x_k > N-1$, $1x_{k+1}$ is not a node in supercube. Suppose that $k = m$. Then, we have $\alpha_{m-1} = \alpha_{k-1} \geqslant 1$. Note that α_{m-1} is the smallest index i of $H(u,v)$ such that $u_{(i)} = 0$, α_m is the smallest index j such that $u_{(j)} = 1$, and $0 \in H(u,v)$. Thus, by definition we have $\alpha_m = 0$. Thus $1x_{k+1} = 1x_m^{(\alpha_m)} = 11\cdots1^{(0)} = 11\cdots10 > N$ and we have $1x_{k+1} \notin S_N$. Hence, statement 1 holds.

(2) We prove that statement 2 holds in the following two cases: (i) $1x_{k-1} = N-1$ and (ii) $1x_{k-1} < N-1$. Note that $\{\alpha_i\}_0^{m-1}$ is decreasing, $\{0, 1, \ldots, s-2\} \subseteq H(u,v)$, and $\alpha_{k-1} \geqslant 1$. Thus, index γ will be in $H(u,v) - \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$ if γ is 0 or less than α_{k-1}. Obviously, if we can show that $1x_{k-1}^{(\gamma)} \leqslant N-1$ and $1x_k^{(\gamma)} \geqslant N$, then the fact that $1x_{k-1}^{(\gamma)}$ is a node in Z^1 and $1x_k^{(\gamma)}$ is not a node of supercube will ensue. Thus, statement 2 holds. For simplicity, we show only $1x_{k-1}^{(\gamma)} \leqslant N-1$ and $1x_k^{(\gamma)} \geqslant N$.

Case 1. $1x_{k-1} = N-1$: Note that

$$N - 1 \neq 11\cdots1\overbrace{00\cdots0}^{i} \quad \text{for } 0 \leqslant i \leqslant s-1.$$

Since $1x_{k-1} = 11\cdots10u_{(\alpha_{k-1}-1)}u_{(\alpha_{k-1}-2)}\cdots u_{(0)}$, there exists one index γ with $0 \leqslant \gamma < \alpha_{k-1}$ such that $u_{(\gamma)} = 1$. Therefore, $1x_{k-1}^{(\gamma)} = 1x_{k-1} - 2^{\gamma} < N-1$ and $1x_k^{(\gamma)} = 1x_k - 2^{\gamma} = 1x_{k-1} + 2^{\alpha_{k-1}} - 2^{\gamma} > N-1$.

Case 2. $1x_{k-1} < N-1$: Suppose that $1x_k > N$. We set $\gamma = 0$. Then, $1x_{k-1}^{(\gamma)} \leqslant 1x_{k-1} + 1 \leqslant N-1$ and $1x_k^{(\gamma)} \geqslant 1x_k - 1 \geqslant N$. Suppose $1x_k = N$. Since $1x_k = 11\cdots1u_{(\alpha_{k-1}-1)}u_{(\alpha_{k-1}-2)}\cdots u_{(0)}$ and $N \leqslant 11\cdots10$, there exists one index γ

with $0 \leqslant \gamma < \alpha_{k-1}$ such that $u_{(\gamma)} = 0$. Therefore, $1x_{k-1}^{(\gamma)} = 1x_k - 2^{\alpha_{k-1}} + 2^{\gamma} < N$ and $1x_k^{(\gamma)} = 1x_k + 2^{\gamma} > N$.

(3) Finally, we claim that statement 3 holds. It can be checked that: (i) $1y_i = 1x_{i+1} - 2^{\alpha_0}$ for $1 \leqslant i \leqslant k-2$; (ii) $1y_1$ is $1u^{(\alpha_1)}$; and (iii) $1y_i \neq 1x_j$ for any $1 \leqslant i, j \leqslant k-1$. Now we show that $1y_1, 1y_1, \ldots, 1y_{k-2}$ are vertices in Z^1: Since $1y_i = 1x_{i+1} - 2^{\alpha_0}$ and $1x_{i+1}$ is a node of Z^1 for $1 \leqslant i \leqslant k-2$, we have $1y_i < 1x_{i+1} < N$. Thus $1y_i$ is a node in Z^1. Now we consider $1y_{k-1}$, which is defined to be $1y_{k-2}^{(\alpha_{h(u,v)-1})}$. Note that the $\alpha_{h(u,v)-1}$th bit of $1y_{k-2}$ is equal to $u_{(\alpha_{h(u,v)-1})}$. Here we consider two cases:

Case 1. If $u_{(\alpha_{h(u,v)-1})} = 1$: Since $1y_{k-2} < N$, we have $1y_{k-1} = 1y_{k-2} - 2^{\alpha_{h(u,v)-1}} < N$. Thus $1y_{k-1}$ is a node in Z^1.

