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Brief Paper

Modified stochastic Luenberger observers☆

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Abstract

A modified stochastic Luenberger observer (MSLO) structure is proposed to recover the optimal performance of the coventional SLO for obtaining full-state estimates in linear discrete-time stochastic systems. The optimal MSLO (OMSLO) which gives the MMSE estimates is derived by using the general two-stage Kalman filter. A reduced-order form of the OMSLO is also proposed for systems having singular measurement noises. The connection between the OMSLO and the optimal minimal-order observer of Leondes and Novak is also shown. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Stochastic Luenberger observer; Minimal-order observer; Two-stage filter

1. Introduction

The general state estimation problem in linear discrete-time stochastic systems may be solved by the stochastic Luenberger observer (SLO) (Aoki & Huddle, 1967) which is used as an alternative to the well-known Kalman filter (KF) (Kalman, 1960). The advantages of using the SLO are due to its numerical and computational superiority associated with the real-time implementation. However, the SLO is in general, a suboptimal estimator (Leondes & Novak, 1972). The suboptimal essence of the SLO is due to the fact that it uses the current measurement as part of system state, and then tries to estimate the remaining state optimally. The SLO is optimal in the minimum-mean-square-error (MMSE) sense when applied to deterministic environment since there is no need to estimate those states which are known perfectly. However, when applied to stochastic systems, the SLO may suffer from performance degradation. The reason is that the current measurement no longer stands for the optimal estimate of the corresponding system state. Thus, the current measurement should be processed to compensate for the noise effect in order to get the optimal performance.

To seek out optimal performance of the SLO, researchers have tried to apply reduced-order stochastic observers (for example, Leondes & Novak, 1974; Tse & Athans, 1970; Tse, 1973; Fairman, 1977; Fogel & Huang, 1980; Halevi, 1989) to handle singular measurement problems. The above works adopt the structure of the observer–estimator where the Luenberger observer (LO) (Luenberger, 1964) is first defined and then the optimal observer parameters are chosen by some optimization methods to minimize the estimation error. Another problem of developing a reduced-order filter to estimate only a part of the state vector can also be solved (Sims & Asher, 1978; Nagpal, Helmick & Sims, 1987). In these results, the LO for the state to be estimated is firstly formed, and then the remaining parameters are optimized with respect to the noise in the system. It was shown that the LO and the KF are special cases of the obtained minimal-order observer (Dwarakanath, 1982) or optimum reduced-order filter (Nagpal et al., 1987). A different approach which attempts to supplement the current measurement with additional delayed measurements to derive an optimal minimal-order observer in general conditions can be found in Priel, Soroka and Shaked (1991). This new reduced-order observer is derived by using the differentiation-transformation block (DTB) construction method of Soroka and Shaked

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(1988), which was developed originally for the continuous-time case. The measurement vector is preprocessed by applying forward shift operator and linear combinations of the new measurements in order to derive a vector of maximum number of independent noise-free measurements. The obtained observer may have a lower order than previous ones, but it is basically a fixed-lag smoother.

All the above-mentioned efforts to improve the performance of the SLO are concerned with the reduced-order observer designing problem. To the best of the knowledge of the authors, few results are focused on reducing the complexity of the SLO for obtaining full-state estimates in linear discrete-time stochastic systems. In this paper, we wish to address the complexity issue of the SLO in view of the computational burden. Furthermore, we consider observer designing problems in a completely new direction: to recover the optimal estimation performance of the SLO for obtaining full-state estimates in linear discrete-time stochastic systems. In this regard, we introduce a modified SLO (MSLO) scheme that is based on modifying the current measurement via a dynamical equation to reduce the effect of the measurement noise. This idea of reducing the estimating error can also be found in Priel et al. (1991). Through a linear transformation of the KF, the optimal MSLO (OMSLO) which gives the MMSE can be derived. Unlike the derivations in the previously mentioned optimization approach, this new approach is characterized by its simplicity in derivation, which is the result of the intuitive two-stage decoupling method (Friedland, 1969; Hsieh & Chen, 1999).

Thus, the aims of this paper are (1) to derive the OMSLO which gives the optimal performance in the MMSE sense when applied to linear discrete-time stochastic systems; (2) to derive the optimal reduced-order MSLO (OROMSLO) which gives the MMSE estimate when applied to systems having singular measurement noises, and (3) to address the complexity issue of the proposed OMSLO. By means of a proper preprocessing of the measurement matrix, the OMSLO can be readily derived from the previously proposed GTSKF (general two-stage Kalman filter) (Hsieh & Chen, 2000). The connection between the OMSLO and the optimal minimal-order observer of Leondes and Novak (1972) is also shown. With the OMSLO at hand, the OROMSLO can be easily derived by a suitable preprocessing of the measurement error covariance matrix and some matrix manipulations. This approach of deriving the singular case by using the nonsingular result can also be found in Bekir's paper (Bekir, 1988). Owing to the decoupling structure of the GTSKF, the proposed OMSLO is shown to be less complex than the conventional SLO.

