On a General Optimal Algorithm for Multirate Output Feedback Controllers of Linear Stochastic Periodic Systems

Nie-Zen Yen and Yung-Chun Wu

Abstract—**A modified optimal algorithm for multirate output feedback controllers of linear stochastic periodic systems is developed. By combining the discrete-time linear quadratic regulation (LQR) control problem and the discrete-time stochastic linear quadratic regulation (SLQR) control problem as the extended linear quadratic regulation (ELQR) control problem, one derives a general optimal algorithm to balance the advantages between the optimal transient response of LQR control problem and the optimal steady-state noise regulation of SLQR control problem. In general, the solution of this algorithm is solved from a set of coupled matrix equations. Special cases for which the coupled matrix equations can be reduced to a discrete-time algebraic Riccati equation are also discussed. A reducible case is the well-known optimal algorithm derived by AL-Rahmani and Franklin [1], where the system has complete state information, and the discrete-time quadratic performance index is transformed from a continuous-time quadratic performance index.**

I. Introduction

In a recent paper [1], AL-Rahmani and Franklin presented an optimal multirate state feedback control scheme for linear periodic systems based on the linear quadratic regulation (LQR) control problem. They show that the continuous-time LQR control problem subject to the multirate structure can be transformed into a discrete-time LQR control problem, and the solution can simply be solved from a discrete-time algebraic Riccati equation. As a matter of fact, the use of multirate sampling not only makes it easy to solve the LQR control problem, but also produces better response characteristics and less LQR cost than that of single rate sampling (if the state is sampled at the same rate). Colaneri and De Nicolao [8] have presented an optimal filter-based multirate control scheme for linear stochastic systems with incomplete state measurements based on the linear quadratic and Guassian (LQG) control problem. Due to the possibility of more freedom in sampling period selection, it is easy to implement a multirate control scheme by digital computers.

In this note, a modified multirate control scheme for linear stochastic periodic systems is developed. Such a control scheme is constructed by using instantaneous output feedback (i.e., a zero-order controller). A fact to be noted is that in the optimal periodic control theory using the classical separation principle (e.g., Bittanti *et al.* [6] and Colaneri and De Nicolao [8]), the LQG control problem can be considered as a direct extension of the LQR control problem. However, this fact is not true in our approach unless the system has complete state information. To overcome this difficulty, one combines the discrete-time linear quadratic regulation (LQR) control problem and the discrete-time stochastic linear quadratic regulation (SLQR) control problem as the extended linear quadratic regulation (ELQR) control problem, then by solving this problem, one derives a general algorithm to extend the LQR and the SLQR algorithms simultaneously. This algorithm allows one to choose ratios on the LQR and the SLQR control problems, so that the presented multirate output feedback controller can balance the advantages between the optimal transient response of LQR control problem and the optimal steady-state noise regulation of SLQR control problem.

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II. Preliminary

A. Periodic System

Consider a multivariable linear sotchastic periodic system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \xi(t) \tag{1.a}$$

$$y_s(t) = C(t)x(t) \tag{1.b}$$

$$y(kT) = y_s(kT) + v(kT) \tag{1.c}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the input, $y_s(t) \in R^r$ is the output, $y(kT)$ (T is a positive real, and $k = 0, 1, 2, \cdots$) is the practical output measurement, $\xi(t) \in R^n$ and $v(kT) \in R^r$ are independent white noise with zero means and correlations described by

$$E(\xi(t)\xi^{\tau}(t_0)) = G(t)\delta(t - t_0) \tag{2.a}$$

$$E(v(kT)v^{\tau}(k_0T)) = Rq(k - k_0) \tag{2.b}$$

where τ denotes the transpose operation of a matrix, $G(t) \in R^{n\times n}$ and $R \in R^{r\times r}$ are positive semidefinite matrices, $\delta(\cdot)$ denotes the continuous-time delta function, $q(\cdot)$ denotes the discrete-time pulse function, $\xi(t)$ and $v(kT)$ are independent of $x(0)$, and the parameters $A(t)$, $B(t)$, $C(t)$, and $G(t)$ are piecewise continuous and satisfy the periodical property that $A(t) = A(t - T)$, $B(t) = B(t - T)$, $C(t) = C(t - T)$, and $G(t) = G(t - T)$.

