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Exact a posteriori error analysis of the least squares finite element method [1]

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Abstract

A residual type a posteriori error estimator is presented for the least squares finite element method. The estimator is proved to equal the exact error in a norm induced by some least squares functional. The error indicator of each element is equal to the exact error norm restricted to the element as well. In other words, the estimator is perfectly effective and reliable for error control and for adaptive mesh refinement. The exactness property requires virtually no assumptions on the regularity of the solution and on the finite element order in the approximation or in the estimation. The least squares method is in a very general setting that applies to various linear boundary-value problems such as the elliptic systems of first-order and of even-order and the mixed type partial differential equations. Numerical results are given to demonstrate the exactness. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Exact error estimator; Least squares finite elements

1. Introduction

A posteriori error estimation is one of the most important components in adaptive methodology [19,22,28]. It seems that there are no exact a posteriori error analyses available for all the finite element, finite volume, and boundary element methods in the literature. An estimator for the least squares finite

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element method (LSFEM) is presented here and is proved to be not only globally but also locally exact.

Since the pioneering work of Babuška and Rheinboldt [6], it has been an important subject to study the reliability of error estimators for adaptive numerical methods in various applications. The reliability is usually quantified by the so-called effectivity index θ defined as

$$\theta = \frac{\varepsilon}{\|\mathbf{e}\|},$$

where ε denotes an error estimator and the denominator is the exact error $\mathbf{e}$ measured in a suitable norm $\|\cdot\|$. Theoretically as well as practically, the quantity is often desired to be either bounded below and above by some positive constants, say C_1 and C_2, which are independent of the mesh parameter h of the approximation, i.e.,

$$C_1 \leqslant \theta \leqslant C_2$$

or asymptotically exact, i.e.,

$$\lim_{h\to 0} \theta = 1$$

or both, see e.g., [1,2,6,7,10,15,16,21,24,26,27].

The error estimator described here is for LSFEM in a very general setting. It is obtained by calculating the residual in L^2 norm and is proved not only globally exact, i.e.,

$$\theta = 1,$$

but also locally exact, i.e.,

$$\theta_i := \frac{\varepsilon_i}{\|\mathbf{e}\|_i} = 1,$$

where ε_i denotes the residual norm (an error indicator) of an element i in a particular triangulation of the domain associated with the parameter h and $\|\mathbf{e}\|_i$ is the exact error norm restricted to that element. Here, the norm $\|\cdot\|$ is induced by the bilinear form derived from a least squares functional which in turn is associated with a given boundary value problem. In other words, this is a perfect error estimator if the error of numerical integration is not taken into account. Even more surprisingly, the proof of the exactness in the context of least squares formulation is almost trivial when compared with the previous a posteriori error analysis. Furthermore, no extra assumptions on the regularity of the solution or on the finite element spaces used in the approximation are required for this exactness.

The residual type error estimation is, of course, not new [6,15,18,19,22,24] and is presented in various formulations for various applications. The present analysis shows that the exactness of the estimator may be an additional outstanding feature for LSFEM which has been recognized as an attractive method in many applications in recent years, see e.g., [3,5,8,9,11–14,18,20].

2. An exact estimator

The boundary value problems considered herein are in a very general setting and are expressed in the form

$$\mathscr{L}_j\mathbf{u} = f_j, \quad j = 1, 2, \ldots, m, \quad \text{in } \Omega, \tag{1}$$

$$\mathscr{B}_k\mathbf{u} = g_k, \quad k = 1, 2, \ldots, n, \quad \text{on } \partial\Omega, \tag{2}$$

where $\Omega \subset \mathbf{R}^d$, $d = 1, 2, 3$, is a bounded domain with the boundary $\partial\Omega$, $\mathbf{u} = (u_1, \ldots, u_m)^{\mathrm{t}}$, $\mathscr{L}_j$ are linear differential operators and $\mathscr{B}_j$ are linear boundary operators. We always assume that the problem (1) and (2) has a unique solution $\mathbf{u}$ in some (m-tuple) Cartesian product of function spaces denoted by $H(\Omega)$ with the given functions $f_j \in L^2(\Omega)$, $g_k \in L^2(\partial\Omega)$. The setting is so general that it applies to a wide class of linear boundary value problems such as the elliptic systems of Refs. [3,25], the first-order systems of [8,9,11–14] and the mixed type PDEs of [4,5,17,23] in the first-order formulation, etc.