This paper is organized as follows. The problem of interest and the structure of the MSLO are stated in Section 2. In Section 3, the GTSKF of Hsieh and Chen (2000) is presented to facilitate the derivation of the proposed optimal observers. In Section 4, the OMSLO serves as a special case of the GTSKF. The relationship between this new observer and the observer of Leondes and Novak (1972) is also shown. In Section 5, a reduced-order form of the OMSLO is presented as an optimal reduced-order observer when applied to singular measurement cases. In Section 6, the computational load of the OMSLO is analyzed to demonstrate the feasibility of the new observer structure as compared to the conventional one. In Section 7, a simulation example is used to illustrate the performance of the proposed observer. Section 8 gives the conclusions.

2. Modified stochastic Luenberger observer

Consider the following discrete-time system:

$$x_{k+1} = A_k x_k + B_k u_k + w_k, \tag{1}$$

$$y_k = C_k x_k + \eta_k, \tag{2}$$

where $x_k \in R^n$ is the system state, $u_k \in R^q$ is the control vector, and $y_k \in R^m$ is the measurement vector. Matrices A_k, B_k, and C_k have the appropriate dimensions (rank of C_k is $m < n$). The process noises w_k and the measurement noise η_k are zero-mean white Gaussian sequences with the following covariances: $E\{w_k(w_l)'\} = Q_k\delta_{kl}$, $E\{\eta_k(\eta_l)'\} = R_k\delta_{kl}$, and $E\{w_k(\eta_l)'\} = 0$, where $'$ denotes transpose and δ_{kl} denotes the Kronecker delta function. The initial state x_0 is assumed to be uncorrelated with the white noise sequences w_k and η_k, and is assumed to be Gaussian random variables with $E\{x_0\} = \bar{x}_0$ and $Cov\{x_0\} = \bar{P}_0$.

The following stochastic Luenberger observer (SLO) may be used as a reduced-order estimator to estimate the system state:

$$z_{k+1} = F_k z_k + D_k y_k + G_k u_k, \tag{3}$$

$$\hat{x}_k = T_k z_k + E_k y_k, \tag{4}$$

where

$$F_k = \bar{T}_{k+1} A_k T_k, \tag{5}$$

$$D_k = \bar{T}_{k+1} A_k E_k, \tag{6}$$

$$G_k = \bar{T}_{k+1} B_k \tag{7}$$

and $\bar{T}_k$, T_k, and E_k are chosen to satisfy the constraint $T_k\bar{T}_k + E_k C_k = I_n$ (Leondes & Novak, 1974). However, the above SLO is suboptimal in general since it uses noisy measurements as part of system state estimates and then tries to optimize the remaining part. To obtain the optimal performance, the SLO should be modified to account for the noise effect. This is achieved by replacing y_k with $\tilde{y}_k$ which is augmented by an mth order system with y_k as input. Hence, the modified SLO (MSLO) is expressed by

$$z_{k+1} = F_k z_k + D_k \tilde{y}_k + G_k u_k, \tag{8}$$

$$\hat{x}_k = T_k z_k + E_k \tilde{y}_k. \tag{9}$$

Then, the problem remains to determine (1) the dynamics of $\tilde{y}_k$, and (2) the optimal observer gain such that the MSLO can give an MMSE estimate.

Recently (2000), the authors have applied a two-stage decoupling technique to extend Friedland's two-stage filter (1969) in general conditions; furthermore, we show in this paper that the obtained two-stage filter which will be presented in the next section can also be used to solve the above-mentioned modified observer design problem. The complexity introduced by those modified terms is shown to be not excessive as compared to the computational load of the conventional SLO.

3. The general two-stage Kalman filter

The key idea for developing a two-stage filter is based on the two-stage transformation that makes the covariance matrices of the KF block diagonal (Hsieh & Chen, 1999). First, consider the stochastic system given by (1) and (2). Then, the two-stage filter is obtained by applying the following two-stage transformation:

$$T(M) = \begin{bmatrix} I_{n-p} & M \\ 0 & I_p \end{bmatrix}$$

to the corresponding KF, where p is a partition parameter, and its range is given by $m \le p < n$. Thus, one has

$$\bar{x}_{k|k-1} = T(-U_k) x_{k|k-1}, \tag{10}$$

$$\bar{x}_{k|k} = T(-V_k) x_{k|k}, \tag{11}$$

$$\bar{P}_{k|k-1} = T(-U_k) P_{k|k-1} T'(-U_k), \tag{12}$$

$$\bar{K}_k = T(-V_k) K_k, \tag{13}$$

$$\bar{P}_{k|k} = T(-V_k) P_{k|k} T'(-V_k), \tag{14}$$

where

$$\bar{x} = \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix}, \qquad \bar{K} = \begin{bmatrix} \bar{K}^1 \\ \bar{K}^2 \end{bmatrix}, \qquad \bar{P} = \begin{bmatrix} \bar{P}^1 & 0 \\ 0 & \bar{P}^2 \end{bmatrix}$$

denote the states, the gains, and the error covariances of the two-stage decoupled subfilters, respectively. The blending matrices U_k and V_k are left to be determined to ensure that (12) and (14) are satisfied, respectively.