Case 2. If $u_{(\alpha_{h(u,v)-1})} = 0$: Because $\{\alpha_i\}_{i=0}^{m-1}$ is the decreasing sequence of indices in $H(u,v)$ such that $u_{(\alpha_i)} = 0$, we have $m = h(u,v)$. Thus, $\alpha_{h(u,v)-1}$ is the smallest index in $H(u,v)$, i.e., $\alpha_{h(u,v)-1} = 0$. Because $1y_{k-1} = 1y_{k-2} + 2^0 = 1x_{k-1} - 2^{\alpha_0} + 2^0$, we have $1y_{k-1} < 1x_{k-1} < N$ and thus $1y_{k-1}$ is a node in Z^1.

Therefore, this lemma is proved. □

In Lemma 6, we have proposed $s+1$ disjoint paths of length no more than $s+1$ between any two node $1u$ and $1v$ for $h(u,v) < s$. In the following lemma, we will propose $s+1$ disjoint paths P_i, $0 \leqslant i \leqslant s$, of length at most $s+1$ from $1u$ to $1v$ for $h(u,v) = s$. We first construct P_0 and P_1, and then construct the other $s-1$ paths $P_2, P_3, \ldots, P_s$ such that these $s-1$ paths are "parallel" to the two paths P_0 and P_1, i.e., all the $s+1$ paths are disjoint.

Lemma 10. *Let* $1u = 1u_{(s-1)}u_{(s-2)} \cdots u_{(0)}$ *and* $1v = 1v_{(s-1)}v_{(s-2)} \cdots v_{(0)}$ *be two nodes in* Z^1. *Then, there exists a container* $C(1u, 1v)$ *with* $w(C(1u,1v)) = s+1$ *and* $l(C(1u,1v)) \leqslant s+1$.

Proof. By Lemma 6, this lemma holds if $h(u,v) \neq s$. Thus, we prove the lemma for $h(u,v) = s$, i.e., $H(u,v) = \{0, 1, \ldots, s-1\}$. Without loss of generality, we assume that $u > v$. Then $u_{(s-1)} = 1, v_{(s-1)} = 0$, and $u_{(i)} = \bar{v}_{(i)}$ for $0 \leqslant i \leqslant s-1$.

Let $\{\alpha_i\}_0^{m-1}$ be the decreasing sequence of indices such that $u_{(\alpha_i)} = 0$. Let $x_0 = u, x_1 = x_0^{(\alpha_0)}, x_2 = x_1^{(\alpha_1)}, x_3 = x_2^{(\alpha_2)}, \ldots, x_m = x_{m-1}^{(\alpha_{m-1})}$. By Lemma 9, there exists an index k with $1 \leqslant k \leqslant m$ such that $1x_k$ is not contained in S_N and all $1x_i$, for $0 \leqslant i < k$, are vertices in Z^1. By definition of supercube, $(1x_{k-1}, 0x_k)$ is an E_3 edge. Moreover, by Lemma 9 there exists an index γ in $H(u,v) - \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$ such that $1x_{k-1}^{(\gamma)}$ is a node in Z^1 and $1x_k^{(\gamma)}$ is not contained in S_N. By Lemma 4, we have $(1x_k^{(\gamma)})^{(s-1)}$ is in Z^1. Thus, $(0x_k^{(\gamma)}, (1x_k^{(\gamma)})^{(s-1)})$ is an E_3 edge. From Lemma 1, we can find a path W of length $s-k-2$ joining $(1x_k^{(\gamma)})^{(s-1)}$ to $1v$ in Z^1. Let W be $1z_0 = (1x_k^{(\gamma)})^{(s-1)}, 1z_1, 1z_2, \ldots, 1z_{s-k-2} = 1v$. We set W' as $0z_0, 0z_1, 0z_2, \ldots, 0z_{s-k-2} = 0v$. Note that W can also be written as $\langle 1z_0 \mid d_0, d_1, \ldots, d_{s-k-3} \rangle$ where d_i, for $0 \leqslant i \leqslant s-k-3$, is a one to one correspondence to the $s-k-2$ indices in $H(u,v) - \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \gamma, s-1\}$.

Now we construct the required $s+1$ disjoint paths from $1u$ to $1v$ as follows. We set P_0 to be $1u, 1x_1, 1x_2, \ldots, 1x_{k-1}, 0x_k, 0x_k^{(s-1)}$, $(0x_k^{(s-1)})^{(\gamma)} = 0z_0$, W', $0v$, $1v$ and P_1 to be $1u, 0u, 0x_1, 0x_2, \ldots, 0x_{k-1}$, $0x_{k-1}^{(\gamma)}$, $0x_k^{(\gamma)}$, $(1x_k^{(\gamma)})^{(s-1)} = 1z_0, W, 1v$ (see Fig. 3). It is clear that P_0 and P_1 are of length $s+1$ and disjoint.