The problem (1) and (2) will be approximated by the LSFEM. The method itself will also be presented in a very general setting by this we mean that the associated least squares functional contains equation residuals and boundary residuals all weighted by positive parameters. It is convenient to weight (1) and (2) before defining the functional. Let (1) and (2) be weighted as follows:

$$\mathscr{L}_w\mathbf{u} = \mathbf{f}_w, \quad \text{in } i\Omega, \tag{3}$$

$$\mathscr{B}_w\mathbf{u} = \mathbf{g}_w, \quad \text{on } \partial\Omega, \tag{4}$$

where

$$\mathscr{L}_w := (l_1\mathscr{L}_1, \ldots, l_m\mathscr{L}_m)^{\mathrm{t}},$$

$$\mathscr{B}_w := (b_1\mathscr{B}_1, \ldots, b_n\mathscr{B}_n)^{\mathrm{t}},$$

$$\mathbf{f}_w := (l_1 f_1, \ldots, l_m f_m)^{\mathrm{t}},$$

$$\mathbf{g}_w := (b_1 g_1, \ldots, b_n g_n)^{\mathrm{t}},$$

and l_j and b_k are the weighting parameters.

The functional, denoted by $\mathscr{J} : H(\Omega) \to \mathbf{R}$, is defined as

$$\begin{aligned}\mathscr{J}(\mathbf{v}) &= \sum_{j=1}^{m} \int_{i\Omega} l_j^2(\mathscr{L}_j\mathbf{v} - f_j)^2 \,\mathrm{d}\Omega + \sum_{k=1}^{n} \int_{\partial\Omega} b_k^2(\mathscr{B}_k\mathbf{v} - g_k)^2 \,\mathrm{d}s \\ &=: \|\mathscr{L}_w\mathbf{v} - \mathbf{f}_w\|_{0,\Omega}^2 + \|\mathscr{B}_w\mathbf{v} - \mathbf{g}_w\|_{0,\partial\Omega}^2.\end{aligned} \tag{5}$$

Here, the norms $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{0,\partial\Omega}$ are associated with the inner products $(\cdot,\cdot)_{0,\Omega}$ and $(\cdot,\cdot)_{0,\partial\Omega}$, respectively. Evidently, the solution $\mathbf{u} \in H(\Omega)$ of (1) and (2) minimizes the functional and vice versa, i.e.,

$$\mathscr{J}(\mathbf{u}) = \min_{\mathbf{v}\in H(\Omega)} \mathscr{J}(\mathbf{v}).$$

Taking the first variation, the solution equivalently satisfies the equation

$$B(\mathbf{u},\mathbf{v}) = F(\mathbf{v}) \quad \forall \mathbf{v} \in H(\Omega), \tag{6}$$

where

$$B(\mathbf{u},\mathbf{v}) := (\mathscr{L}_w\mathbf{u}, \mathscr{L}_w\mathbf{v})_{0,\Omega} + (\mathscr{B}_w\mathbf{u}, \mathscr{B}_w\mathbf{v})_{0,\partial\Omega},$$

$$F(\mathbf{v}) := (\mathbf{f}_w, \mathscr{L}_w\mathbf{v})_{0,\Omega} + (\mathbf{g}_w, \mathscr{B}_w\mathbf{v})_{0,\partial\Omega}.$$

The LSFEM for (1) and (2) is to solve (6) in a finite element subspace S_h of $H(\Omega)$, that is, to find $\mathbf{u}_h \in S_h$ such that

$$B(\mathbf{u}_h,\mathbf{v}_h) = F(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in S_h, \tag{7}$$

where h is the mesh size of a regular triangulation $\mathscr{T}_h$ of $\overline{\Omega}$. Note that if the boundary condition (2) is computed exactly, i.e., if the finite element space S_h consists of functions that satisfy (2), the weighting parameters b_k are indifferent. Hence, the least squares functional (5) leads to a very general setting of LSFEMs with weighted or non-weighted differential and/or boundary residuals. It thus applies to all the above mentioned least squares problems.

Evidently, the uniqueness assumption of the solution of (1) and (2) implies that the bilinear form $B(\cdot,\cdot)$ induces a norm by which we denote $\|\cdot\|_B$. For most first-order systems, the norm is equivalent to the H^1 norm [8,11–14]. It also implies the following result for which a proof can be found in [5].