Second, using the *two-step iterative substitution* method (Hsieh & Chen, 1999), (10)–(14) become

$$\bar{x}_{k|k-1} = T(-U_k)(A_{k-1} T(V_{k-1}) \bar{x}_{k-1|k-1} + B_{k-1} u_{k-1}), \tag{15}$$

$$\bar{x}_{k|k} = T(U_k - V_k) \bar{x}_{k|k-1} + \bar{K}_k (y_k - C_k T(U_k) \bar{x}_{k|k-1}), \tag{16}$$

$$\bar{P}_{k|k-1} = T(-U_k)(A_{k-1} T(V_{k-1}) \bar{P}_{k-1|k-1} T'(V_{k-1}) A'_{k-1} + Q_{k-1}) T'(-U_k), \tag{17}$$

$$\bar{K}_k = T(U_k - V_k) \bar{P}_{k|k-1} (C_k T(U_k))' \{C_k T(U_k) \bar{P}_{k|k-1} T'(U_k) C'_k + R_k\}^{-1}, \tag{18}$$

$$\bar{P}_{k|k} = (T(U_k - V_k) - \bar{K}_k C_k T(U_k)) \bar{P}_{k|k-1} T'(U_k - V_k). \tag{19}$$

Defining the following notations:

$$T_k = [I_{n-p} \quad 0]', \qquad E_k = [U'_k \quad I_p]',$$
$$\bar{T}_k = [I_{n-p} \quad -U_k], \qquad \tilde{T}_k = [0 \quad I_p] \tag{20}$$

one can expand (15)–(19) into the following two-stage decoupled subfilter 1:

$$\bar{x}^1_{k|k-1} = F_{k-1} \bar{x}^1_{k-1|k-1} + D_{k-1} \bar{x}^2_{k-1|k-1} + G_{k-1} u_{k-1}, \tag{21}$$

$$\bar{x}^1_{k|k} = \bar{x}^1_{k|k-1} + \bar{K}^1_k (y_k - C_k T_k \bar{x}^1_{k|k-1}), \tag{22}$$

$$\bar{P}^1_{k|k-1} = F_{k-1} \bar{P}^1_{k-1|k-1} F'_{k-1} + D_{k-1} \bar{P}^2_{k-1|k-1} D'_{k-1} + \bar{T}_k Q_{k-1} \bar{T}'_k, \tag{23}$$

$$\bar{K}^1_k = \bar{P}^1_{k|k-1} (C_k T_k)' \{C_k T_k \bar{P}^1_{k|k-1} (C_k T_k)' + R_k\}^{-1}, \tag{24}$$

$$\bar{P}^1_{k|k} = (I - \bar{K}^1_k C_k T_k) \bar{P}^1_{k|k-1}, \tag{25}$$

and the following two-stage decoupled subfilter 2:

$$\bar{x}^2_{k|k-1} = L_{k-1} \bar{x}^1_{k-1|k-1} + M_{k-1} \bar{x}^2_{k-1|k-1} + N_{k-1} u_{k-1}, \tag{26}$$

$$\bar{x}^2_{k|k} = \bar{x}^2_{k|k-1} + \bar{K}^2_k (y_k - C_k T_k \bar{x}^1_{k|k-1} - C_k E_k \bar{x}^2_{k|k-1}), \tag{27}$$

$$\bar{P}^2_{k|k-1} = L_{k-1} \bar{P}^1_{k-1|k-1} L'_{k-1} + M_{k-1} \bar{P}^2_{k-1|k-1} M'_{k-1} + \tilde{T}_k Q_{k-1} \tilde{T}'_k, \tag{28}$$

$$\bar{K}^2_k = \bar{P}^2_{k|k-1} (C_k E_k)' \{C_k E_k \bar{P}^2_{k|k-1} (C_k E_k)' + C_k T_k \bar{P}^1_{k|k-1} (C_k T_k)' + R_k\}^{-1}, \tag{29}$$

$$\bar{P}^2_{k|k} = (I - \bar{K}^2_k C_k E_k) \bar{P}^2_{k|k-1}, \tag{30}$$

where

$$[F_k \quad D_k] = \bar{T}_{k+1} \begin{bmatrix} H_k & S_k \\ L_k & M_k \end{bmatrix}, \tag{31}$$

$$\begin{bmatrix} H_k & S_k \\ L_k & M_k \end{bmatrix} = A_k T(V_k), \tag{32}$$

$$G_k = \bar{T}_{k+1} B_k, \tag{33}$$

$$N_k = \tilde{T}_{k+1} B_k. \tag{34}$$

The blending matrices U_k and V_k are determined to satisfy the following two constraints:

$$0 = F_{k-1}\bar{P}^1_{k-1|k-1}L'_{k-1} + D_{k-1}\bar{P}^2_{k-1|k-1}M'_{k-1} + \bar{T}_k Q_{k-1}\tilde{T}'_k, \quad (35)$$

$$0 = U_k - V_k - \bar{K}^1_k C_k E_k, \quad (36)$$

and can be calculated as follows:

$$U_k = P^{12}_{k|k-1}(\bar{P}^2_{k|k-1})^{-1}, \quad (37)$$

$$V_k = U_k - \bar{K}^1_k C_k E_k, \quad (38)$$

where

$$P^{12}_{k|k-1} = H_{k-1}\bar{P}^1_{k-1|k-1}L'_{k-1} + S_{k-1}\bar{P}^2_{k-1|k-1}M'_{k-1} + T'_k Q_{k-1}\tilde{T}'_k. \quad (39)$$