B. Multirate Output Feedback Controller

A multirate output feedback controller of the linear stochastic periodic system (1) (see Fig. 1) is given by

$$u(kT + iT/f + \theta) = L_i y(kT) \tag{3}$$

where f is a selected positive integer, $\theta \in [0 \;\; T/f)$, and $L_i \in R^{m\times r}$ for $i = 0, 1, 2, \cdots, f-1$ are the controller gains. By substituting the multirate control (3) into the system (1), one can obtain the following closed-loop sampled-data system:

$$x((k+1)T) = (\bar{A} + \Omega LC)x(kT) + \eta(kT) \tag{4}$$

in which $\bar{A} = \phi(T, 0)$, $C = C(0)$, and

$$\Omega = [\Omega_0 \;\; \Omega_1 \;\; \cdots \;\; \Omega_{f-1}] \in R^{n\times mf},$$

$$L = \begin{pmatrix} L_0 \\ L_1 \\ \vdots \\ L_{f-1} \end{pmatrix} \in R^{mf\times r} \tag{5}$$

$$\eta(kT) = \int_0^T \phi(T, s)\xi(kT + s)\, ds + \Omega L v(kT) \tag{6}$$

where $\phi(t, s)$ denotes the state transition matrix, and

$$\Omega_i = \int_{iT/f}^{(i+1)T/f} \phi(T, s)B(s)\, ds. \tag{7}$$

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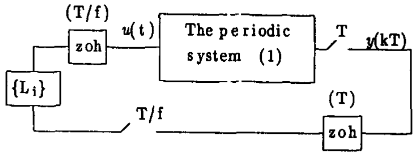

Fig. 1. A multirate output feedback controller of linear periodic systems.

Notice that $\eta(kT)$ is also a white noise with zero mean and correlation described by

$$E(\eta(kT)\eta^\tau(k_0T)) = (S + \Omega LRL^\tau\Omega^\tau)q(k - k_0) \tag{8}$$

where

$$S = \int_0^T \phi(T,s)G(s)\phi(T,s)^\tau\, ds. \tag{9}$$

Remark 1: In the presented approach, the output is sampled at the rate $1/T$ (if necessary, the initial time $t = 0$ can be moved). A caution to such sampling is that the detectability may be lost (that is, $C(t), A(t)$) is detectable but $(C, \bar{A})$ is not detectable, see [2], [10]). If this condition occurs, one needs to sample the output faster than $1/T$. In general, a faster output sampling would result in a more complicate (e.g., time varying or higher order) closed-loop sampled-data system which is beyond the scope of this note.

C. The ELQR Control Problem

Let $x_0(t) = E[x(t)|x(0)]$ denote the conditional mean of $x(t)$ on $x(0)$. By the stochastic closed-loop system (4), one has

$$x_0((k+1)T) = (\bar{A} + \Omega LC)x_0(kT). \tag{10}$$

Thus, from the viewpoint of the conditional mean, the transient response due to $x(0)$ can be considered as a deterministic regulation problem. One can define the following two performance indexes

$$J_d = \sum_{k=0}^{\infty} E[x_0^\tau(kT)(Q_1 + C^\tau L^\tau Q_2^\tau + Q_2LC + C^\tau L^\tau Q_3LC)x_0(kT)] \tag{11.a}$$

$$J_s = \lim_{k\to\infty} E[x^\tau(kT)(Q_1 + C^\tau L^\tau Q_2^\tau + Q_2LC + C^\tau L^\tau Q_3LC)x(kT)] \tag{11.b}$$

where $Q_1 \in R^{n\times n}$, $Q_2 \in R^{nxmf}$, $Q_3 \in R^{mf\times mf}$, and Q_1 and Q_3 are positive semidefinite. Notice that J_d serves as a measure of the transient response due to $x(0)$, and J_s serves as a measure of the steady-state noise regulation. Corresponding to the indexes, the following two control problems can be considered.

LQR control problem: Find a gain L, such that $\bar{A} + \Omega LC$ is stable (i.e., all eigenvalues lie inside the unit complex circle), and the index J_d subject to the deterministic closed-loop system (10) is minimized.

