Weak-field expansion for processes in a homogeneous background magnetic field

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The weak-field expansion of the charged fermion propagator under a uniform magnetic field is studied. Starting from Schwinger's proper-time representation, we express the charged fermion propagator as an infinite series corresponding to different Landau levels. This infinite series is then reorganized according to the powers of the external field strength B. For illustration, we apply this expansion to $\gamma \rightarrow \nu\bar{\nu}$ and $\nu \rightarrow \nu\gamma$ decays, which involve charged fermions in the internal loop. The leading and subleading magnetic-field effects to the above processes are computed.

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I. INTRODUCTION

Particle reactions taking place in the early universe or astrophysical environments are often affected by the background magnetic field or excitations in the medium [1]. A typical example is the modification of the neutrino index of refraction in the early universe or supernova [2]. There one needs to compute the neutrino self-energy in the medium or the background electromagnetic field or both. The neutrino index of refraction is then extracted from the modified dispersion relation of the neutrino. Another example is the plasmon decay $\gamma^* \rightarrow \nu\bar{\nu}$ [1] where the decaying photon acquires an effective mass through the effects of the medium or the background magnetic field. With such an effective mass, the above decay is kinematically permissible. Furthermore, the behavior of electron propagators occurring in the internal loop of the above decay is also affected by the medium or the magnetic field. This also leads to a modification to the plasmon decay amplitude. Finally, a more recent example is the enhancement of neutrino-photon scatterings due to the background magnetic field [3,4]. At the lowest order in the weak interaction, it is known that the amplitude for $\gamma\gamma \rightarrow \nu\bar{\nu}$ is proportional to the neutrino mass [5]. Hence the resulting scattering cross section is rather suppressed. On the other hand, the presence of the background magnetic field alters the structures of internal electron propagators, such that $\gamma\gamma \rightarrow \nu\bar{\nu}$ is non-vanishing at $O(G_F)$ even in the massless limit of neutrinos. Specifically, the $\gamma\gamma \rightarrow \nu\bar{\nu}$ cross section is enhanced by a factor $(m_W/m)^4(B/B_c)^2$ due to a background magnetic field B [3,4], where m_W and m are the masses of W boson and electron respectively; $B_c \equiv m^2/e$ is the critical magnetic field.

In the above processes, the relevant magnetic-field strengths are often smaller than the critical value B_c. Therefore it is appropriate to expand the decay width, cross section or other physical quantities in powers of B/B_c. In the literature, such an expansion is usually performed after the relevant amplitude is obtained [6]. For a more complicated process, it is not always convenient to do so since the amplitude to be expanded may be very cumbersome. In this article, we shall propose a more straightforward weak-field expansion, which is performed directly on the charged fermion propagator participating in the process. With the charged fermion propagators expanded, the physical amplitude can be easily expressed in powers of B/B_c. To perform such an expansion on propagators, we shall begin with Schwinger's proper-time representation for a charged fermion propagator under a uniform background magnetic field [7]. It is useful to realize that Schwinger's representation can be recast into a series expansion in terms of Landau levels [8]. In the weak field limit $B \ll B_c$, we shall demonstrate that one can reorganize the infinite series in powers of the field strength B. This is the expansion we are after.

This article is organized as follows: In Sec. II, we will review Schwinger's derivation of charged fermion propagator in a homogeneous background magnetic field. Since the convention used by Schwinger differs from the currently popular convention, we shall repeat some relevant details of the derivation for clarification. We shall also illustrate how to rewrite Schwinger's result as an infinite series where each term is associated with specific Landau levels [8]. In the weak-field limit, we shall demonstrate how to rearrange the above series in powers of the magnetic-field strength B. Finally, some technical issues relevant to the phase factor in Schwinger's proper-time representation will be discussed in this section. In Sec. III, we begin with a brief discussion on our earlier work [4], where the weak-field expansion technique is applied to $\gamma\gamma \rightarrow \nu\bar{\nu}$ and its crossed processes in a background magnetic field [3,9]. To further illustrate the technique of weak-field expansion, we also calculate the decay rates of $\gamma \rightarrow \nu\bar{\nu}$ and the neutrino Cherenkov process $\nu \rightarrow \nu\gamma$ in a background magnetic field. Our results will be compared to previous calculations which are performed using exact charged-fermion propagators in the background magnetic field [10–12]. A few concluding remarks are presented in Sec. IV.

II. CHARGED-FERMION PROPAGATOR IN A HOMOGENEOUS BACKGROUND MAGNETIC FIELD

A. The exact propagator solution

The Green's function $G(x,x')$ of the Dirac field in the presence of a gauge field A_μ satisfies the following equation:

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$$(i\partial\!\!\!/+eA\!\!\!/-m)G(x,x')=\delta(x-x'), \tag{1}$$

where $\delta(x-x')$ is the Dirac's delta function and m stands for the mass of the Dirac field. We will follow the technique employed in Schwinger's paper [7] which regards $G(x,x')$ as the matrix element of an operator G, namely $G(x,x')=\langle x'|G|x\rangle$. Therefore, Eq. (1) may be written as

$$(\Pi\!\!\!/-m)G=1, \tag{2}$$

with $\Pi_\mu=P_\mu+eA_\mu$ denoting the conjugated momentum, which obeys the following commutation relations:

$$[\Pi_\mu,x_\nu]=ig_{\mu\nu}, \tag{3}$$

$$[\Pi_\mu,\Pi_\nu]=ieF_{\mu\nu}, \tag{4}$$

with $F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu$ denoting the field-strength tensors of the gauge field. Eq. (2) can be formally solved by writing

$$G=\frac{1}{\Pi\!\!\!/-m}=-i\int_0^\infty ds(\Pi\!\!\!/+m)\exp[-i(m^2-\Pi\!\!\!/^2)s]. \tag{5}$$

This integral representation for G implies that

$$G(x,x')=-i\int_0^\infty ds e^{-im^2s}\langle x'|(\Pi\!\!\!/+m)U(s)|x\rangle, \tag{6}$$

where $U(s)=e^{-iHs}$ with $H\equiv-(\Pi\!\!\!/)^2=-\Pi^2-\frac{1}{2}e\sigma_{\mu\nu}F^{\mu\nu}$. We observe that $U(s)$ can be viewed as the unitary time-evolution factor if one takes H as the effective Hamiltonian that evolves the state $|x\rangle$ according to

$$|x(s)\rangle=U(s)|x(0)\rangle, \tag{7}$$

where s is the proper time variable. One can now rewrite $G(x,x')$ as

$$G(x,x')=-i\int_0^\infty ds e^{-im^2s}[\gamma^\mu\langle x'(0)|\Pi_\mu(0)|x(s)\rangle +m\langle x'(0)|x(s)\rangle], \tag{8}$$

where we have assumed $\Pi_\mu(s)$ operates on $|x(s)\rangle$ and $\Pi_\mu(0)$ operates on $|x(0)\rangle$. We note that the operators x_μ and Π_μ satisfy