Remark 1. To guarantee a unique solution of (35), it is assumed that the error covariance $\tilde{T}_k Q_{k-1}\tilde{T}'_k$ of the process noise of the two-stage decoupled subfilter 2 is positive definite. Thus, the covariance matrix $\bar{P}^2_{k|k-1} > 0$ and the matrix inverse in (37) exists. This is mostly encountered in physical problems due to the presence of modeling error. However, for the case where the measurement noise is a time-wise correlated sequence, then the covariance matrix may be singular. In such cases, (37) should be modified by

$$U_k = P^{12}_{k|k-1}(\bar{P}^2_{k|k-1})^{+}, \quad (40)$$

where $(M)^+$ is the *Moore-penrose* pseudo-inverse of M as suggested in Leondes and Novak (1972). To simplify the following discussions, we only consider the former case in this paper.

To simplify the complexity of the above two-stage decoupled subfilters, some necessary assumptions must be made about the measurement matrix C_k. Without loss of generality, it will be assumed that the measurement matrix C_k has been preprocessed to have the form

$$C_k = [0 \quad I_m]. \quad (41)$$

If it is not, then a linear transformation can be made to achieve this under the assumption that C_k is of full-rank (Leondes & Novak, 1972). Then, using the specific form of the measurement equation (41), one obtains

$$C_k T_k = 0, \qquad C_k E_k = \bar{C}_k = [0_{m\times(p-m)} \quad I_m]. \quad (42)$$

Using (42), the two-stage decoupled subfilter 1 given by (21)–(25) is simplified to

$$\bar{x}^1_{k|k} = F_{k-1}\bar{x}^1_{k-1|k-1} + D_{k-1}\bar{x}^2_{k-1|k-1} + G_{k-1}u_{k-1}, \quad (43)$$

$$\bar{P}^1_{k|k} = F_{k-1}\bar{P}^1_{k-1|k-1}F'_{k-1} + D_{k-1}\bar{P}^2_{k-1|k-1}D'_{k-1} + \bar{T}_k Q_{k-1}\bar{T}'_k, \quad (44)$$

the two-stage decoupled subfilter 2 given by (26)–(30) is simplified to

$$\bar{x}^2_{k|k} = (I - \bar{K}^2_k\bar{C}_k)(L_{k-1}\bar{x}^1_{k-1|k-1} + M_{k-1}\bar{x}^2_{k-1|k-1} + N_{k-1}u_{k-1}) + \bar{K}^2_k y_k, \quad (45)$$

$$\bar{P}^2_{k|k} = (I - \bar{K}^2_k\bar{C}_k)\bar{P}^2_{k|k-1}, \quad (46)$$

$$\bar{K}^2_k = \bar{P}^2_{k|k-1}\bar{C}'_k(\bar{C}_k\bar{P}^2_{k|k-1}\bar{C}'_k + R_k)^{-1}, \quad (47)$$

$$\bar{P}^2_{k|k-1} = L_{k-1}\bar{P}^1_{k-1|k-1}L'_{k-1} + M_{k-1}\bar{P}^2_{k-1|k-1}M'_{k-1} + \tilde{T}_k Q_{k-1}\tilde{T}'_k, \quad (48)$$

and the blending matrices are given by

$$V_k = U_k = P^{12}_{k|k-1}(\bar{P}^2_{k|k-1})^{-1}. \quad (49)$$

Finally, using (11), (14), and (49), the Kalman estimate can be reconstructed by the following general two-stage Kalman filter (GTSKF):

$$x_{k|k} = T_k\bar{x}^1_{k|k} + E_k\bar{x}^2_{k|k}, \quad (50)$$

$$P_{k|k} = T_k\bar{P}^1_{k|k}T'_k + E_k\bar{P}^2_{k|k}E'_k, \quad (51)$$

with the following initial conditions:

$$\bar{x}_0 = T_0\bar{x}^1_{0|0} + E_0\bar{x}^2_{0|0},$$
$$\bar{P}_0 = T_0\bar{P}^1_{0|0}T'_0 + E_0\bar{P}^2_{0|0}E'_0. \quad (52)$$

4. Optimal modified stochastic Luenberger observers

The proposed modified stochastic Luenberger observer (MSLO) is a new structure which intends to solve the suboptimal problem presented in the conventional stochastic Luenberger observer (SLO). The basic idea to solve this problem is to replace noisy measurements, which stands for part of the state estimate, with an optimal filter, which has order "m", to compensate for the noise effect. Through the aid of the GTSKF presented in the preceding section, the optimal MSLO (OMSLO) which gives the MMSE estimate of the system state can be derived.