Theorem 1. *If the problem* (1) *and* (2) *has a unique solution* $\mathbf{u} \in H(\Omega)$, *then there exists a unique function* $\mathbf{u}_h \in S_h \subset H(\Omega)$ *satisfying Eq.* (7).

Once the approximate solution $\mathbf{u}_h$ is available, one of the major concerns in practice is to assess the reliability of this approximation, i.e., to estimate the exact error $\mathbf{u} - \mathbf{u}_h$ in some suitable norm. A posteriori error estimators represent an important means towards the assessment. For LSFE solution, we introduce a residual error estimator ε defined as

$$\varepsilon^2 = \|\mathscr{L}_w\mathbf{u}_h - \mathbf{f}_w\|^2_{0,\Omega} + \|\mathscr{B}_w\mathbf{u}_h - \mathbf{g}_w\|^2_{0,\partial\Omega}. \tag{8}$$

The estimator is readily computable. In fact, if we compute the residual in each element τ_i of the current triangulation $\mathscr{T}_h$, we obtain an error indicator ε_i for that element, i.e.,

$$\varepsilon_i^2 = \|\mathscr{L}_w\mathbf{u}_h - \mathbf{f}_w\|^2_{0,\tau_i} + \|\mathscr{B}_w\mathbf{u}_h - \mathbf{g}_w\|^2_{0,\partial\Omega\cap\bar{\tau}_i}. \tag{9}$$

The square of the estimator is thus the sum of all squares of the indicators. Define the effectivity index

$$\theta = \frac{\varepsilon}{\|\mathbf{e}\|_B}. \tag{10}$$

We also define local effectivity indices by

$$\theta_i = \frac{\varepsilon_i}{\|\mathbf{e}\|_{B,\bar{\tau}_i}}, \tag{11}$$

where $\|\mathbf{e}\|_{B,\bar{\tau}_i}$ is the restriction of $\|\mathbf{e}\|_B$ on $\bar{\tau}_i$.

Theorem 2. *Let* $\mathbf{u} \in H(\Omega)$ *and* $\mathbf{u}_h \in S_h \subset H(\Omega)$ *be the solutions of* (6) *and* (7), *respectively. Then*

$$\theta = \theta_i = 1 \tag{12}$$

for all $\tau_i \in \mathscr{T}_h$.

Proof.

$$\begin{aligned}\varepsilon_i^2 &= \|\mathscr{L}_w\mathbf{u}_h - \mathbf{f}_w\|^2_{0,\tau_i} + \|\mathscr{B}_w\mathbf{u}_h - \mathbf{g}_w\|^2_{0,\partial\Omega\cap\bar{\tau}_i}\\ &= \|\mathscr{L}_w\mathbf{u}_h - \mathscr{L}_w\mathbf{u}\|^2_{0,\tau_i} + \|\mathscr{B}_w\mathbf{u}_h - \mathscr{B}_w\mathbf{u}\|^2_{0,\partial\Omega\cap\bar{\tau}_i}\\ &= (\mathscr{L}_w\mathbf{e}, \mathscr{L}_w\mathbf{e})_{0,\tau_i} + (\mathscr{B}_w\mathbf{e}, \mathscr{B}_w\mathbf{e})_{0,\partial\Omega\cap\bar{\tau}_i}\\ &= B(\mathbf{e},\mathbf{e})|_{\bar{\tau}_i}.\end{aligned}$$

Therefore, we have the exactness of the local effectivity indices. The global exactness is again immediate. □

Several remarks are in order.

Remark 1. The global as well as local exactness of (12) appears to be first presented in the literature. The proof of this statement is almost trivial when compared with the previous a posteriori error analysis, see e.g. [1,2,6,7,10, 16,21,24,26,27]. Furthermore, there are virtually no assumptions for this exactness, i.e., no saturation assumptions like those of [7,21], no extra regularity assumptions on the exact solution like those of [1,2,6,10,16,24,26] and no restrictions on the finite element orders used in the approximation. In fact, we even do not require the approximate solutions $\mathbf{u}_h$ of (7) to be convergent at all (cf. Theorem 1).

Remark 2. It is well-known that the LSFEM inherently provides very attractive properties in applications. For example, the trial and test functions are not required to satisfy the boundary conditions, its discretization results in symmetric and positive definite algebraic system, a single piecewise polynomial finite element space may be used for all test and trial functions and it does not require the inf–sup condition to be satisfied when compared with the mixed finite element method etc., see loc. cit. The exactness of the error estimator may yet provide an additional outstanding feature of the LSFEM versus other methods, since it is perfectly reliable and effective.