$$\frac{dx_\mu}{ds}=-i[x_\mu,H]=2\Pi_\mu,$$

$$\frac{d\Pi_\mu}{ds}=-i[\Pi_\mu,H]=-2eF_{\mu\nu}\Pi^\nu, \tag{9}$$

for a constant field strength $F_{\mu\nu}$. In the matrix notation, we may write $dx/ds=2\Pi$, and $d\Pi/ds=-2eF\Pi$. Furthermore the transformation function $\langle x'(0)|x(s)\rangle$ can be characterized by the following equations:

$$i\partial_s\langle x'(0)|x(s)\rangle=\langle x'(0)|H|x(s)\rangle,$$

$$(i\partial_\mu+eA_\mu(x))\langle x'(0)|x(s)\rangle=\langle x'(0)|\Pi_\mu(s)|x(s)\rangle,$$

$$(-i\partial'_\mu+eA_\mu(x'))\langle x'(0)|x(s)\rangle=\langle x'(0)|\Pi_\mu(0)|x(s)\rangle, \tag{10}$$

with the boundary condition: $\langle x'(0)|x(s)\rangle\to\delta^4(x-x')$ as $s\to0$. To evaluate Eq. (8), we first solve Eq. (9) and obtain

$$\Pi(s)=e^{-2eFs}\Pi(0),$$

$$x(s)-x(0)=(1-e^{-2eFs})(eF)^{-1}\Pi(0). \tag{11}$$

This solution implies

$$\Pi^2\equiv-H-\frac{1}{2}e\sigma_{\mu\nu}F^{\mu\nu} =(x(s)-x(0))K(x(s)-x(0)),$$

$$[x_\mu(s),x_\nu(0)]=i(1-e^{-2eFs})(eF)^{-1}, \tag{12}$$

where $K\equiv\frac{1}{4}(eF)^2\sinh^{-2}eFs$. Therefore, one has

$$\langle x'(0)|H|x(s)\rangle=-\frac{1}{2}e\sigma F-(x-x')K(x-x') -\frac{i}{2}\,\mathrm{tr}(eF\coth eFs)\langle x'(0)|x(s)\rangle. \tag{13}$$

With this result, one can solve the first equation in Eq. (10), which gives

$$\langle x'(0)|x(s)\rangle=C(x,x')s^{-2}\times\exp\left[-\frac{1}{2}\,\mathrm{tr}\,\ln[(eFs)^{-1}\sinh(eFs)]\right]\times\exp\left[-\frac{i}{4}(x-x')eF\coth(eFs)(x-x')+\frac{i}{2}e\sigma_{\mu\nu}F^{\mu\nu}s\right]. \tag{14}$$

The factor $C(x,x')$ can be determined by substituting Eq. (14) into the second and third equations in Eq. (10). Since the RHS of these two equations are given by

$$\langle x'(0)|\Pi(s)|x(s)\rangle=\frac{1}{2}[eF\coth(eFs)-eF](x-x')\times\langle x'(0)|x(s)\rangle,$$

$$\langle x'(0)|\Pi(0)|x(s)\rangle=\frac{1}{2}[eF\coth(eFs)+eF](x-x')\times\langle x'(0)|x(s)\rangle, \tag{15}$$

one then arrives at

$$\left[i\partial_\mu + eA_\mu(x) - \frac{1}{2}eF_{\mu\nu}(x'-x)^\nu\right]C(x,x')=0,$$

$$\left[-i\partial'_\mu + eA_\mu(x') + \frac{1}{2}eF_{\mu\nu}(x'-x)^\nu\right]C(x,x')=0. \qquad (16)$$

Therefore $C(x,x')$ is found to be

$$C(x,x')=C'(x')\exp\left[ie\int_{x'}^{x}d\xi^\mu\left(A_\mu+\frac{1}{2}F_{\mu\nu}(\xi-x')^\nu\right)\right]$$
$$=C(x)\exp\left[ie\int_{x'}^{x}d\xi^\mu\left(A_\mu+\frac{1}{2}F_{\mu\nu}(\xi-x)^\nu\right)\right]. \qquad (17)$$

Here $C'(x')$ and $C(x)$ denote integration constants in x' and x respectively. Note that the integral $A_\mu+\frac{1}{2}F_{\mu\nu}(\xi-x')^\nu$ is a total derivative in the presence of a homogeneous field if the first homology group of the space-time M is trivial, i.e., $H_1(M)=0$ [13]. Hence the phase factor is independent of the integration path connecting x and x'. One can further show that $C(x')=C'(x)$. Therefore $C(x')$ or $C'(x)$ has to be a constant independent of x and x'. This constant can be determined by applying the boundary condition $\langle x(s)|x'(0)\rangle\to\delta^4(x-x')$ as $s\to 0$. One obtains

$$C=-i(4\pi)^{-2} \qquad (18)$$

with the help of the identity

$$\int_{-\infty}^{\infty}e^{ia^2x^2}dx=\sqrt{\frac{i\pi}{a^2}}. \qquad (19)$$

From Eqs. (8), (14), (15) and (18), one arrives at

$$G(x,x')=\Phi(x,x')\mathcal{G}(x,x'), \qquad (20)$$

where

$$\mathcal{G}(x,x')\equiv-(4\pi)^{-2}\int_0^\infty\frac{ds}{s^2}$$
$$\times\left[m+\frac{1}{2}\gamma\cdot(eF\coth(eFs)+eF)(x-x')\right]$$
$$\times\exp\left(-im^2s+\frac{i}{2}e\sigma_{\mu\nu}F^{\mu\nu}s\right)$$
$$\times\exp\left[-\frac{1}{2}\,\mathrm{tr}\,\ln[(eFs)^{-1}\sinh(eFs)]\right.$$
$$\left.-\frac{i}{4}(x-x')(eF\coth(eFs))(x-x')\right], \qquad (21)$$

$$\Phi(x,x')\equiv\exp\left\{ie\int_{x'}^{x}d\xi^\mu\left[A_\mu+\frac{1}{2}F_{\mu\nu}(\xi-x')^\nu\right]\right\}. \qquad (22)$$