By setting the partition parameter p of the GTSKF to m, one obtains from (42) that $\bar{C}_k = I_m$. Using this result and the following substitutions:

$$z_k = \bar{x}^1_{k|k}, \quad \tilde{y}_k = \bar{x}^2_{k|k}, \quad \Phi_k = \bar{K}^2_k, \quad \Gamma^{12}_k = P^{12}_{k|k-1},$$
$$\Gamma^{22}_k = \bar{P}^2_{k|k-1}, \quad \Pi_k = \bar{P}^1_{k|k}, \quad \Phi_k R_k = \bar{P}^2_{k|k} \quad (53)$$

we propose the OMSLO via the simplified two-stage decoupled subfilters (43)–(48) and Eqs. (39), (49), and (50) as

$$z_{k+1} = F_k z_k + D_k\tilde{y}_k + G_k u_k, \quad (54)$$

$$\tilde{y}_{k+1} = (I - \Phi_{k+1})(L_k z_k + M_k\tilde{y}_k + N_k u_k - y_{k+1}) + y_{k+1}, \quad (55)$$

$$x_{k|k} = T_k z_k + E_k\tilde{y}_k, \quad (56)$$

$$U_k = \Gamma_k^{12}(\Gamma_k^{22})^{-1}, \tag{57}$$

$$\begin{bmatrix} \Gamma_{k+1}^{12} \\ \Gamma_{k+1}^{22} \end{bmatrix} = \begin{bmatrix} H_k & S_k \\ L_k & M_k \end{bmatrix} \begin{bmatrix} \Pi_k L_k' \\ \Phi_k R_k M_k' \end{bmatrix} + \begin{bmatrix} Q_k^{12} \\ Q_k^{22} \end{bmatrix}, \tag{58}$$

$$\Pi_{k+1} = H_k \Pi_k H_k' + S_k \Phi_k R_k S_k' + Q_k^{11} - U_{k+1}(\Gamma_{k+1}^{12})', \tag{59}$$

$$\Phi_k = \Gamma_k^{22}(\Gamma_k^{22} + R_k)^{-1}, \tag{60}$$

where

$$Q_k^{11} = T_{k+1}' Q_k T_{k+1}, \qquad Q_k^{12} = T_{k+1}' Q_k \tilde{T}_{k+1}',$$
$$Q_k^{22} = \tilde{T}_{k+1} Q_k \tilde{T}_{k+1}'. \tag{61}$$

Note that the difference between the structure of the above OMSLO and that of the conventional one is that $\tilde{y}_k$ in the OMSLO is a filtered version of the measurement, while the corresponding term in a conventional one is just the current measurement; furthermore, the matrix U_k, which serves as the new observer gain, is derived from the two-stage decoupling method. This is different from the conventional one which is obtained mainly by minimizing the estimation error. From (13) and using the relationships: $\bar{K}_k^1 = 0$ and $\bar{K}_k^2 = \Phi_k$, it is clear that the relation of this new observer gain with the Kalman gain is

$$K_k = \begin{bmatrix} U_k \Phi_k \\ \Phi_k \end{bmatrix}. \tag{62}$$

Owing to the optimality of the GTSKF (Hsieh & Chen, 2000), the substitutions in (53), the arguments in remark 1, and the last notation in (61), we have directly the following theorem.

Theorem 1. *If the error covariance Q_k^{22} is positive definite, then the OMSLO given by* (54)–(60) *gives the MMSE estimate of the system state.*

To give a connection with the suboptimal SLO of Leondes and Novak (1972), we reformulate gain calculations (57)–(59) as follows:

$$U_k = \Gamma_k^{12}\{\Phi_k(\Gamma_k^{22} + R_k)\}^{-1}, \tag{63}$$

$$\Gamma_{k+1} = A_k \begin{bmatrix} \bar{T}_k \Gamma_k \bar{T}_k' + U_k \Phi_k R_k U_k' & U_k \Phi_k R_k \\ (U_k \Phi_k R_k)' & \Phi_k R_k \end{bmatrix} A_k' + Q_k, \tag{64}$$

where

$$\Gamma_k = \begin{bmatrix} \Gamma_k^{11} & \Gamma_k^{12} \\ (\Gamma_k^{12})' & \Gamma_k^{22} \end{bmatrix}, \qquad \Gamma_k^{11} = \Pi_k + U_k \Gamma_k^{22} U_k'. \tag{65}$$

Eq. (64) is derived in the appendix. Then, if the measurement noise intensity is small compared to that of the system noise, Φ_k in (60) is close to an identity matrix. Thus, substituting this special form of Φ_k, i.e., $\Phi_k = I_m$, into the OMSLO which is given by (54)–(56) and (63)–(64), we obtain the following optimal reduced-order observer which was constrained to be of order "$n - m$":

$$z_{k+1} = F_k z_k + D_k y_k + G_k u_k, \tag{66}$$

$$x_{k|k} = T_k z_k + E_k y_k, \tag{67}$$

$$U_{k+1} = \Gamma_{k+1}^{12}(\Gamma_{k+1}^{22} + R_{k+1})^{-1}, \tag{68}$$

$$\Gamma_{k+1} = A_k \begin{bmatrix} \bar{T}_k \Gamma_k \bar{T}_k' + U_k R_k U_k' & U_k R_k \\ (U_k R_k)' & R_k \end{bmatrix} A_k' + Q_k. \tag{69}$$

The above reduced-order observer is equivalent to the optimal minimal-order observer of Leondes and Novak (1972), and its optimality is stated in the following Theorem 2.