SLQR control problem: Find a gain L, such that $\bar{A} + \Omega LC$ is stable, and the index J_s subject to the stochastic closed-loop system (4) is minimized.

In general, the LQR and the SLQR control problems cannot have the same solution unless the system has complete state information (as will be clear later). To balance the advantages of LQR and SLQR control problems, an extended control problem can be introduced as follows.

ELQR control problem: Find a gain L, such that $\bar{A} + \Omega LC$ is stable, and the index $J = \gamma_d J_d + \gamma_s J_s$ (γ_d and γ_s are nonnegative real numbers) subject to the deterministic closed-loop system (10) and the stochastic closed-loop system (4) is minimized.

Remark 2: It can be shown (see [1]) that a continuous-time quadratic performance index subject to the multirate structure can be transformed to an equivalent discrete-time quadratic performance index, so that it does not lose the generality to select discrete-time quadratic performance indices for multirate controller designs of a linear continuous-time periodic system.

III. Solving the ELQR Control Problem

By the deterministic closed-loop system (10) and the stochastic closed-loop system (4), the general index $J = \gamma_d J_d + \gamma_s J_s$ equals to (see [12], [14], [11])

$$J = \gamma_d \operatorname{Tr} V_d E[x(0)x^\tau(0)] + \gamma_s \operatorname{Tr} \lambda_s(Q_1 + C^\tau L^\tau Q_2^\tau + Q_2LC + C^\tau L^\tau Q_3LC) \tag{12}$$

where V_d and λ_s are positive semidefinite matrices solved from the following Lyapunov equations, respectively,

$$(\bar{A} + \Omega LC)^\tau V_d(\bar{A} + \Omega LC) - V_d + (Q_1 + C^\tau L^\tau Q_2^\tau + Q_2LC + C^\tau L^\tau Q_3LC) = 0 \tag{13.a}$$

$$(\bar{A} + \Omega LC)\lambda_s(\bar{A} + \Omega LC)^\tau - \lambda_s + (S + \Omega LRL^\tau\Omega^\tau) = 0. \tag{13.b}$$

Now, let $\lambda_d \in R^{n\times n}$ be a positive semidefinite matrix solved from the following Lyapunov equation (if $\bar{A} + \Omega LC$ is stable, such a solution must exist):

$$(\bar{A} + \Omega LC)\lambda_d(\bar{A} + \Omega LC)^\tau - \lambda_d + E[x(0)x^\tau(0)] = 0. \tag{14}$$

By (14) and (13.a), one can obtain

$$\begin{aligned} \operatorname{Tr} V_d E[x(0)x^\tau(0)] &= \operatorname{Tr}\{-V_d(\bar{A} + \Omega LC)\lambda_d(\bar{A} + \Omega LC)^\tau + V_d\lambda_d\} \\ &= \operatorname{Tr}\{-\lambda_d(\bar{A} + \Omega LC)^\tau V_d(\bar{A} + \Omega LC) + \lambda_d V_d\} \\ &= \operatorname{Tr} \lambda_d(Q_1 + C^\tau L^\tau Q_2^\tau + Q_2LC + C^\tau L^\tau Q_3LC). \end{aligned} \tag{15}$$

Thus, by defining $\lambda = \gamma_d\lambda_d + \gamma_s\lambda_s$, the index (12) can be rearranged by

$$J = \operatorname{Tr} \lambda(Q_1 + C^\tau L^\tau Q_2^\tau + Q_2LC + C^\tau L^\tau Q_3LC). \tag{16}$$

On the other hand, by combining (13.b) and (14), one can obtain

$$(\bar{A} + \Omega LC)\lambda(\bar{A} + \Omega LC)^\tau - \lambda + \gamma_d E[x(0)x^\tau(0)] + \gamma_s(S + \Omega LRL^\tau\Omega^\tau) = 0. \tag{17}$$