Note that the translation invariance is broken by the phase factor $\Phi(x,x')$. Note also that the phase factor $\Phi(x,x')$ vanishes if the path connecting x and x' is chosen to be a straight-line. In addition, if the background gauge field is a homogeneous magnetic field such that $F_{12}=-F_{21}=B$, one can show that

$$\sigma_{\mu\nu}F^{\mu\nu}=2F_{12}\sigma_3\equiv2F_{12}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix},$$

$$\exp\left[-\frac{1}{2}\,\mathrm{tr}\,\ln(F^{-1}\sinh F)\right]=\frac{B}{\sin B},$$

$$\gamma(F\coth F-F)x=(\gamma\cdot x)_\parallel-\frac{B}{\sin B}(\gamma\cdot x)_\perp e^{iF_{12}\sigma_3},$$

$$x(F\coth F)x=x_\parallel^2-B\cot Bx_\perp^2, \qquad (23)$$

with $(a\cdot b)_\parallel=a^0b^0-a^3b^3$ and $(a\cdot b)_\perp=a^1b^1+a^2b^2$ for arbitrary 4-vectors a^μ and b^μ. Hence $a_\parallel^2=a^0a^0-a^3a^3$, and $a_\perp^2=a^1a^1+a^2a^2$. To simplify the notations, we shall denote $(\gamma\cdot p)_{\parallel(\perp)}$ as $\gamma\cdot p_{\parallel(\perp)}$. From the relations in Eq. (23), the propagator function $\mathcal{G}(x,x')$, which respects the translation invariance, becomes

$$\mathcal{G}(x)=-(4\pi)^{-2}\int_0^\infty\frac{ds}{s^2}\frac{eBs}{\sin(eBs)}\exp(-im^2s+ieBs\sigma_3)$$
$$\times\exp\left[-\frac{i}{4s}(x_\parallel^2-eBs\cot(eBs)x_\perp^2)\right]$$
$$\times\left[m+\frac{1}{2s}\left(\gamma\cdot x_\parallel-\frac{eBs}{\sin(eBs)}\right.\right.$$
$$\left.\left.\times\exp(-ieBs\sigma_3)\gamma\cdot x_\perp\right)\right]. \qquad (24)$$

B. Weak field limit

We find it is more convenient to cast Eq. (24) in the form [6]

$$\mathcal{G}(x,x')=\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-x')}\mathcal{G}(p), \qquad (25)$$

with

$$\mathcal{G}(p)=\int d^4xe^{ipx}\mathcal{G}(x)$$
$$=-i\int_0^\infty\frac{ds}{\cos(eBs)}\exp\left[-is\left(m^2-p_\parallel^2\right.\right.$$
$$\left.\left.+\frac{\tan(eBs)}{eBs}p_\perp^2\right)\right]\left[\exp(-ieBs\sigma_3)(m+\gamma\cdot p_\parallel)\right.$$
$$\left.-\frac{\gamma\cdot p_\perp}{\cos(eBs)}\right]. \qquad (26)$$

One can further show that

$$\mathcal{G}(p) = -i\int_0^\infty \frac{ds}{\cos(eBs)} \exp\left[-is\left(m^2 - p_\parallel^2 + \frac{\tan(eBs)}{eBs}p_\perp^2\right)\right] \times\left[[\cos(eBs) + \gamma_1\gamma_2 \sin(eBs)](m + \gamma\cdot p_\parallel) - \frac{\gamma\cdot p_\perp}{\cos(eBs)}\right] \tag{27}$$

when the following identities are applied:

$$\exp(iz\sigma_3) = \cos z\mathbf{I} + i\sin z\sigma_3, \tag{28}$$

$$\sigma_3 \equiv \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = i\gamma_1\gamma_2. \tag{29}$$

If we define a new variable $v \equiv eBs$, then Eq. (27) can be rewritten as [8]

$$\mathcal{G}(p) \equiv -i\int_0^\infty dv\, \exp(-iv\rho)\frac{1}{eB}[(m + \gamma\cdot p_\parallel)I_1 + \gamma_1\gamma_2 \times(m + \gamma\cdot p_\parallel)I_2 - (\gamma\cdot p_\perp)I_3], \tag{30}$$

where

$$I_1 = \exp(-i\alpha\tan v),$$

$$I_2 = \exp(-i\alpha\tan v)\tan v,$$

$$I_3 = \exp(-i\alpha\tan v)\frac{1}{\cos^2 v}, \tag{31}$$

with $\rho \equiv (m^2 - p_\parallel^2)/eB$ and $\alpha \equiv p_\perp^2/eB$. Because $I_j(v) = I_j(v + n\pi)$ for $j = 1,2,3$, we get

$$\int_0^\infty dv \exp(-iv\rho)I_j = \sum_{n=0}^\infty \exp(-i\rho n\pi)\int_0^\pi dv \times \exp(-i\rho v)I_j(v) = \frac{1}{1 - e^{-i\rho\pi}}\int_0^\pi dv\, \exp(-i\rho v)I_j \equiv \frac{1}{1 - e^{-i\rho\pi}}A_j. \tag{32}$$

It is sufficient to evaluate A_1 since the other integrals are obtained using

$$A_2 = i\frac{\partial}{\partial\alpha}A_1,$$

$$A_3 = \frac{-i}{\alpha}(1 - e^{-i\rho\pi}) - \frac{\rho}{\alpha}A_1. \tag{33}$$

To evaluate $A_1 \equiv \int_0^\pi dv\, \exp[-i\alpha\tan v]\exp(-i\rho v)$, we rewrite

$$\exp[-i\alpha\tan v] = \exp\left[\alpha\frac{-e^{-2iv} + 1}{-e^{-2iv} - 1}\right]. \tag{34}$$

The RHS of this equation can be expanded using the Laguerre polynomials. Specifically, the Laguerre polynomials $L_n(x)$ are generated by the following generating function:

$$\frac{\exp[-xZ/(1-Z)]}{1-Z} = \sum_{n=0}^\infty L_n(x)Z^n \tag{35}$$

for $|Z| \leqslant 1$. Upon multiplying Z on both sides of Eq. (35) and subtracting Eq. (35), one arrives at

$$\exp\left[\frac{-xZ}{1-Z}\right] = \sum_{n=0}^\infty (L_n(x) - L_{n-1}(x))Z^n, \tag{36}$$

where one sets $L_{-1}(x) = 0$. Using the identity

$$\exp\left(\frac{x}{2}\frac{Z+1}{Z-1}\right) = \exp\left[-\frac{xZ}{1-Z}\right]\cdot\exp\left(-\frac{x}{2}\right) \tag{37}$$

with the identifications $Z \equiv -e^{-2iv}$, $x \equiv 2\alpha$, and combining Eqs. (32), (34), and (36), one obtains

$$A_1 = \int_0^\pi dv e^{-\alpha}\sum_{n=0}^\infty (L_n(2\alpha) - L_{n-1}(2\alpha)) \times\exp(-2inv)(-1)^n\exp(-i\rho v) = e^{-\alpha}\sum_{n=0}^\infty C_n(2\alpha)(-1)^n\int_0^\pi dv\, \exp[-i(\rho + 2n)v] = -ie^{\alpha}(1 - e^{-i\rho\pi})\sum_{n=0}^\infty \frac{(-1)^n C_n(2\alpha)}{\rho + 2n}. \tag{38}$$