Theorem 2. *If measurements are noise-free, then the optimal minimal-order observer given by* (66)–(69) *gives the MMSE estimate of the system state.*

Proof. Substitute $R_k = 0$ into (60), and then use the result of Theorem 1.

In view of Theorem 2, the proposed OMSLO may serve as an alternative to derive Leondes and Novak's minimal-order observer (1972).

5. Optimal reduced-order MSLO

In this section, the authors give a reduced-order form of the OMSLO to apply to singular measurement cases, and the obtained filter will be named as the optimal reduced-order MSLO (OROMSLO). Without loss of generality, it will be assumed that the measurement error covariance matrix R_k is of the form

$$R_k = \begin{bmatrix} \tilde{R}_k & 0 \\ 0 & 0_{\tilde{m}} \end{bmatrix}, \tag{70}$$

where $\tilde{R}_k > 0$ (if exist) and $0 < \tilde{m} \le m$. If it is not, then a linear transformation can be made to achieve this (Leondes & Novak, 1974). Defining

$$\Gamma_k^{22} = \begin{bmatrix} \alpha_k & \beta_k \\ \beta_k' & \gamma_k \end{bmatrix},$$

one obtains

$$\Phi_k = \begin{bmatrix} \alpha_k & \beta_k \\ \beta_k' & \gamma_k \end{bmatrix} \begin{bmatrix} \alpha_k + \tilde{R}_k & \beta_k \\ \beta_k' & \gamma_k \end{bmatrix}^{-1} = \begin{bmatrix} \Phi_k^1 & \Phi_k^2 \\ 0 & I_{\tilde{m}} \end{bmatrix}, \tag{71}$$

where

$$\Phi_k^1 = (\alpha_k - \tilde{U}_k \beta_k')\{\alpha_k - \tilde{U}_k \beta_k' + \tilde{R}_k\}^{-1}, \tag{72}$$

$$\Phi_k^2 = (I - \Phi_k^1)\tilde{U}_k, \tag{73}$$

$$\tilde{U}_k = \beta_k \gamma_k^{-1}. \tag{74}$$

Using (70) and (71), one obtains

$$\Phi_k R_k = \begin{bmatrix} \Phi_k^1 \tilde{R}_k & 0 \\ 0 & 0_{\tilde{m}} \end{bmatrix}. \tag{75}$$

Using (75) and the following partitions: $S_k = [S_k^1 \ S_k^2]$ and $M_k = [M_k^1 \ M_k^2]$, where $S_k^2 \in R^{n-m,\tilde{m}}$ and $M_k^2 \in R^{m,\tilde{m}}$, one can reformulate the OMSLO as the following OROMSLO:

$$z_{k+1} = F_k z_k + D_k \tilde{y}_k + G_k u_k, \tag{76}$$

$$\tilde{y}_{k+1} = \begin{bmatrix} [I - \Phi_{k+1}^1 \quad -\Phi_{k+1}^2](L_k z_k + M_k \tilde{y}_k + N_k u_k - y_{k+1}) \\ 0_{\tilde{m} \times 1} \end{bmatrix} + y_{k+1}, \tag{77}$$

$$x_{k|k} = T_k z_k + E_k \tilde{y}_k, \tag{78}$$

$$U_k = \Gamma_k^{12}(\Gamma_k^{22})^{-1}, \tag{79}$$

$$\begin{bmatrix} \Gamma_{k+1}^{12} \\ \Gamma_{k+1}^{22} \end{bmatrix} = \begin{bmatrix} H_k & S_k^1 \\ L_k & M_k^1 \end{bmatrix} \begin{bmatrix} \Pi_k L_k' \\ \Phi_k^1 \tilde{R}_k (M_k^1)' \end{bmatrix} + \begin{bmatrix} Q_k^{12} \\ Q_k^{22} \end{bmatrix}, \tag{80}$$

$$\Pi_{k+1} = H_k \Pi_k H_k' + S_k^1 \Phi_k^1 \tilde{R}_k (S_k^1)' + Q_k^{11} - U_{k+1}(\Gamma_{k+1}^{12})'. \tag{81}$$

The above OROMSLO has order $n - \tilde{m}$ and may serve as an optimal minimal-order MSLO. Specifically, in the special case of no measurement noise, i.e., $\tilde{m} = m$ and $\tilde{R}_k$ is null, one has $\tilde{y}_k = y_k$ and $\Phi_k^1 \tilde{R}_k$ in (80) and (81) is vanished, and hence the optimal reduced-order MSLO will be equivalent to the optimal reduced-order observer (66)–(69). On the other hand, in the nonsingular measurement case, i.e., $\tilde{m} = 0$ and $\tilde{R}_k = R_k > 0$, one has $\Phi_k^1 = \Phi_k$. Then, the OROMSLO will be equivalent to the OMSLO.