Let $\theta_0 = \alpha_0, \theta_1 = \alpha_1, \ldots, \theta_{k-1} = \alpha_{k-1}, \theta_k = \gamma, \theta_{k+1} = s-1, \theta_{k+2} = d_0, \theta_{k+3} = d_1, \ldots, \theta_{s-1} = d_{s-k-3}$. Then, for $2 \leqslant i \leqslant s$, we construct P_i as $\langle 1u \mid \theta_{0+i}, \theta_{1+i}, \ldots, \theta_{s-1+i} \rangle$ where the addition of subscripts being performed under modulo of s. Note that there might exist some nodes which are not contained in supercube. For any $1q \notin S_N$ of any Q_i, we replace $1q$ by $0q$. By definition of supercube, we can get a feasible path of the same length in supercube. Thus, all internal nodes (except for $0x_{k-1}^{(\gamma)}$ and $0x_k^{(s-1)}$) of P_0 and P_1 are disjoint from those of $P_2, P_3, \ldots, P_s$. Thus, we have to prove that $0x_{k-1}^{(\gamma)}$ and $0x_k^{(s-1)}$ are not contained in $P_2, P_3, \ldots, P_s$. Obviously, an internal node $0q \in Z^0$ appears in P_i, for $2 \leqslant i \leqslant s$, if and only if $1q \notin S_N$. Because both $1x_{k-1}^{(\gamma)}$ and $1x_k^{(s-1)}$ are in Z^1, neither $0x_{k-1}^{(\gamma)}$ nor $0x_k^{(s-1)}$ is in P_i for $2 \leqslant i \leqslant s$. Hence, $P_0, P_1, \ldots, P_s$ are disjoint and satisfy our requirement. □

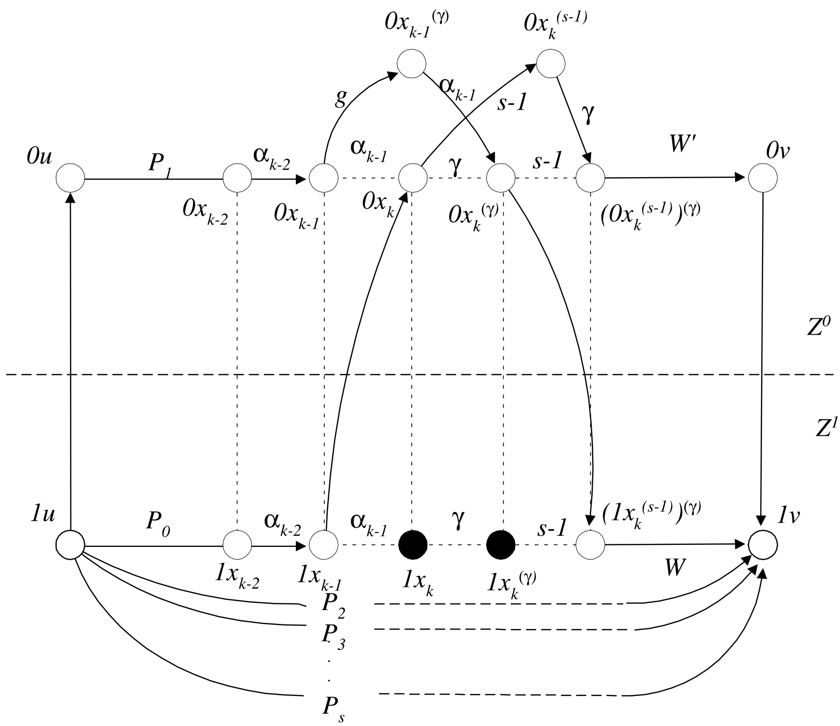


Fig. 3. Illustration for Lemma 10.

In Lemma 7, we have proposed $s+1$ disjoint paths of length no more than $s+1$ between any two node $0u$ and $0v$ except for two cases: (a) $h(u,v)=s$ and both $1u$ and $1v$ are nodes in Z^1, and (b) $h(u,v)=s-1$, $u_{(s-1)}=v_{(s-1)}=1$, and either $1u$ or $1v$ is in S_N. In the following lemma, we will propose $s+1$ disjoint paths P_i, $0\leqslant i\leqslant s$, of length at most $s+1$ from $1u$ to $1v$ for these two cases. We first construct s disjoint paths $P_0, P_1, \ldots, P_{s-1}$. And then we will construct two new paths, P_0^0 and P_0^1, by modifying P_0.

Lemma 11. *Le* $0u = 0u_{(s-1)}u_{(s-2)}\cdots u_{(0)}$ *and* $0v = 0v_{(s-1)}v_{(s-2)}\cdots v_{(0)}$ *be two nodes of* S_N *with* $0u\in Z^0$ and $0v\in Z^0$. *Then, there exists a container* $C(0u,0v)$ with $w(C(0u,0v))=s+1$ *and* $l(C(0u,0v))\leqslant s+1$.