Using Eqs. (30), (33) and (38), one rewrites the propagator function $\mathcal{G}(p)$ into a simple form [8]

$$i\mathcal{G}(p) = \sum_{n=0}^\infty \frac{-id_n(\alpha)D + d_n'(\alpha)\bar{D}}{p_L^2 + 2neB} + i\frac{\gamma\cdot p_\perp}{p_\perp^2}, \tag{39}$$

where $d_n(\alpha) \equiv (-1)^n e^{-\alpha}C_n(2\alpha)$, $d_n' = \partial d_n/\partial\alpha, p_L^2 = m^2 - p_\parallel^2$, and

$$D = (m + \gamma\cdot p_\parallel) + \gamma\cdot p_\perp\frac{m^2 - p_\parallel^2}{p_\perp^2},$$

$$\bar{D} = \gamma_1\gamma_2(m + \gamma\cdot p_\parallel). \tag{40}$$

We note that, in the limit of extreme field strength, i.e., $B \gg B_c$ or $B \ll B_c$, only part of the terms in Eq. (39) are relevant. In the strong field limit $B \gg B_c$, only contributions from the lowest Landau level $n = 0$ need to be kept. For the weak field limit $B \ll B_c$, we shall demonstrate that the infinite series in Eq. (39) may be reorganized in powers of the magnetic field B. Therefore those terms with lower powers of B are more important in this limit. To reorganize the series, we first observe that

$$\sum_{n=0}^{\infty}\frac{-id_nD+d_n'\bar{D}}{p_L^2+2neB}=\frac{1}{p_L^2}\sum_{n=0}^{\infty}\frac{-id_nD+d_n'\bar{D}}{1+\frac{2neB}{p_L^2}}=\frac{1}{p_L^2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-id_nD+d_n'\bar{D})\left(\frac{-2neB}{p_L^2}\right)^k=\sum_{k=0}^{\infty}\frac{1}{p_L^2}\left(\frac{-2eB}{p_L^2}\right)^k\times\left(-iD\sum_{n=0}^{\infty}n^kd_n(\alpha)+\bar{D}\sum_{n=0}^{\infty}n^kd_n'(\alpha)\right). \tag{41}$$

The infinite series $\Sigma_{n=0}^{\infty}n^kd_n(\alpha)$ and $\Sigma_{n=0}^{\infty}n^kd_n'(\alpha)$ can be evaluated with the the identity

$$\sum_{n=0}^{\infty}d_n(\alpha)\exp(-2inv)=\exp[-i\alpha\tan v], \tag{42}$$

which follows from Eqs. (34), (36), and (37). Let us proceed by taking a derivative $\partial/\partial v$ on both sides of Eq. (42). This gives

$$(-2i)^1\sum_{n=0}^{\infty}n^1d_n(\alpha)\exp(-2inv)=\frac{-i\alpha}{\cos^2 v}\exp[-i\alpha\tan v].$$

Taking this derivative k times, we find that

$$(-2i)^k\sum_{n=0}^{\infty}n^kd_n(\alpha)\exp(-2inv)=\left\{\left(\frac{-i\alpha}{\cos^2 v}\right)^k+O(\alpha^{k-1})\right\}\exp[-i\alpha\tan v]. \tag{43}$$

To be more specific, one can define $U(v)\equiv\exp[-i\alpha\tan v]$ following Eq. (42). It can be shown that $\partial_vU=FU$ with $F\equiv-i\alpha/\cos^2 v$. Hence one can show that

$$\partial_v^kU=\sum_{l=0}^{k-1}C_l^{k-1}\partial_v^{k-l-1}F\partial_v^lU=[F^k+C_2^kF^{k-2}\partial_vF+C_3^kF^{k-3}\partial_v^2F+C_2^3C_4^kF^{k-4}\times(\partial_vF)^2]+\kappa_3(\alpha)+\kappa_4(\alpha)+O(\alpha^{k-5}). \tag{44}$$

Here $C_b^a\equiv a!/[b!(a-b)!]$ denotes the number of combinations of size b from a collection of size a. In addition, κ_3 and κ_4 denote the third and fourth derivative terms. They can be shown to be

$$\kappa_3(\alpha)=C_4^kF^{k-4}\partial_v^3F+C_5^kC_2^5F^{k-5}\partial_vF\partial_v^2F+C_2^6C_6^kF^{k-6}(\partial_vF)^3,$$

$$\kappa_4(\alpha)=C_5^kF^{k-5}\partial_v^4F+C_6^kC_4^6F^{k-6}\partial_vF\partial_v^3F+C_2^5C_6^kF^{k-6}(\partial_v^2F)^2+C_7^kC_4^7C_2^3F^{k-7}\partial_v^2F(\partial_vF)^2+C_8^kC_3^7C_2^3F^{k-8}(\partial_vF)^4. \tag{45}$$

Note that above formula for the expansion of ∂_v^kU can either be proved by method of induction or can be read off directly from the combinatorial factor in the the expansion of $(\partial_v+F)^k\cdot 1$ [14]. It is worthy pointing out that $\partial_v^kF(v=0)=0$ for all odd number k and the value of $\partial_v^kF(v=0)$ when k is even can be computed directly. For example one can show that $\partial_v^2F(v=0)=2F(v=0)$, and $\partial_v^4F(v=0)=16F(v=0)$. Hence the order of α^{k-2} and the order of α^{k-4} terms read $\partial_v^kU(v=0)=2C_3^k(-i\alpha)^{k-2}+[16C_5^k+40C_6^k](-i\alpha)^{k-4}$. Similarly, one can also show that the order of α^{k-n} term for the $\bar{D}$ term vanishes when n is an even integer while D term vanishes for all odd integer n. Hence, by setting $v=0$ on both sides of Eq. (43), we obtain

$$\sum_{n=0}^{\infty}n^kd_n(\alpha)=\left(\frac{\alpha}{2}\right)^k-\frac{1}{2}C_3^k\left(\frac{\alpha}{2}\right)^{k-2}+\left[C_5^k+\frac{5}{2}C_6^k\right]\left(\frac{\alpha}{2}\right)^{k-4},$$

$$\sum_{n=0}^{\infty}n^kd_n'(\alpha)=\frac{k}{2}\left(\frac{\alpha}{2}\right)^{k-1}-\frac{k-2}{4}C_3^k\left(\frac{\alpha}{2}\right)^{k-3}+\frac{k-4}{2}\times\left[C_5^k+\frac{5}{2}C_6^k\right]\left(\frac{\alpha}{2}\right)^{k-5}+O(\alpha^{k-6}). \tag{46}$$