Remark 2. It should be remarked that the implementation of the OROMSLO is obtained by simplifying (55), (58), (59), and (60) into (77), (80), (81), and (71), respectively. These equations imply that the structures of the OROMSLO and the OMSLO are similar and can be processed using the same framework. This is due to the fact that the dimension of z_k is unchanged. This is obviously different from the previous optimal reduced-order observer's results (for example, Leondes & Novak, 1974; Tse, 1970; Fairman, 1977), where the dimension of z_k is dependent on the noise-free measurement's dimension, i.e., $\tilde{m}$. Thus, the data structure of these reduced-order observers will vary with $\tilde{m}$. This may have some disadvantages when the singular measurement equation is not known exactly.

6. Computational considerations

To illustrate that the computational load of the proposed OMSLO [(54)–(60)] is superior than that of the conventional SLO [(66)–(69)], the authors used floating-point operations, or "flops", in Matlab as a measure of the computational complexity. Each multiplication and each addition contributed one to the flops count.

First, the authors listed the flops counts of the SLO and the OMSLO as follows:

$$flops(SLO) = 4n^3 + 6(m+1)n^2 - (6m^2 - m - 2q - 4)n + 2m^3 - 3m^2 - 3m, \tag{82}$$

$$flops(OMSLO) = 4n^3 - (2m-5)n^2 + (2m^2 + 2m + 2q + 4)n + 8m^3 + m^2 - m, \tag{83}$$

with (54) and (66) being implemented respectively as

$$z_{k+1} = \bar{T}_{k+1}(A_k[T_k \quad E_k][z_k' \quad \tilde{y}_k']' + B_k u_k), \tag{84}$$

$$z_{k+1} = \bar{T}_{k+1}(A_k[T_k \quad E_k][z_k' \quad y_k']' + B_k u_k). \tag{85}$$

Note that to simplify the discussion, the symmetric property of the covariance matrix is not used to reduce the complexity of the considered algorithms.

Using (82) and (83), the flops saving, denoted by $\triangle flops$, of the OMSLO as compared to the SLO is given as

$$\triangle flops_{SLO}(OMSLO) = (8m+1)n^2 - (8m^2 + m)n - 6m^3 - 4m^2 - 2m. \tag{86}$$

It is clear from (86) that the saving will be evident when $n \gg m$. This computational efficiency is mainly due to the fact that the original covariance update (69) is simplified by using a decoupled one which is characterized by (58) and (59). However, if we implement the OMSLO by using the conventional covariance updating structure, i.e., (64), the flops of the OMSLO will become

$$flops(OMSLO) = flops(SLO) + 6m^3 + 3m^2 + 2m. \tag{87}$$

In view of (87), the overhead, denoted by $\hat{O}$, of the OMSLO as compared to the load of the SLO is given as

$$\hat{O}_{SLO}(OMSLO) = \frac{flops(OMSLO) - flops(SLO)}{flops(SLO)} = \frac{6m^3 + 3m^2 + 2m}{4n^3 + 6(m+1)n^2 - (6m^2 - m - 2q - 4)n + 2m^3 - 3m^2 - 3m}. \tag{88}$$

From (88), it is clear that the overhead of this OMSLO is negligible as compared to the load of the SLO for $n \gg m$.

In summary, the OMSLO is a feasible solution to solve the modified observer design problem, which intends to recover the optimal performance of the conventional SLO.

7. Simulation example

To verify the previous analytical results, the following target tracking simulation was conducted. Consider a target maneuvers with accelerations $x^a = y^a = 0.075$ m/s^2. The initial position and velocity of the target were $x_0^p = 2000$ m, $x_0^v = 0$ m/s, $y_0^p = 10{,}000$ m, and $y_0^v = -15$ m/s. The sampling interval was $T = 10$ s; the simulation time was 500 s. The target position was measured. The system matrices were given by

$$A_k = \begin{bmatrix} 1 & 10 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 10 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 10 & 50 & 0 & 0 & 1 & 0 \\ 0 & 0 & 10 & 50 & 0 & 1 \end{bmatrix}, \quad B_k = 0,$$

$$C_k = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Q_k = \begin{bmatrix} 20 & 2 & 0 & 0 & 100 & 0 \\ 2 & 0.2 & 0 & 0 & 10 & 0 \\ 0 & 0 & 20 & 2 & 0 & 100 \\ 0 & 0 & 2 & 0.2 & 0 & 10 \\ 100 & 10 & 0 & 0 & 500 & 0 \\ 0 & 0 & 100 & 10 & 0 & 500 \end{bmatrix},$$

$$R_k = \begin{bmatrix} 10000 & 0 \\ 0 & 10000 \end{bmatrix}$$

and the state vector was $x_k = [x_k^v \; x_k^a \; y_k^v \; y_k^a \; x_k^p \; y_k^p]'$.