Main Theorem: A necessary condition for L to solve the ELQR control problem is that L satisfies the following coupled matrix equations:

$$(\bar{A} + \Omega LC)^\tau V(\bar{A} + \Omega LC) - V + (Q_1 + C^\tau L^\tau Q_2^\tau + Q_2LC + C^\tau L^\tau Q_3LC) = 0 \tag{18.a}$$

$$(\bar{A} + \Omega LC)\lambda(\bar{A} + \Omega LC)^\tau - \lambda + \gamma_d E[x(0)x^\tau(0)] + \gamma_s(S + \Omega LRL^\tau\Omega^\tau) = 0 \tag{18.b}$$

$$\Omega^\tau V(\bar{A} + \Omega LC)\lambda C^\tau + \gamma_s\Omega^\tau V\Omega LR + (Q_2^\tau\lambda C^\tau + Q_3LC\lambda C^\tau) = 0 \tag{18.c}$$

where $V \in R^{n\times n}$ and $\lambda \in R^{n\times n}$ are positive semidefinite matrices.

Proof: To minimize the index (16) subject to (17), one can give an augmented cost function as follows (see [7]):

$$J_c = \mathrm{Tr}\,\lambda(Q_1 + C^\tau L^\tau Q_2^\tau + Q_2 LC + C^\tau L^\tau Q_3 LC) + \mathrm{Tr}\,V\big\{(\bar{A} + \Omega LC)\lambda(\bar{A} + \Omega LC)^\tau - \lambda + \gamma_d E[x(0)x^\tau(0)] + \gamma_s(S + \Omega LRL^\tau\Omega^\tau)\big\} \quad (19)$$

where the multiplier $V \in R^{n\times n}$ is positive semidefinite. By taking $dJ_c/d\lambda = 0$ and $dJ_c/dV = 0$, one obtains (18.a) and (18.b), respectively. Also, by taking $dJ_c/dL = 0$, one obtains (18.c). Hence, the theorem is proved. □

Now, let $\gamma_d E[x(0)x^\tau(0)] + \gamma_s S$ be positive definite, the following two corollaries are easily obtained from main theorem.

Corollary 1: If $\mathrm{rank}[C] = n$, $R = 0$, Q_3 is positive definite, and $(Q_1, \bar{A}, \Omega)$ is stabilizable and detectable, then there exists a unique solution of the ELQR control problem. Moreover, the unique solution is given by

$$L = -(\Omega^\tau V\Omega + Q_3)^{-1}(\Omega^\tau V\bar{A} + Q_2^\tau)C^{-1} \quad (20.a)$$

where V is a positive semidefinite matrix solved from the following discrete-time algebraic Riccati equation:

$$\bar{A}^\tau V\bar{A} - V + Q_1 - (\Omega^\tau V\bar{A} + Q_2^\tau)^\tau(\Omega^\tau V\Omega + Q_3)^{-1}\cdot(\Omega^\tau V\bar{A} + Q_2^\tau) = 0. \quad (20.b)$$

Proof: Since $\mathrm{rank}[\lambda C^\tau] = n$ and $R = 0$, (18.c) can be reduced to

$$\Omega^\tau V(\bar{A} + \Omega LC) + Q_2^\tau + Q_3 LC = 0 \quad (21)$$

which leads to (20.a). By substituting (20.a) into (18.a), one obtains (20.b). Thus, the necessity of (20) is proved. Besides, by (18.a) and (18.b), one has $J = \mathrm{Tr}\,\lambda(Q_1 + C^\tau L^\tau Q_2^\tau + Q_2 LC + C^\tau L^\tau Q_3 LC) = \mathrm{Tr}\,V(\gamma_d E[x(0)x^\tau(0)] + \gamma_s S)$. It is known (see [9], [15]) that the algebraic Riccati equation (20) exists a unique stable minimal solution. Hence, the corollary is proved. □

Corollary 2: If $\mathrm{rank}[\Omega] = n$, $\mathrm{rank}[C] = r$, Q_1 is positive definite, $Q_2 = 0$, $Q_3 = 0$, and $(C, \bar{A})$ is detectable, then the ELQR control problem can be solved by

$$L = -\Omega^\dagger\bar{A}\lambda C^\tau(C\lambda C^\tau + \gamma_s R)^{-1} \quad (22.a)$$

where $\Omega^\dagger \in R^{mf\times n}$ is a matrix to satisfy $\Omega\Omega^\dagger = I_n$, and λ is a positive definite matrix solved from the following discrete-time algebraic Riccati equation:

$$\bar{A}\lambda\bar{A}^\tau - \lambda + \gamma_d E[x(0)x^\tau(0)] + \gamma_s S - \bar{A}\lambda C^\tau(C\lambda C^\tau + \gamma_s R)^{-1}C\lambda\bar{A}^\tau = 0. \quad (22.b)$$