Here we only keep terms to the order of $O(\alpha^{k-5})$. Since $\alpha=p_\perp^2/eB$, the leading terms on the RHS of the above equation give up to order of $O(e^3B^3)$ contributions to $\mathcal{G}(p)$, as can be seen from Eq. (41). Precisely we have

$$\sum_{n=0}^{\infty}\frac{-id_nD+d_n'\bar{D}}{p_L^2+2neB}=\sum_{k=0}^{\infty}\frac{1}{p_L^2}\left(\frac{-2eB}{p_L^2}\right)^k\left\{-iD\left[\left(\frac{\alpha}{2}\right)^k-\frac{1}{2}C_3^k\left(\frac{\alpha}{2}\right)^{k-2}\right]+\bar{D}\left[\frac{k}{2}\left(\frac{\alpha}{2}\right)^{k-1}-\frac{k-2}{4}C_3^k\left(\frac{\alpha}{2}\right)^{k-3}\right]\right\}+i\mathcal{G}_4(p)$$
$$=\sum_{k=0}^{\infty}\frac{1}{p_L^2}\left[-iD\left(\frac{-p_\perp^2}{p_L^2}\right)^k+\bar{D}\left(\frac{-p_\perp^2}{p_L^2}\right)^{k-1}\left(\frac{-k}{p_L^2}\right)eB\right]+\sum_{k=0}^{\infty}\frac{1}{p_L^2}\left[i2C_3^kD\left(\frac{eB}{p_L^2}\right)^2\left(\frac{p_\perp^2}{p_L^2}\right)^{k-2}\right.$$
$$\left.+2(k-2)C_3^k\bar{D}\left(\frac{eB}{p_L^2}\right)^3\left(\frac{-p_\perp^2}{p_L^2}\right)^{k-3}\right]+i\mathcal{G}_4(p)$$
$$=\frac{-iD}{p_L^2}\frac{1}{1+\frac{p_\perp^2}{p_L^2}}-\frac{\bar{D}}{(p_L^2)^2}\frac{1}{\left(1+\frac{p_\perp^2}{p_L^2}\right)^2}eB+i\mathcal{G}_2(p)+i\mathcal{G}_4(p)$$
$$=\frac{iD}{p^2-m^2}-\frac{\bar{D}}{(p^2-m^2)^2}eB+i\mathcal{G}_2(p)+i\mathcal{G}_4(p), \tag{47}$$

where $i\mathcal{G}_2(p)$ denotes terms of order e^2B^2 and e^3B^3 and $i\mathcal{G}_4(p)$ denotes terms of order e^4B^4 and e^5B^5. Therefore, by Eqs. (40) and (39), we arrive at

$$i\mathcal{G}(p)=\frac{iD}{p^2-m^2}-\frac{\bar{D}}{(p^2-m^2)^2}eB+i\frac{\gamma\cdot p_\perp}{p_\perp^2}+i\mathcal{G}_2(p)+i\mathcal{G}_4(p)$$
$$=\frac{i(\not{p}+m)}{p^2-m^2}-\frac{\gamma_1\gamma_2(\gamma\cdot p_\parallel+m)}{(p^2-m^2)^2}eB+i\mathcal{G}_2(p)+i\mathcal{G}_4(p). \tag{48}$$

The first term of Eq. (48) is just the electron propagator in the vacuum, while the second term is its correction to $O(eB)$. The corrections with higher powers in eB can be calculated in a similar way. For example, to evaluate the term $\mathcal{G}_2(p)$ and $\mathcal{G}_4(p)$, we need to compute a few identities. Note that $C_3^k=k(k-1)(k-2)/6$ hence one can show that $\Sigma_{k=0}^{\infty}C_3^k(-x)^{k-3}/6=1/(1+x)^4$ for all $|x|<1$ from consecutive differentiating the identity $\Sigma_{k=0}^{\infty}(-x)^k=1/(1+x)$ which is valid for all $|x|<1$. Similarly, one can also show that $\Sigma_{k=0}^{\infty}(k-2)C_3^k(-x)^{k-3}=1/(1+x)^4-4x/(1+x)^5$ for all $|x|<1$. Therefore, one can extract the $O(e^2B^2)$ and $O(e^3B^3)$ terms from the series expansion in Eq. (41). The result leads to the following contribution:

$$i\mathcal{G}_2(p)=-\frac{2ie^2B^2p_\perp^2}{(p^2-m^2)^4}D+2e^3B^3\times\left[\frac{1}{(p^2-m^2)^4}+\frac{4p_\perp^2}{(p^2-m^2)^5}\right]\bar{D}, \tag{49}$$

where D and $\bar{D}$ have been defined in Eq. (40). Similarly, one can show that $\Sigma_{k=0}^{\infty}C_5^k(-x)^{k-5}=1/(1+x)^6$, $\Sigma_{k=0}^{\infty}C_6^k(-x)^{k-6}=1/(1+x)^7$, $\Sigma_{k=0}^{\infty}(k-4)C_5^k(-x)^{k-5}=1/(1+x)^6-6x/(1+x)^7$ and $\Sigma_{k=0}^{\infty}(k-4)C_6^k(-x)^{k-6}=2/(1+x)^7-7x/(1+x)^8$ for all $|x|<1$. Hence the fourth and fifth order propagator $i\mathcal{G}_4(p)$ can be shown to be

$$i\mathcal{G}_4(p)=-[8ie^4B^4]\left[\frac{2p_\perp^2p_L^2-3(p_\perp^2)^2}{(p^2-m^2)^7}\right]D-8e^5B^5\left[\frac{15(p_\perp^2)^2-16p_\perp^2p_L^2+2p_L^2}{(p^2-m^2)^8}\right]\bar{D}. \tag{50}$$

C. Phase factor

In this subsection, we discuss how to treat the phase factor $\Phi(x,x')$ as defined in Eq. (22). First, we note that $\Phi(x,x')$ is reduced to

$$\Phi(x,x')=\exp\left\{ie\int_{x'}^{x}d\xi^\mu A_\mu(\xi)\right\}, \tag{51}$$

if the integration path connecting x and x' is a straight line. This choice of integration path is only for convenience since the integration in Eq. (22) is path independent provided that the vector potential $A_\mu(\xi)$ is nonsingular. Second, for a particular type of Coulomb gauge:

$$A_0(\xi)=0,$$

$$\mathbf{A}(\xi)=\frac{B}{2}(x_2'-\xi_2,\xi_1-x_1',0),$$

the exponent $\int_{x'}^{x}d\xi^\mu A_\mu(\xi)$ vanishes, hence $\Phi(x,x')=1$. Therefore, by choosing the above Coulomb gauge, the phase factor $\Phi(x,x')$ in the electron Green's function can be disregarded. Such a simplification is, however, no longer valid for more complicated processes where more than one electron propagators are involved in the process. To illustrate, let us consider a one-loop triangular diagram composed of three electron propagators. We denote vertices of the diagram as