The OMSLO^1 [(54)–(60)], the OMSLO^2 [(54)–(57), (64), and (60)], and the SLO [(66)–(69)] are considered. All filters were initialized by taking the initial state estimate $\bar{x}_0$ and the corresponding covariance matrix $\bar{P}_0$ as $\bar{x}_0 \sim N(x_0, \bar{P}_0)$ and $\bar{P}_0 = Q_0$, respectively, where x_0 was the initial target state. The tracking error is defined as the root-mean-square of the state estimating error.

A Monte-Carlo simulation of 50 runs (using Matlab) was performed. The simulation results in Table 1 show the tracking error and the corresponding flops generated by Matlab. Table 1 shows that the tracking error of the OMSLO is smaller than that of the SLO. It also shows that the flops of the OMSLO^1 is fewer than that of the SLO. Although the flops of the conventional SLO is slightly fewer than that of the OMSLO^2, the estimate of the SLO is degraded. This simulation result also shows that the covariance update in the conventional observer design can be simplified by using the proposed one [(58) and (59)]. Note that if one substitutes $n = 6$, $m = 2$, and $q = 0$ into (82), (83), and (87), the flops of the SLO, the OMSLO^1, and the OMSLO^2 are 1402, 1062, and 1466, respectively. These results are exactly the same as the simulation results.

Table 1
Performances of the OMSLO and the SLO filters

Performances	OMSLO^1	OMSLO^2	SLO
Tracking error	125.85	125.85	142.42
flops	1062	1466	1402

8. Conclusions

This paper presents a modified stochastic Luenberger observer (MSLO) structure to recover the optimal performance of the conventional SLO. Specifically, the optimal MSLO (OMSLO), which is optimal in the MMSE sense, is derived directly from the general two-stage Kalman filter (GTSKF). It is illustrated by analytical and simulation results that the computational complexity of the proposed OMSLO is less than that of the conventional SLO. This paper also shows that how to modify the SLO's structure to get the optimal performance. In view of this fact, it is also shown by analytical and simulation results that the overhead of this OMSLO is not excessive as compared to the load of the conventional SLO. A reduced-order form of the OMSLO is also presented for singular measurements. Our results suggest that the proposed OMSLO can be used as a general model to replace the SLO for obtaining state estimates in time-varying, linear discrete-time stochastic and deterministic systems.

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Appendix

Derivation of Eq. (64). Using (20), (65), and (57), we obtain

$$\bar{T}_k \Gamma_k \bar{T}_k' = \Gamma_k^{11} - U_k \Gamma_k^{22} U_k' = \Pi_k. \tag{A.1}$$

Using (65), (59), and (57), we obtain

$$\Gamma^{11}_{k+1} = H_k \Pi_k H'_k + S_k \Phi_k R_k S'_k + Q^{11}_k + U_{k+1}(U_{k+1}\Gamma^{22}_{k+1} - \Gamma^{12}_{k+1})' = H_k \Pi_k H'_k + S_k \Phi_k R_k S'_k + Q^{11}_k = [H_k \quad S_k]\begin{bmatrix} \Pi_k & 0 \\ 0 & \Phi_k R_k \end{bmatrix}\begin{bmatrix} H'_k \\ S'_k \end{bmatrix} + Q^{11}_k. \tag{A.2}$$

Replacing (58) with the following one

$$\begin{bmatrix} \Gamma^{12}_{k+1} \\ \Gamma^{22}_{k+1} \end{bmatrix} = \begin{bmatrix} H_k & S_k \\ L_k & M_k \end{bmatrix}\begin{bmatrix} \Pi_k & 0 \\ 0 & \Phi_k R_k \end{bmatrix}\begin{bmatrix} L'_k \\ M'_k \end{bmatrix} + \begin{bmatrix} Q^{12}_k \\ Q^{22}_k \end{bmatrix}. \tag{A.3}$$

Using (65), (A.2), (A.3), (32), (49), (20), and (A.1), we obtain

$$\Gamma_{k+1} = \begin{bmatrix} H_k & S_k \\ L_k & M_k \end{bmatrix}\begin{bmatrix} \Pi_k & 0 \\ 0 & \Phi_k R_k \end{bmatrix}\begin{bmatrix} H_k & S_k \\ L_k & M_k \end{bmatrix}' + Q_k = A_k[T_k \quad E_k]\begin{bmatrix} \Pi_k & 0 \\ 0 & \Phi_k R_k \end{bmatrix}\begin{bmatrix} T'_k \\ E'_k \end{bmatrix}A'_k + Q_k = A_k\begin{bmatrix} \Pi_k + U_k\Phi_k R_k U'_k & U_k \Phi_k R_k \\ (U_k \Phi_k R_k)' & \Phi_k R_k \end{bmatrix}A'_k + Q_k = A_k\begin{bmatrix} \bar{T}_k \Gamma_k \bar{T}'_k + U_k\Phi_k R_k U'_k & U_k \Phi_k R_k \\ (U_k \Phi_k R_k)' & \Phi_k R_k \end{bmatrix}A'_k + Q_k. \tag{A.4}$$

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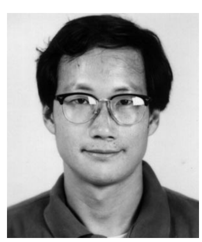

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