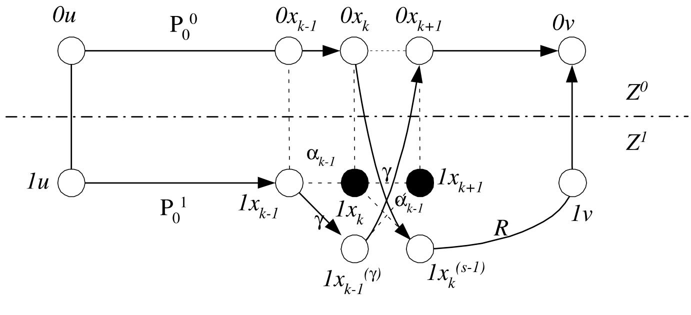
Proof. By Lemma 7, this lemma is true except: (a) $h(u,v)=s$ and both $1u$ and $1v$ are in S_N, and (b) $h(u,v)=s-1$, $u_{(s-1)}=v_{(s-1)}=1$, and either $1u$ or $1v$ is in S_N. We assume that $1u$ is a vertex in Z^1. Let $\{\alpha_i\}_0^{m-1}$ be the decreasing sequence of indices in $H(u,v)$ such that $u_{(\alpha_i)}=0$. Set $x_0=u$ and $x_{i+1}=x_i^{(\alpha_i)}$ for $0\leqslant i\leqslant m-1$. By Lemma 9, there exists an integer k, $k\leqslant m$, such that $1x_1, 1x_2, \ldots, 1x_{k-1}$ are all vertices in supercube and $1x_k$ is not a node in supercube. Moreover, there exists an index γ in $H(u,v)-\{\alpha_0,\alpha_1,\ldots,\alpha_{k-1}\}$ such that $1x_{k-1}^{(\gamma)}$ is a vertex in Z^1 and $1x_{k+1}=1x_k^{(\gamma)}$ is not a node of S_N. Because $1x_{k-1}^{(\gamma)}$ differs from $1x_{k+1}$ only at (α_{k-1})th bit, $(1x_{k-1}^{(\gamma)}, 0x_{k+1})$ is an E_3 edge.

Set α_i for $k+1\leqslant i\leqslant h(u,v)-1$ as a one to one correspondence to the $h(u,v)-k-1$ indices in $H(u,v)-\{\alpha_0,\alpha_1,\ldots,\alpha_{k-1},\gamma\}$. We construct $P_i=\langle 0u \mid \alpha_{0+i},\alpha_{1+i},\ldots,\alpha_{h(u,v)-1+i}\rangle$ with the addition of subscripts being performed under modulo $h(u,v)$ for $0\leqslant i\leqslant h(u,v)-1$. Obviously, P_0 is the path $0u=0x_0, 0x_1, 0x_2, \ldots, 0x_{h(u,v)}=0v$, where $x_{i+1}=x_i^{(\alpha_i)}$ for $0\leqslant i\leqslant h(u,v)-1$.

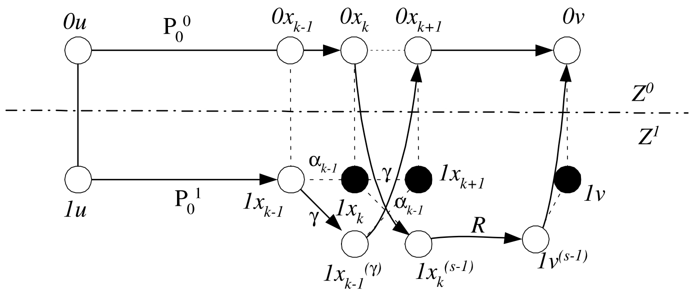
We construct two new paths based on P_0 in the following cases:

Case A. $h(u,v)=s$ and both $1u$ and $1v$ are in S_N: Without loss of generality, we assume that $u>v$ and thus $u_{(s-1)}=1$, $v_{(s-1)}=0$. Thus, $s-1$ is not contained in $\{\alpha_0,\alpha_1,\ldots,\alpha_{k-1}\}$ and the $(s-1)$th bit of $1x_k$ is the same as $u_{(s-1)}$. By Lemma 4, $1x_k^{(s-1)}$ is a node in Z^1. According to definition of supercube, $(0x_k, 1x_k^{(s-1)})$ is an E_3 edge. By Lemma 1, there exists a path R of length $h(1x_k^{(s-1)}, 1v)=s-k-1$ in Z^1 between $1x_k^{(s-1)}$ and $1v$. Obviously, each node q of R satisfies $q_{(s-1)}=v_{(s-1)}=0$. We set P_0^0 to be $0u, 0x_1, 0x_2, \ldots, 0x_k, 1x_k^{(s-1)}, R, 1v, 0v$. Thus, P_0^0 is of length $s+1$. We set P_0^1 to be $0u, 1u, 1x_1, 1x_2, \ldots, 1x_{k-1}, 1x_{k-1}^{(\gamma)}, 0x_{k+1}, 0x_{k+2}, \ldots, 0x_s=0v$. Obviously, P_0^1 is of length $s+1$ and each internal node p in Z^1 satisfies $p_{(s-1)}=u_{(s-1)}=1$. Therefore, P_0^0 and P_0^1 are disjoint (see Fig. 4).