P, Q and R respectively. It is useful to recall that the full phase factor between two points P and Q is

$$\Phi(P,Q)=\exp\left\{i\int_Q^P dx^\mu\left[A_\mu(x)+\frac{1}{2}F_{\mu\nu}(x-Q)^\nu\right]\right\} \tag{52}$$

according to Eq. (22). Here we use P^μ to denote the coordinate of the point P. Similarly Q^μ and R^μ denote coordinates of the points Q and R respectively. As discussed before, one can set $\Phi(P,Q)=1$ by choosing the special gauge

$$\mathbf{A}_Q(\mathbf{x})\equiv\mathbf{A}_0(\mathbf{x})+\tilde{\mathbf{A}}_Q(\mathbf{x}), \tag{53}$$

with $\mathbf{A}_0(\mathbf{x})=B/2\cdot(-x_2,x_1,0)$ and $\tilde{\mathbf{A}}_Q(\mathbf{x})=B/2\cdot(Q_2,-Q_1,0)$. Similarly, one can respectively set $\Phi(R,P)$ and $\Phi(Q,R)$ to unity by choosing the gauges

$$\mathbf{A}_P(\mathbf{x})\equiv\mathbf{A}_0(\mathbf{x})+\tilde{\mathbf{A}}_P(\mathbf{x}),$$

$$\mathbf{A}_R(\mathbf{x})\equiv\mathbf{A}_0(\mathbf{x})+\tilde{\mathbf{A}}_R(\mathbf{x}), \tag{54}$$

with $\tilde{\mathbf{A}}_P(\mathbf{x})=B/2\cdot(P_2,-P_1,0)$ and $\tilde{\mathbf{A}}_R(\mathbf{x})=B/2\cdot(R_2,-R_1,0)$ respectively. Apparently, $A_Q(\mathbf{x})$, $A_P(\mathbf{x})$, and $A_R(\mathbf{x})$ are distinct from one another. Hence they cannot be adopted simultaneously to set all phase factors to unity. In other words, the phase factors shall give rise to a nontrivial contribution to the three-point amplitude. In fact this nontrivial contribution can be understood in an alternative view. Taking Eq. (52) as an example, the integrand on the RHS of the equation can be written as $\mathcal{A}\equiv A+\frac{1}{2}F\Delta x$ where $\mathcal{A}\equiv\mathcal{A}_\mu dx^\mu$, $A\equiv A_\mu dx^\mu$, and $F\Delta x\equiv F_{\mu\nu}dx^\mu(x-Q)^\nu$ are all one-form. One can easily show that $\mathcal{A}$ is a closed form, i.e.,

$$d\mathcal{A}=0. \tag{55}$$

Note that $\mathcal{A}$ is exact if the first homology group is trivial, namely, $H_1(M)=0$. To be more specific, if the gauge function $A_\mu(x)$ is regular everywhere, then the one-form $\mathcal{A}$ is also regular. Therefore there exists a zero-form ω such that $\mathcal{A}=d\omega$ is an exact form. As a result, the line integration which defines $\Phi(P,Q)$ is path independent. In our problem, we need to compute the product of three phases: $\Phi(P,Q)\cdot\Phi(R,P)\cdot\Phi(Q,R)$. It is then important to note that the one-form $\mathcal{A}$ in each of the above phases depends on the boundary point of the path, despite the fact that the gauge function $A_\mu(x)$ is regular. In other words, the gauge of $\mathcal{A}$ is chosen differently in each path, which then gives rise to a non-trivial phase for a three-point amplitude. Precisely one may isolate the boundary dependencies of $\mathcal{A}$ by writing, for example, $\mathcal{A}=\mathcal{A}'-F_{\mu\nu}dx^\mu Q^\nu$ in the case of $\Phi(P,Q)$. Apparently, $\mathcal{A}'$ is an exact form universal to the three phases, while $F_{\mu\nu}dx^\mu Q^\nu$ depends on the boundary point Q. Using this separation, one may rewrite each phase as

$$\Phi(x,x')=\exp\left[ie\int_{x'}^x d\xi^\mu\left(A_\mu+\frac{1}{2}F_{\mu\nu}x^\nu\right)\right]$$
$$\times\exp\left[-i\frac{e}{2}F_{\mu\nu}\int_{x'}^x d\xi^\mu x'^\nu\right]. \tag{56}$$

Let us denote the first factor $\exp[ie\int_{x'}^x d\xi^\mu(A_\mu+\frac{1}{2}F_{\mu\nu}x^\nu)]$ as $\Phi'(x,x')$. Since $\Phi'(P,Q)\cdot\Phi'(R,P)\cdot\Phi'(Q,R)=\Phi'(Q,Q)=1$, we conclude from Eq. (56) that

$$\Phi(P,Q)\cdot\Phi(R,P)\cdot\Phi(Q,R)$$
$$=\exp\left[-\frac{ie}{2}(R-P)^\mu F_{\mu\nu}(P-Q)^\nu\right]. \tag{57}$$

This is the nontrivial phase contribution one must attach to the amplitude of a three-point process when we write all weak field charged propagator according to Eq. (48).

III. APPLICATIONS

A. $\gamma\gamma\to\nu\bar{\nu}$

The weak-field expansion derived in the last section has been applied to calculate the amplitudes of $\gamma\gamma\to\nu\bar{\nu}$ and its crossed processes in a homogeneous magnetic field less than B_c [4]. According to the discussion in the previous section, the magnetic-field dependencies of above amplitudes reside in two places: the first place is in the electron propagator which is affected by the external magnetic field, while the second place is in the overall phase which is a function of the field strength tensor $F_{\mu\nu}$. Let us now take $\gamma\gamma\to\nu\bar{\nu}$ as an example for illustration. Since the incoming photon energies are much less than m_W, we can calculate the scattering amplitudes using the following effective four-fermion interactions between leptons and neutrinos:

$$\mathcal{L}=-\frac{G_F}{\sqrt{2}}(\bar{\nu}_l\gamma_\alpha(1-\gamma_5)\nu_l)(\bar{e}\gamma^\alpha(g_V-g_A\gamma_5)e), \tag{58}$$

where $g_V=1/2+2\sin^2\theta_w$ and $g_A=1/2$ for $l=e$; $g_V=-1/2+2\sin^2\theta_w$ and $g_A=-1/2$ for $l=\mu,\tau$. The Feynman diagram contributing to $\gamma\gamma\to\nu\bar{\nu}$ is shown in Fig. 1 of Ref. [4]. We should remark that the contribution due to g_A is proportional to the neutrino mass in the limit of vanishing magnetic field. At $O(eB)$ in the limit $B\ll B_c$, it gives no contribution to the amplitude by the charge conjugation invariance. Therefore we shall neglect the contribution by g_A. Likewise, we shall also neglect contributions by g_V for $l=\mu,\tau$, since $-1/2+2\sin^2\theta_w=0.04\ll1$.

To $O(eB)$, the amplitude for $\gamma\gamma\to\nu\bar{\nu}$ can be written as $M\equiv M_1+M_2$, where M_1 arises from inserting the external magnetic field to electron propagators according to Eq. (48), whereas M_2 comes from expanding the overall phase factor for the three-point function as shown in Eq. (57). Therefore one has