Proof: Since $\mathrm{rank}[\Omega^\tau V] = n$, $Q_2 = 0$ and $Q_3 = 0$, (18.c) can be reduced to

$$(\bar{A} + \Omega LC)\lambda C^\tau + \gamma_s\Omega LR = 0 \quad (23)$$

which leads to (22.a). By substituting (22.a) into (18.b), one obtains (22.b). Hence, the necessity of (22) is proved. On the other hand, the algebraic Riccati equation (22) exists a stable minimal solution. Hence, the corollary is proved.

IV. Solving the LQR and SLQR Problems

A. LQR Control Problem

Let $\gamma_d = 1$ and $\gamma_s = 0$, the main theorem leads to the following result.

Corollary 3: A necessary condition for L to solve the LQR control problem is that L satisfies the following coupled matrix equations:

$$(\bar{A} + \Omega LC)^\tau V_d(\bar{A} + \Omega LC) - V_d + (Q_1 + C^\tau L^\tau Q_2^\tau + Q_2 LC + C^\tau L^\tau Q_3 LC) = 0 \quad (24.a)$$

$$(\bar{A} + \Omega LC)\lambda_d(\bar{A} + \Omega LC)^\tau - \lambda_d + E[x(0)x^\tau(0)] = 0 \quad (24.b)$$

$$\Omega^\tau V_d(\bar{A} + \Omega LC)\lambda_d C^\tau + Q_2^\tau\lambda_d C^\tau + Q_3 LC\lambda_d C^\tau = 0 \quad (24.c)$$

where $V_d \in R^{n\times n}$ and $\lambda_d \in R^{n\times n}$ are positive semidefinite matrices.

Let $E[x(0)x^\tau(0)]$ be positive definite, corollaries 1 and 2 lead to the following results.

Corollary 4: If $\mathrm{rank}[C] = n$, Q_3 is positive definite, and $(Q_1, \bar{A}, \Omega)$ is stabilizable and detectable, then there exists a unique solution of LQR control problem. Moreover, the unique solution is given by

$$L = -(\Omega^\tau V_d\Omega + Q_3)^{-1}(\Omega^\tau V_d\bar{A} + Q_2^\tau)C^{-1} \quad (25.a)$$

where V_d is a positive semidefinite matrix solved from the following discrete-time algebraic Riccati equation:

$$\bar{A}^\tau V_d\bar{A} - V_d + Q_1 - (\Omega^\tau V_d\bar{A} + Q_2^\tau)^\tau(\Omega^\tau V_d\Omega + Q_3)^{-1}\cdot(\Omega^\tau V_d\bar{A} + Q_2^\tau) = 0. \quad (25.b)$$

Corollary 5: If $\mathrm{rank}[\Omega] = n$, $\mathrm{rank}[C] = r$, Q_1 is positive definite, $Q_2 = 0$, $Q_3 = 0$, and $(C, \bar{A})$ is detectable, then the LQR control problem can be solved by

$$L = -\Omega^\dagger\bar{A}\lambda_d C^\tau(C\lambda_d C^\tau)^{-1} \quad (26.a)$$

where $\Omega^\dagger \in R^{mf\times n}$ is a matrix to satisfy $\Omega\Omega^\dagger = I_n$, and λ_d is a positive definite matrix solved from the following discrete-time algebraic Riccati equation:

$$\bar{A}\lambda_d\bar{A}^\tau - \lambda_d + E[x(0)x^\tau(0)] - \bar{A}\lambda_d C^\tau(C\lambda_d C^\tau)^{-1}C\lambda_d\bar{A}^\tau = 0. \quad (26.b)$$

B. SLQR Control Problem

Let $\gamma_d = 0$ and $\gamma_s = 1$, the main theorem leads to the following result.