Case B. $u_{(s-1)}=v_{(s-1)}=1$, $1u\in Z^1$ and $1v\notin S_N$, and $H(u,v)=\{0,1,\ldots,s-2\}$: Obviously, $1x_k^{(s-1)}$ is in Z^1 and $(0x_k, 1x_k^{(s-1)})$ is an E_3 edge. By Lemma 1, there exists a shortest path R of length $s-k-1$ in Z^1 from $1x_k^{(s-1)}$ to $1v^{(s-1)}$ with each node q satisfies $q_{(s-1)}=v_{(s-1)}=0$. We set P_0^0 to be $0u, 0x_1, 0x_2, \ldots, 0x_k, 1x_k^{(s-1)}, R, 1v^{(s-1)}, 0v$. Then, P_0^0 is of length $s+1$. We set P_0^1



Case A



Case B

Fig. 4. Illustration for Lemma 11.

to be $0u, 1u, 1x_1, 1x_2, \ldots, 1x_{k-1}, 1x_{k-1}^{(\gamma)}, 0x_{k+1}, 0x_{k+2}, \ldots, 0x_{0,s-1} = 0v$. Obviously, P_0^1 is of length s and each internal node p in Z^1 satisfies $p_{(s-1)} = u_{(s-1)} = 1$. Therefore, P_0^0 and P_0^1 are disjoint.

Hence, $P_0^0, P_0^1, P_1, P_2, \ldots, P_{s-1}$ satisfy our requirement (see Fig. 4). Thus, this lemma is proved. □

Then, we discuss the final case. In Lemma 8, we have proposed $s+1$ disjoint paths, $Q_0, Q_1, \ldots, Q_s$, of length no more than $s+1$ between any two nodes $0u$ and $1v$ for $h(0u, 1v) \neq s$. Note that if $h(0u, 1v) = s$, then all these paths except for Q_{s-1} are of length $s+1$. In the following lemma, we will propose $s+1$ disjoint paths of length at most $s+1$ from $1u$ to $0v$ for $h(0u, 1v) = s$. We will discuss the following two cases: (1) If $0u^{(t)} \vee 1v^{(t)} \geqslant N$ and (2) $0u^{(t)} \vee 1v^{(t)} < N$. For case (1), we will construct a new path Q_{s-1} such that Q_{s-1} is of length at most $s+1$ and disjoint from the s paths $Q_0, Q_1, \ldots, Q_{s-2}, Q_s$ constructed in Lemma 8. For case (2), we will reconstruct the $s+1$ disjoint paths as follows: We first build $s-1$ disjoint paths, $Q_0, Q_1, \ldots, Q_{s-2}$, of length $s-1$ from $0v$ to

$0u$ in Z^0. Then based on the path Q_0, we construct two paths P_0 and P_{s-1}. And we set P_i by modifying Q_i for $1 \leqslant i \leqslant s-2$. Finally, we construct the path P_s such that all these $s+1$ paths $P_0, P_1, \ldots, P_s$ are disjoint and of length at most $s+1$.

Lemma 12. *Let* $0u = 0u_{(s-1)}u_{(s-2)} \cdots u_{(0)}$ *and* $1v = 1v_{(s-1)}v_{(s-2)} \cdots v_{(0)}$ *be two nodes of* S_N *with* $0u \in Z^0$ and $1v \in Z^1$. *Then, there exists a container* $C(0u, 1v)$ *with* $w(C(0u,1v)) = s+1$ *and* $l(C(0u,1v)) \leqslant s+1$.

Proof. By Lemma 8, this lemma is true if $h(0u, 1v) \neq s$. Hence, we focus our attention on $h(0u, 1v) = s$. Let t be the unique index such that $u_{(t)} = v_{(t)}$.

According to Lemma 8, we can find s disjoint paths, $Q_0, Q_1, \ldots, Q_{s-2}, Q_s$, from $0u$ to $1v$ such that: (a) the length of Q_i, $0 \leqslant i \leqslant s-2$, is at most s and each node q of Q_i satisfies $q_{(t)} = u_{(t)}$; (b) the length of Q_s is at most $s+1$ and every internal node p of Q_s is in Z^1; and (c) $p_{(t)} = \bar{u}_{(t)}$ only if $t = s-1$ and $u_{(t)} = 1$. Now we will prove this lemma by the following two cases:

(1) *If* $0u^{(t)} \vee 1v^{(t)} \geqslant N$: Note that

$$0u^{(t)} \vee 1v^{(t)} = 11 \cdots 1\bar{u}_{(t)} \overbrace{11 \cdots 1}^{t}.$$

Assume that $t = s-1$ and $u_{(t)} = v_{(t)} = 1$. Then, $0u^{(t)} \vee 1v^{(t)} = 1011 \cdots 1 < N$. We get a contradiction. Thus, each node q of $Q_0, Q_1, \ldots, Q_{s-2}, Q_s$ satisfies $q_{(t)} = u_{(t)}$. To construct Q_{s-1}, we first choose a neighboring node z of $1v$. Let z be $1v^{(t)}$ if $1v^{(t)}$ is a node in Z^1, and be $0v^{(t)}$ otherwise. Obviously, $(z, 1v)$ is an edge of supercube. Let Q' be the path constructed by our shortest path routing algorithm from $0u^{(t)}$ to z. Then, we set Q_{s-1} as $0u, 0u^{(t)}, Q', z, 1v$. Obviously, Q_{s-1} is of length $h(u,v)+1$ and each internal node p of Q_{s-1} satisfies that $q_{(t)} = \bar{u}_{(t)}$. Hence $Q_0, Q_1, \ldots, Q_s$ are disjoint and satisfy our requirement.