$$M_1=i4\pi\alpha eB\frac{G_Fg_V}{\sqrt{2}}\overline{u}(p_2)\gamma^{\alpha}(1-\gamma_5)v(p_1)\epsilon_1^{\mu}\epsilon_2^{\nu}\int\frac{d^4l}{(2\pi)^4}$$
$$\times\mathrm{tr}\Bigg\{\gamma_{\alpha}(1-\gamma_5)\Bigg[\frac{\gamma_1\gamma_2[\gamma\cdot(l+k_1)_{\parallel}+m]}{[(l+k_1)^2-m^2]^2}\gamma^{\mu}\frac{i(\not{l}+m)}{l^2-m^2}\gamma^{\nu}\frac{i(\not{l}-\not{k}_2+m)}{(l-k_2)^2-m^2}$$
$$+\frac{i(\not{l}+\not{k}_1+m)}{(l+k_1)^2-m^2}\gamma^{\mu}\frac{\gamma_1\gamma_2[\gamma\cdot l_{\parallel}+m]}{(l^2-m^2)^2}\gamma^{\nu}\frac{i(\not{l}-\not{k}_2+m)}{(l-k_2)^2-m^2}$$
$$+\frac{i(\not{l}+\not{k}_1+m)}{(l+k_1)^2-m^2}\gamma^{\mu}\frac{i(\not{l}+m)}{l^2-m^2}\gamma^{\nu}\frac{\gamma_1\gamma_2[\gamma\cdot(l-k_2)_{\parallel}+m]}{[(l-k_2)^2-m^2]^2}\Bigg]\Bigg\}, \tag{59}$$

where $g_V=1-(1-4\sin^2\theta_w)/2$ for ν_e and m is the mass of the electron. The first and second term in g_V are the contributions from the W and Z exchanges, respectively. To write down the amplitude M_2, we recall Eq. (57) which states that the overall phase factor for $\gamma\gamma\to\nu\overline{\nu}$ is

$$\Phi(X,Y)\cdot\Phi(Z,X)\cdot\Phi(Y,Z)=\exp\left[-\frac{ie}{2}(Z-X)^{\mu}F_{\mu\nu}(X-Y)^{\nu}\right]. \tag{60}$$

With **B** in the forward z direction and choosing $X_{\mu}=(0,0,0,0)$, we arrive at

$$\Phi(X,Y)\cdot\Phi(Z,X)\cdot\Phi(Y,Z)=\exp\left\{\left(\frac{-iB}{2}\right)(Y_1Z_2-Y_2Z_1)\right\}$$
$$\simeq 1-\frac{ieB}{2}(Y_1Z_2-Y_2Z_1). \tag{61}$$

Since we calculate the amplitude only to $O(eB)$, the first term of the above expansion gives rise to M_1; while the second term gives rise to M_2 which reads

$$M_2=i4\pi\alpha eB\frac{G_Fg_V}{\sqrt{2}}\int d^4Yd^4Z\frac{-i}{2}(Y_1Z_2-Y_2Z_1)\int\frac{d^4ld^4qd^4r}{(2\pi)^{12}}\epsilon_1^{\mu}\epsilon_2^{\nu}$$
$$\times\exp[-i(q-l-k_1)\cdot Y]\exp[-i(r-q-k_2)\cdot Z]\overline{u}(p_2)\gamma^{\alpha}(1-\gamma_5)v(p_1)$$
$$\times\mathrm{tr}\left\{\gamma_{\alpha}(1-\gamma_5)\frac{i(\not{l}+m)}{l^2-m^2}(-ie\gamma_{\mu})\frac{i(\not{q}+m)}{q^2-m^2}(-ie\gamma_{\nu})\frac{i(\not{r}+m)}{r^2-m^2}\right\}. \tag{62}$$

We can recast the amplitude M_2 by using the equations

$$Y_i\exp[-i(q-l-k_1)\cdot Y]=-i\frac{\partial}{\partial l_i}\exp[-i(q-l-k_1)\cdot Y],$$

$$Z_i\exp[-i(r-q-k_2)\cdot Z]=i\frac{\partial}{\partial r_i}\exp[-i(r-q-k_2)\cdot Z],$$

and the integration by part, such that

$$M_2=i4\pi\alpha eB\frac{G_Fg_V}{\sqrt{2}}\int d^4Yd^4Z\frac{-i}{2}\int\frac{d^4ld^4qd^4r}{(2\pi)^{12}}\epsilon_1^{\mu}\epsilon_2^{\nu}\exp[-i(q-l-k_1)\cdot Y]\exp[-i(r-q-k_2)\cdot Z]\overline{u}(p_2)\gamma^{\alpha}(1-\gamma_5)v(p_1)$$
$$\times\left[\frac{\partial}{\partial l_1}\frac{\partial}{\partial r_2}-\frac{\partial}{\partial l_2}\frac{\partial}{\partial r_1}\right]\mathrm{tr}\left\{\gamma_{\alpha}(1-\gamma_5)\frac{i(\not{l}+m)}{l^2-m^2}(-ie\gamma_{\mu})\frac{i(\not{q}+m)}{q^2-m^2}(-ie\gamma_{\nu})\frac{i(\not{r}+m)}{r^2-m^2}\right\}. \tag{63}$$

Before we proceed to compute M_1 and M_2, we wish to reiterate the validity of the above expansion. As we have pointed out in Ref. [4] that, by dimensional analysis, any given power of eB in the expansion of M is accompanied by an equal power of $1/m^2$ (for $m>p$) or $1/p^2$ (for $p>m$). Here p denotes the typical energy scale of external particles. Therefore, both eB/m^2 and eB/p^2 are much smaller than unity for $B\ll B_c\equiv m^2/e$. Now performing the integration in M_1 and M_2, we obtain

$$M \equiv M_1 + M_2 = \frac{G_F g_V \alpha^{3/2}}{\sqrt{2}\sqrt{4\pi}} \overline{u}(p_2)\gamma_\alpha(1-\gamma_5)v(p_1)J^\alpha, \tag{64}$$

where [15]

$$\begin{aligned} J^\alpha = {} & C_1(\epsilon_1 F \epsilon_2)(k_1^\alpha - k_2^\alpha) + C_2[(\epsilon_1 F k_1)(k_1\cdot\epsilon_2)k_2^\alpha + (\epsilon_2 F k_2)(k_2\cdot\epsilon_1)k_1^\alpha] + C_3[(\epsilon_1 F k_1)\epsilon_2^\alpha + (\epsilon_2 F k_2)\epsilon_1^\alpha] + C_4[(\epsilon_1 F k_2) \\ & \times (k_1\cdot\epsilon_2)k_1^\alpha + (\epsilon_2 F k_1)(k_2\cdot\epsilon_1)k_2^\alpha] + C_5[(\epsilon_1 F k_2)(k_1\cdot\epsilon_2)k_2^\alpha + (\epsilon_2 F k_1)(k_2\cdot\epsilon_1)k_1^\alpha] + C_6[(\epsilon_1 F k_2)\epsilon_2^\alpha + (\epsilon_2 F k_1)\epsilon_1^\alpha] \\ & + C_7(k_2\cdot\epsilon_1)(k_1\cdot\epsilon_2)[(Fk_1)^\alpha + (Fk_2)^\alpha] + C_8(\epsilon_1\cdot\epsilon_2)[(Fk_1)^\alpha + (Fk_2)^\alpha] + C_9(k_1 F k_2)(\epsilon_1\cdot\epsilon_2)(k_1^\alpha - k_2^\alpha) + C_{10}(k_1 F k_2) \\ & \times (k_2\cdot\epsilon_1)(k_1\cdot\epsilon_2)(k_1^\alpha - k_2^\alpha) + C_{11}(k_1 F k_2)[(k_2\cdot\epsilon_1)\epsilon_2^\alpha - (k_1\cdot\epsilon_2)\epsilon_1^\alpha], \end{aligned} \tag{65}$$