Corollary 6: A necessary condition for L to solve the SLQR control problem is that L satisfies the following coupled matrix equations:

$$(\bar{A} + \Omega LC)\lambda_s(\bar{A} + \Omega LC)^\tau - \lambda_s + (S + \Omega LRL^\tau\Omega^\tau) = 0 \quad (27.a)$$

$$(\bar{A} + \Omega LC)^\tau V_s(\bar{A} + \Omega LC) - V_s + (Q_1 + C^\tau L^\tau Q_2^\tau + Q_2 LC + C^\tau L^\tau Q_3 LC) = 0 \quad (27.b)$$

$$\Omega^\tau V_s(\bar{A} + \Omega LC)\lambda_s C^\tau + \Omega^\tau V_s\Omega LR + Q_2^\tau\lambda_s C^\tau + Q_3 LC\lambda_s C^\tau = 0 \quad (27.c)$$

where $V_s \in R^{n\times n}$ and $\lambda_s \in R^{n\times n}$ are positive semidefinite matrices.

Let S be positive definite, corollaries 1 and 2 lead to the following results.

Corollary 7: If $\mathrm{rank}[C] = n$, $R = 0$, Q_3 is positive definite, and $(Q_1, \bar{A}, \Omega)$ is stabilizable and detectable, then there exists a unique solution of SLQR control problem. Moreover, the unique solution is given by

$$L = -(\Omega^T V_s \Omega + Q_3)^{-1}(\Omega^T V_s \bar{A} + Q_2^T)C^{-1} \quad (28.a)$$

where V_s is a positive semidefinite matrix solved from the following discrete-time algebraic Riccati equation:

$$\bar{A}^T V_s \bar{A} - V_s + Q_1 - (\Omega^T V_s \bar{A} + Q_2^T)^T (\Omega^T V_s \Omega + Q_3)^{-1} \cdot(\Omega^T V_s \bar{A} + Q_2^T) = 0. \quad (28.b)$$

Corollary 8: If $\operatorname{rank}[\Omega] = n$, $\operatorname{rank}[C] = r$, Q_1 is positive definite, $Q_2 = 0$, $Q_3 = 0$, and $(C, \bar{A})$ is detectable, then the SLQR control problem can be solved by

$$L = -\Omega^\dagger \bar{A}\lambda_s C^T (C\lambda_s C^T + R)^{-1} \quad (29.a)$$

where $\Omega^\dagger \in R^{mf\times n}$ is a matrix to satisfy $\Omega\Omega^\dagger = I_n$, and λ_s is a positive definite matrix solved from the following discrete-time algebraic Riccati equation:

$$\bar{A}\lambda_s \bar{A}^T - \lambda_s + S - \bar{A}\lambda_s C^T (C\lambda_s C^T + R)^{-1} C\lambda_s \bar{A}^T = 0. \quad (29.b)$$

Remark 3: Although it is necessary that all the solutions of LQR, SLQR, and ELQR control problems be solved from a set of coupled matrix equations, however, an admissible solution may not exist even if the set $\aleph = \{L | \bar{A} + \Omega LC \text{ is stable}\}$ is not empty. To guarantee the ELQR algorithm to be solvable, a simple way is by assuming Q_1, Q_3, and $\gamma_d E[x(0)x(0)^T] + \gamma_s S$ to be positive definite. Notice that by (16) and (17), the index J can be expressed by

$$J = \operatorname{Tr} \lambda (Q_1 + C^T L^T Q_2^T + Q_2 LC + C^T L^T Q_3 LC) \quad (30)$$

where

$$\lambda = \sum_{k=0}^{\infty} (\bar{A} + \Omega LC)^k \{\gamma_d E[x(0)x(0)^T] + \gamma_s (S + \Omega LRL^T \Omega^T)\} \cdot (\bar{A}^T + C^T L^T \Omega^T)^k \quad (31)$$

so that by assumption, a gain L with finite index must be bounded and contained in $\aleph$. Since for every compact subset # of $\aleph$, J is continuous and bounded in L over # (a proof can be found in [14]), so that by Weierstrass theorem, # contains a minimal point. Hence, it is clear that if and only if $\aleph$ is not empty, the ELQR control problem is solvable (the LQR and SLQR control problems are two special cases).

Remark 4: By main theorem and corollaries 3 and 6, it is clear that in general, the LQR, SLQR, and ELQR algorithms have distinct solutions. However, by corollaries 1, 4, and 7, all three control problems have the same solution if the system has complete state information.