(2) *If* $0u^{(t)} \vee 1v^{(t)} < N$: In this case, we will reconstruct $s+1$ disjoint paths, $P_0, P_1, \ldots, P_s$, of length at most $s+1$ from $1v$ to $0u$. Since

$$0u^{(t)} \vee 1v^{(t)} = 11 \cdots 1\bar{u}_{(t)} \overbrace{11 \cdots 1}^{t} < N \leqslant \overbrace{11 \cdots 1}^{s} 0,$$

we have $u_{(t)} = v_{(t)} = 1$ and $t \geqslant 1$. Obviously, $1v^{(t)}, 1u^{(t)} < N$ in this case and thus both $1v^{(t)}$ and $1u^{(t)}$ are vertices in Z^1. Let $\{\alpha_i\}_0^{m-1}$ be the decreasing sequence of indices in $H(u,v)$ such that $v_{(\alpha_i)} = 0$. And let $\{\alpha_i\}_m^{s-2}$ be the increasing sequence of indices in $H(u,v)$ such that $v_{(\alpha_i)} = 1$. Then, for $0 \leqslant i \leqslant s-2$ we set Q_i as $\langle 0v \mid \alpha_{0+i}, \alpha_{1+i}, \ldots, \alpha_{s-2+i} \rangle$ with the addition of subscripts being performed under modulo $s-1$. Thus, we get $s-1$ disjoint paths of length $h(0v, 0u) = s-1$ from $0v$ to $0u$ in Z^0.

Let Q_0 be written as $0x_0 = 0v, 0x_1, 0x_2, \ldots, 0x_{s-1} = 0u$. Note that $x_i(t) = v_{(t)} = 1$ for every i. By Lemma 9, there exists an integer k, $1 \leqslant k \leqslant m$, such that $1x_k$ and $1x_{k+1}$ are not in S_N and $1x_i$ for $0 \leqslant i \leqslant k-1$ are all vertices in

Z^1. Moreover, $1y_1, 1y_2, \ldots, 1y_{k-1}$ are nodes in Z^1 and disjoint from $1x_1, 1x_2, \ldots, 1x_{k-1}$, where $1y_1 = 1v^{(\alpha_1)}$, $1y_{k-1} = 1y_{k-2}^{(\alpha_{h(u,v)-1})}$, and $1y_i = 1x_{i+1}^{(\alpha_0)}$, for $1 \leqslant i \leqslant k-2$.

Now we set P_0 and P_{s-1}. Obviously, $1x_i^{(t)}$ is a vertex of Z^1 for every i because $1x_i^{(t)} < (11\cdots1)^{(t)} = 0u^{(t)} \vee 1v^{(t)} < N$. Since $1x_k$ and $1x_{k+1}$ are not vertices of S_N and $1x_{k+1}^{(t)}$ is a node in Z^1, we have $(1x_{k-1}, 0x_k) \in E_3$ and $(1x_{k+1}^{(t)}, 0x_{k+1}) \in E_3$. We set P_0 as $1v, 1v^{(t)}, 1x_1^{(t)}, 1x_2^{(t)}, \ldots, 1x_{k+1}^{(t)}, 0x_{k+1}, 0x_{k+2}, \ldots, 0x_{s-1} = 0u$ and set P_{s-1} as $1v, 1x_1, 1x_2, \ldots, 1x_{k-1}, 0x_k, 0x_k^{(t)}, 0x_{k+1}^{(t)}, \ldots, 0x_{s-1}^{(t)} = 0u^{(t)}, 0u$. Obviously, P_0 and P_{s-1} are of length $s+1$.

We now construct P_i for $1 \leqslant i \leqslant s-2$ by replacing the subpath $0v, 0v^{(\alpha_i)}$ of Q_i as follows: We replace the subpath $0v, 0v^{(\alpha_1)}$ of Q_1 by $1v, 0v, 0v^{(\alpha_1)}$ and set P_1 to be this new path joining $1v$ to $0u$. For $2 \leqslant i \leqslant s-2$, we replace the subpath $0v, 0v^{(\alpha_i)}$ of Q_i by $1v, 1v^{(\alpha_i)}, 0v^{(\alpha_i)}$ if $1v^{(\alpha_i)}$ is a node of Z^1, and by $1v, 1v^{(\alpha_i)}$ otherwise. Then, we set P_i to be this new path joining $1v$ to $0u$.

Then we construct P_s. We choose a neighboring node z of $0u$ as follows: If $1u \in Z^1$, we set $z = 1u$; otherwise, because $1u^{(t)}$ is a node, we set $z = 1u^{(t)}$. By Lemma 1 there exists a path R of length at most $h(u,v) = s-k+1$ joining $1y_{k-1}$ to z in Z^1. We construct P_s as $1v, 1y_1, 1y_2, \ldots, 1y_{k-1}, R, z, 0u$. Therefore, $P_0, P_1, \ldots, P_s$ are disjoint and satisfy our requirement (see Fig. 5). Hence, this lemma is proved. □

According to Lemmas 10–12, we get the following theorem.