Remark 3. Obviously, the implementation of the residual estimator is simple. The exactness is also preserved in terms of the numerical integration. The error indicators can be computed on parallel processors without any communication cost since no jump terms across element boundaries and no local boundary conditions are involved. Together with the symmetric property of the algebraic system, the entire adaptive procedure of least squares computations can be parallel and distributed if a conjugate gradient solver is used since, in this case, there is no need for a global assembly and the iterative process can be done locally [18].

3. Numerical example

By introducing the vorticity $\omega = \partial v/\partial x - \partial u/\partial y$, the 2-D dimensionless Stokes equations can be written as [13]

$$\mathscr{L}\mathbf{u} = \begin{bmatrix} 0 & 0 & \nu\partial/\partial y & \partial/\partial x \\ 0 & 0 & -\nu\partial/\partial x & \partial/\partial y \\ \partial/\partial x & \partial/\partial y & 0 & 0 \\ \partial/\partial y & -\partial/\partial x & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ \omega \\ p \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \end{bmatrix} = \mathbf{f} \quad \text{in } \Omega, \tag{13}$$

where u and v are the x and y components of velocity, p the total head of pressure, ν the inverse of Reynolds number and f_1 and f_2 the given body forces. We consider a model problem [8] of (13) to which the exact solution is given by

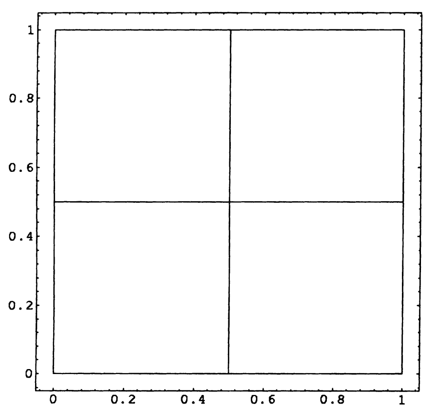

Fig. 1. Initial mesh.

$$u_1 = \left[(x-a)^2 + (y-b)^2\right]^{c/2},$$

$$u_2 = \left[(x-a)^2 + (y-b)^2\right]^{c/2},$$

$$\omega = \left[(x-a)^2 + (y-b)^2\right]^{c/2},$$

$$p = \left[(x-a)^2 + (y-b)^2\right]^{c/2},$$

in a unit square $\{(x,y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$, where $a = b = 0.1234$ and $c = 0.9$. Singularity appears at the point (a,b). The appropriate boundary condition (2) can be constructed via the exact solution. For the least squares

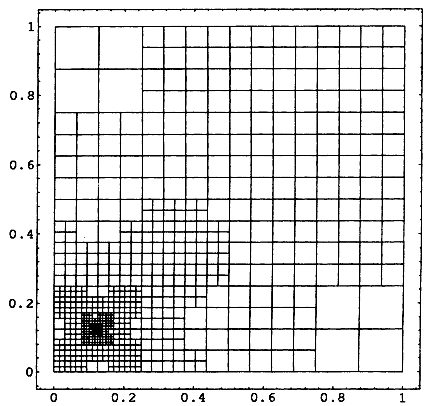

Fig. 2. Final mesh.

Table 1

NN	$\|\mathbf{e}\|_{\mathscr{B}}$	$\theta_{\min}$	$\theta_{\max}$	θ
21	0.4416	1	1	1
30	0.3549	1	1	1
39	0.0962	1	1	1
137	0.0456	1	1	1
165	0.0313	1	1	1
243	0.0221	1	1	1
289	0.0180	1	1	1
294	0.0174	1	1	1
353	0.0131	1	1	1
823	0.0054	1	1	1
1071	0.0038	1	1	1
1522	0.0026	1	1	1
2382	0.0016	1	1	1

approximation (7), we impose the boundary condition in biquadratic finite element spaces S_h and set all weighting parameters to one. An adaptive process using the residual error estimator (8) begins with the initial mesh Fig. 1 and ends with the final mesh Fig. 2. All of the (global and local) effectivity indices of the adaptive computation are equal to one and are shown in Table 1 where NN denotes the number of nodes, $\theta_{\min} = \min_i \theta_i$ and $\theta_{\max} = \max_i \theta_i$.

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