V. Example

Consider a periodic system as follows:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix} u(t) \quad (32.a)$$

$$y_s(t) = [1 \quad 0] x(t) \quad (32.b)$$

$$y(kT) = y_s(kT) + v(kT) \quad (32.c)$$

where $v(kT)$ $k = 0, 1, 2, \cdots$ represents the measurement errors which are uniformly distributed in the interval $(-0.1 \ 0.1)$ (i.e., the correlation is $R = 0.0033$). Assume $T = 1$, $E[x(0)x^T(0)] = I_2$, $f = 2$, and a discrete-time performance index as follows:

$$J = \gamma_d E\left[\sum_{i=0}^{\infty} x_0^T(kT) Q_1 x_0(kT)\right] + \gamma_s \lim_{k\to\infty} E[x^T(kT) Q_1 x(kT)]. \quad (33)$$

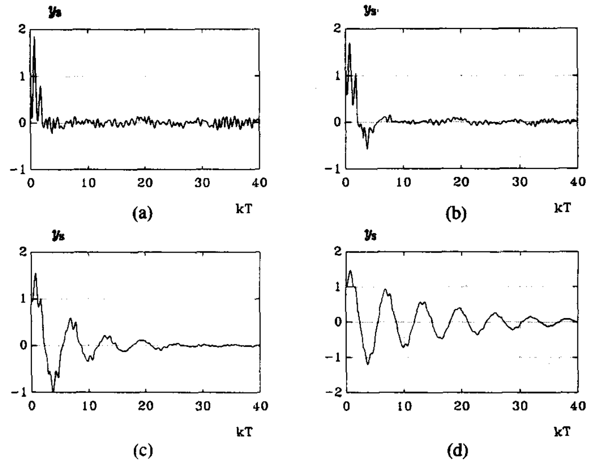

Fig. 2. The output responses of linear periodic system (32) with the multirate output feedback controller (35).

By (5), one has

$$\bar{A} = \begin{pmatrix} 0.5403 & 0.8415 \\ -0.8415 & 0.5403 \end{pmatrix} \quad \Omega = \begin{pmatrix} 0.1952 & -0.0845 \\ 0.1811 & -0.2525 \end{pmatrix}. \quad (34)$$

By (3), the multirate output feedback controller is taken by

$$u(kT + \theta) = \begin{cases} L_0 y(kT); & \text{for } 0 \le \theta < T/2 \\ L_1 y(kT); & \text{for } T/2 \le \theta < T \end{cases} \quad (35)$$

where the gain $L = [L_0 \ L_1]^T \in R^2$ is solved from the discrete-time algebraic Riccati equation (22). Four cases a): $\gamma_d = 1$, $\gamma_s = 0$, b): $\gamma_d = 1$, $\gamma_s = 600$, c): $\gamma_d = 1$, $\gamma_s = 6000$, d): $\gamma_d = 1$, $\gamma_s = 30000$ are solved. Their responses are also plotted in Fig. 2 (where $x(0)$ is given by $[1 \ 1]^T$ for simulation). Notice that case a) is the LQR case. Although this case can yield a fast response, however, noise interruption is the worst. Also notice that by increasing the ratio γ_s, the ability of noise rejection is improved but the response becomes slow.

VI. Conclusion

In this note, a modified optimal algorithm for multirate output feedback controllers of linear stochastic periodic systems is developed. By combining the discrete-time LQR and SLQR control problems as the ELQR control problem, one derives a general optimal algorithm to extend the LQR and the SLQR algorithms, simultaneously. Such an algorithm allows one to choose ratios on the LQR and the SLQR control problems, so that the presented multirate output feedback controller can balance the advantages between the optimal transient response of LQR control problem and the optimal steady-state noise regulation of SLQR control problem.

In general, all the solutions of LQR, SLQR, and ELQR algorithms are solved from a set of coupled matrix equations. Important special cases for which the coupled matrix equations can be reduced to a discrete-time algebraic Riccati equation are also discussed. The most significant case is the multirate state feedback control scheme derived by AL-Rahmani and Franklin [1], where the discrete-time quadratic performance index is transformed from a continuous-time quadratic performance index. However, complete state information may be a necessary condition for the reduction of coupled matrix equations if a continuous-time quadratic performance index is selected. This difficulty can be overcome by using the classical separation principle to give an optimal filter [8].