Theorem 4. *There exists a container $C(u,v)$ between any two nodes u and v in S_N with $w(C(u,v)) = s+1$ and $l(C(u,v)) \leqslant s+1$ if $2^s + 2^{s-1} \leqslant N \leqslant 2^{s+1}$ and $N \notin \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$.*

5. Wide diameter and fault diameter

In this section, we compute the wide diameter and the fault diameter of the supercube S_N. Let F be the set of faulty nodes. We discuss the problem into three cases: (1) $2^s < N < 2^s + 2^{s-1}$; (2) $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$; and (3) the remaining cases.

5.1. $2^s < N < 2^s + 2^{s-1}$

In this case, the connectivity of S_N is s. By Theorem 2, there exists a container $C(u,v)$ between nodes u and v in S_N with $w(C(u,v)) = s$ and $l(C(u,v)) \leqslant s+1$. Therefore, we have the following corollary.

Corollary 1. $D_\kappa(S_N) \leqslant s+1$ *if* $2^s < N < 2^s + 2^{s-1}$.

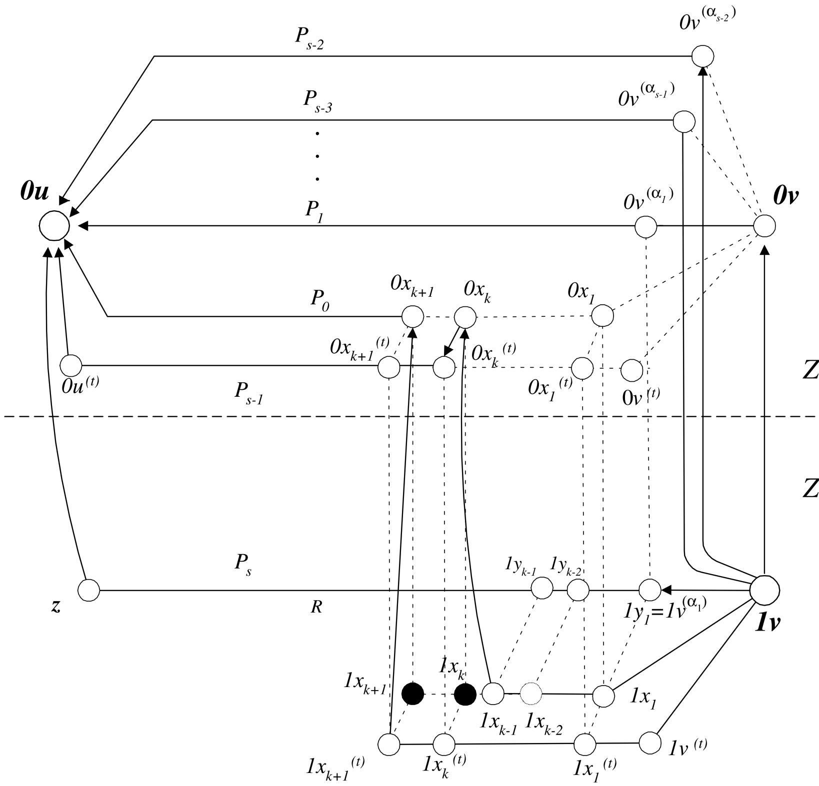


Fig. 5. Illustration for Lemma 12.

Because $N < 2^s + 2^{s-1}$, each vertex in S_N is less than

$$10\overbrace{11\cdots1}^{s-1}.$$

In this following lemma, we discuss the lower bound for the fault diameter of the supercube with at most $s-1$ faulty nodes.

Lemma 13. $D^f_{\kappa-1}(S_N) \geqslant s+1$ if $2^s < N < 2^s + 2^{s-1}$.

Proof. Let

$$0u = 0\overbrace{11\cdots1}^{s} \quad \text{and} \quad 0v = 01\overbrace{00\cdots0}^{s-1}.$$

Obviously, $1u \notin S_N$ and $1u^{(i)} \notin S_N$ for $0 \leqslant i \leqslant s-1$. Therefore, $0u$ has not any adjacent node in Z^1. Thus, the degree of $0u$ in S_N is s. Let $F = \{0u^{(i)} \mid 0 \leqslant$

$i \leqslant s-2\}$ be the set of faulty nodes. Obviously, $|F| = s-1$ and $0u^{(s-1)}$ is the only fault-free node that is adjacent to $0u$. By Lemma 2, the length of each path from $0u^{(s-1)}$ to $0v$ is at least s, i.e., each the fault-free path from $0u$ to $0v$ is of length at least $s+1$. Therefore, $D^f_{\kappa-1}(S_N) \geqslant s+1$. □

By Corollary 1, Lemma 13, and the fact that $D^f_{\kappa-1}(G) \leqslant D_\kappa(G)$, we have the following theorem.

Theorem 5. *$D^f_{\kappa-1}(S_N) = D_\kappa(S_N) = s+1$ if $2^s < N < 2^s + 2^{s-1}$.*

5.2. $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$

In this case, the connectivity of S_N is $s+1$. In Theorem 3, we have proposed a container $C(u,v)$ between nodes u and v in S_N with $w(C(u,v)) = s+1$ and $l(C(u,v)) \leqslant s+2$. Thus, we get the following corollary.

Corollary 2. *$D_\kappa(S_N) \leqslant s+2$ if $2^s + 2^{s-1} \leqslant N \leqslant 2^{s+1}$.*

In the following lemma, we discuss the lower bound for the fault diameter of the supercube with at most s faulty nodes.

Lemma 14. *$D^f_{\kappa-1}(S_N) \geqslant s+2$ if $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$.*

Proof. Because $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$, we have

$$N - 1 = 11\cdots1\overbrace{00\cdots0}^{t}$$

with $0 \leqslant t \leqslant s-1$. Let

$$0u = 011\cdots1\overbrace{00\cdots0}^{t} \quad \text{and} \quad 0v = 00\cdots0\overbrace{11\cdots1}^{t}.$$

Then $0u, 0v \in Z^0$ and $h(u,v) = s$. Let F be the set of all neighboring nodes of $0u$ in Z^0. Thus $|F| = s$. Let $P : 0u, x_1, x_2, \ldots, 0v$ be any path in $S_N - F$ joining $0u$ to $0v$. Then, $x_1 = 11\cdots1\overbrace{00\cdots0}^{t}$ and $x_2 = x_1^{(i)}$ for some i such that x_2 is a node in Z^1. Since $x_1^{(k)} > N-1$ for every k with $0 \leqslant k < t$, $x_1^{(k)}$ is not a node of the supercube. Hence, we have $x_2 = x_1^{(j)}$ for some j with $j \geqslant t$. Therefore, $x_2 \vee 0v = (11\cdots1)^{(j)} < 11\cdots1\overbrace{00\cdots0}^{t} < N$. By Lemma 3, the shortest distance from x_2 to $0v$ is at least $h(x_2, 0v) = s$. Hence, the length of P is at least $s+2$. Therefore $D^f_{\kappa-1}(S_N) \geqslant s+2$. □

By Corollary 2 and Lemma 14, we get the following theorem.

Theorem 6. *$D^f_{\kappa-1}(S_N) = D_\kappa(S_N) = s+2$ if $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$.*

5.3. The remaining cases

Finally, we discuss the case that $2^s + 2^{s-1} \leqslant N \leqslant 2^{s+1}$ and $N \notin \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$. In this case N is at most $11\cdots10$, s is at least 4, and the connectivity of S_N is $s+1$. Moreover, if $N < 2^{s+1} - 3$, then $N - 1 < 11\cdots100$. And if $N = 2^{s+1} - 2$, then $N - 1 = 11\cdots101$.

The following lemma computes the lower bound for the fault diameter of the supercube with at most s faulty nodes.

Lemma 15. *$D^f_{\kappa-1}(S_N) \geqslant s+1$ if $N \notin \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$.*

Proof. Note that $\kappa(S_N) = s+1$. Let $0u = 00\cdots0$, $0v = 011\cdots1$, and F be the set of all neighboring nodes of $0u$ in Z^0. Obviously, $|F| = s$ and $1u$ is a node in Z^1. Let $P: 0u, x_1, x_2, \ldots, 0v$ be any path in $S_N - F$ joining $0u$ to $0v$. Obviously, $x_1 = 1u$. By Lemma 3, the path from $0u$ through $1u$ to $0v$ is of length at least $1 + (h(1u, 0v) - 1) = s+1$. Hence, $D^f_{\kappa-1}(S_N) \geqslant s+1$. □

By Theorem 4, there exists a container $C(u,v)$ between nodes u and v in S_N with $w(C(u,v)) = s+1$ and $l(C(u,v)) \leqslant s+1$ if $N \notin \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$. Thus, we have the following corollary.

Corollary 3. *$D_\kappa(S_N) \leqslant s+1$ if $N \notin \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$.*

Therefore, by Corollary 3 and Lemma 15 we have the following theorem:

Theorem 7. *If $N \in \{i \mid 2^s + 2^{s-1} \leqslant i \leqslant 2^{s+1}\} - \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$, then $D^f_{\kappa-1}(S_N) = D_\kappa(S_N) = s+1$.*

6. Concluding remarks

Various topological properties of supercubes have been studied. The shortest path routing algorithm which can estimate the exactly shortest distance between any two nodes for supercubes has not been discussed though. In this paper, we have proved that $d_{S_N}(u,v) = h(u,v) - 1$ if $u_{(s)} \neq v_{(s)}$ and $u \vee v \geqslant N$, and $d_{S_N}(u,v) = h(u,v)$ otherwise. And we have proposed a new shortest path routing algorithm on supercubes. We also have constructed $\kappa(S_N)$ disjoint paths for any pair of vertices and computed the wide diameter and the fault diameter of the supercubes S_N. As a conclusion, both the wide diameter and the fault diameter are equal to $s+2$ if $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leqslant i \leqslant s-1\}$ and $s+1$ otherwise.

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