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Schur Stability of Interval Polynomials

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***Abstract*—In this note, we shall present a result for checking the Schur stability of interval polynomials. In particular, we are interested in the number of critical vertex and edge polynomials that are sufficient for inferring robust Schur stability.**

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NOTATION:

$\mathbb{C}$	The complex plane.
P^n	A family of real coefficient nth-order interval polynomials.
$P^n(z_o)$	The Nyquist image of P^n at $z_o \in \mathbb{C}$.
V_P^n	The vertex set of P^n.
E_P^n	The exposed edge set of P^n.
$\bar{E}_P^n$	A special subset of E_P^n.
$\bar{n}$	$:= (\lfloor n/2 \rfloor)$, the truncation of $(n/2)$.
D	A simply-connected region which is symmetrical with respect to the real axis.

I. Introduction

The main contribution of the Kharitonov's result [1] is the simplicity with which one can check the Hurwitz property of a family of interval polynomials. Several extensions of the Kharitonov-type results to other stability regions are also interesting and useful [2]–[5]. Unfortunately, the Kharitonov-type result does not apply to Schur stability [4], [5] unless the order of the interval polynomials is less than six [7]. Hence, in general one has to rely on the edge theorem [6], [9] to check the Shur property of interval polynomials. However, the computational load for checking the edges can be prohibitive since one generally has to examine $(n+1)2^n$ edges! Recently, Kraus *et al.* [8] have shown that the number of edges to be checked can be reduced to KH where $K = 2^{n-\bar{n}}$ and $H = (\bar{n}+1)2^{\bar{n}}$. In this note, we shall show that the critical edges can be drastically reduced to the order of $2n^2$.

II. Preliminaries

For real coefficient interval polynomials, our stability region D is symmetrical with respect to the real axis. This means that we only have to consider the Nyquist image of the whole family of polynomials on the upper plane since the Nyquist image on the lower plane is just the mirror image of the Nyquist image on the upper plane. For extensions to a family of complex coefficient interval polynomials, we need to derive two sets of bounding polynomials, one for the upper plane and one for the lower plane. Our first preliminary result states that the Nyquist image of P^n at any $z_o \in \mathbb{C}$ is a $2(n+1)$-sided parpolygon.

Proposition 2.1: Let P^n denote a family of real coefficient nth-order interval polynomials, i.e.,

$$P^n = \left\{ p(z) : p(z) = \sum_{k=0}^{n} a_k z^k \right\} \tag{2.1}$$

where, in general, $a_k \in [a_k^-, a_k^+]$, $k = 0, 1, \cdots, n$, and $0 \notin [a_n^-, a_n^+]$. Then the Nyquist image of P^n at any $z_o \in \mathbb{C}$, i.e., $P^n(z_o)$, is a $2(n+1)$-sided convex parpolygon (may be degenerate) whose edges are at angles $k\theta_{z_o}$, $k = 0, 1, \cdots, n$, with respect to the positive real axis; θ_{z_o} denotes the phase of z_o.

Proof: For any $z_o = \exp(j\theta_{z_o})x \in \mathbb{C}$, the Nyquist image of P^n is given by

$$P(\exp(j\theta_{z_o})x) = \sum_{k=0}^{n} a_k(\cos k\theta_{z_o} + j \sin k\theta_{z_o})x^k.$$

Clearly, the coefficients of $P(\exp(j\theta_{z_o})x)$ always lie on phase lines with angles $k\theta_{z_o}$, $k = 0, 1, \cdots, n$, in the complex plane. Hence, $P(\exp(j\theta_{z_o})x)$ is a $2(n+1)$-sided parpolygon whose edges are at angles $k\theta_{z_o}$, $k = 0, 1, \cdots, n$, with respect to the positive real axis. To identify the critical vertexes, we basically select a_k to be either a_k^- or a_k^+ to maximize (minimize) the imaginary part of $\exp(-jk_1\theta_{z_o})P^n(\exp(j\theta_{z_o})x)$ for each $k_1 \in$