with, for instance, $(\epsilon_1 F \epsilon_2) \equiv \epsilon_1^\mu F_{\mu\nu} \epsilon_2^\nu$ and $(Fk_1)^\alpha \equiv F^{\alpha\beta} k_{1\beta}$. The coefficient functions C_i's are given as follows:

$$\begin{aligned} C_1 = {} & -\frac{8}{m^2}(I[0,0,1] + I[0,0,2] - 4I[1,1,1] - 5I[1,1,2] + 2I[2,1,1] + 2I[2,1,2] + tI[2,1,2] + 2I[2,2,1] + 2I[2,2,2] \\ & - 5tI[3,2,2] + 2tI[4,2,2] + 2tI[4,3,2]), \end{aligned}$$

$$C_2 = -\frac{8}{m^4}(I[1,1,2] - 2I[2,1,2] - 3I[2,2,2] + 4I[3,2,2] + 2I[3,3,2] - 4I[4,3,2]),$$

$$\begin{aligned} C_3 = {} & -\frac{4}{m^2}(2I[0,0,2] - 4I[1,1,1] - 4I[1,1,2] - tI[1,1,2] + 2cI[2,1,2] + 2I[2,2,1] + 2I[2,2,2] + 3tI[2,2,2] - 4tI[3,2,2] \\ & - 2tI[3,3,2] + 2tI[4,3,2]), \end{aligned}$$

$$C_4 = -\frac{16}{m^4}(5I[3,2,2] - 2I[4,2,2] - 4I[4,3,2]),$$

$$C_5 = -\frac{8}{m^4}(I[1,1,2] + 2I[2,1,2] - I[2,2,2] - 10I[3,2,2] + 8I[4,2,2] + 4I[4,3,2]),$$

$$\begin{aligned} C_6 = {} & -\frac{4}{m^2}(2I[0,0,1] + 2I[0,0,2] - 4I[1,1,1] - 4I[1,1,2] - tI[1,1,2] - 4I[2,1,1] - 4I[2,1,2] - 2I[2,2,1] - 2I[2,2,2] \\ & + tI[2,2,2] + 2tI[3,2,2] - 4tI[4,2,2] - 2tI[4,3,2]), \end{aligned}$$

$$C_7 = \frac{8}{m^4}(I[1,1,2] - 2I[2,1,2] - I[2,2,2] + 4I[3,2,2] - 4I[4,3,2]),$$

$$\begin{aligned} C_8 = {} & \frac{4}{m^2}(2I[0,0,2] - 4I[1,1,1] - 4I[1,1,2] - tI[1,1,2] + 2tI[2,1,2] + 2I[2,2,1] + 2I[2,2,2] + tI[2,2,2] - 4tI[3,2,2] \\ & + 2tI[4,3,2]), \end{aligned}$$

$$\begin{aligned} C_9 = {} & -\frac{8}{m^4}(I[1,1,2] + 2I[2,1,2] + 4I[2,1,3] - I[2,2,2] - 10I[3,2,2] - 12I[3,2,3] + 4I[4,2,2] + 4I[4,2,3] + 4tI[4,2,3] \\ & + 4I[4,3,2] + 4I[4,3,3] - 12tI[5,3,3] + 4tI[6,3,3] + 4tI[6,4,3]), \end{aligned}$$

$$C_{10} = \frac{64}{m^6}(I[4,2,3] - 4I[5,3,3] + 2I[6,3,3] + 2I[6,4,3]),$$

$$C_{11}=-\frac{8}{m^4}(I[1,1,2]+2I[2,1,2]+4I[2,1,3]-I[2,2,2]-4I[3,2,3]-4I[4,2,2]-4I[4,2,3]-4I[4,3,2]-4I[4,3,3]+4tI[5,3,3]-4tI[6,3,3]-4tI[6,4,3]), \tag{66}$$

where

$$I[a,b,c]\equiv\int_0^1 dx\int_0^{1-x} dy\frac{x^b y^{a-b}}{(1-txy-i\varepsilon)^c}, \tag{67}$$

with $t\equiv 2(k_1\cdot k_2/m^2)$. Our result is an extension of the calculation in Ref. [3] which considers only the low energy limit $k_1\cdot k_2\ll m^2$. In such a limit, one can calculate M using the effective Lagrangian for $\gamma\gamma\to\nu\bar{\nu}\gamma$ [9] and replacing one of the photons by the external magnetic field.

With the amplitude M, it is straightforward to compute the scattering cross section $\sigma_B(\gamma\gamma\to\nu\bar{\nu})$ in the background magnetic field. Since $\gamma\gamma\to\nu\bar{\nu}$ could contribute to the energy-loss of a magnetized star, it is useful to compute the stellar energy-loss rate Q, which is related to σ_B through [16]

$$Q=\frac{1}{2(2\pi)^6}\int\frac{2d^3\vec{k}_1}{e^{\omega_1/T}-1}\times\int\frac{2d^3\vec{k}_2}{e^{\omega_2/T}-1}\frac{(k_1\cdot k_2)}{\omega_1\omega_2}(\omega_1+\omega_2)\times\sigma_B(\gamma\gamma\to\nu\bar{\nu}). \tag{68}$$

In Ref. [3], Q is calculated based upon an approximated cross section obtained in the limit $E_\gamma\ll m$. Such a calculation is repeated in our earlier work [4] which is based upon the cross section $\sigma_B(\gamma\gamma\to\nu\bar{\nu})$ obtained from the amplitude M in Eq. (64). We found that, for temperatures below 0.01 MeV, the effective-Lagrangian approach employed in Ref. [3] works very well. On the other hand, this approach becomes rather inaccurate for temperatures greater than 1 MeV. At $T=0.1$ MeV, our calculation gives an energy-loss rate almost two orders of magnitude greater than the result from the effective Lagrangian. Such a behavior can be understood from the energy dependence of the scattering cross section, as shown in Fig. 2 of Ref. [4]. It is clear that, for $T=0.1$ MeV, Q must have received significant contributions from scatterings with $\omega\approx m$. At this energy, the full calculation gives a much larger scattering cross section than that given by the effective Lagrangian. By comparing the predictions of the full calculation and the effective-Lagrangian approach [3], we conclude that the applicability of the latter to the energy-loss rate is quite restricted. While the effective Lagrangian works reasonably well with $\omega<0.1m$, it would give a poor approximation on Q unless $T<0.01m$.

B. $\gamma\to\nu\bar{\nu}$, $\nu\to\nu\gamma$

In order to compare our approach with previous ones, we consider the simple two-body decay modes $\gamma\to\nu\bar{\nu}$ [10–12] and $\nu\to\nu\gamma$ [17] in a background magnetic field. We shall limit the energies of incoming particles to be less than the pair-production threshold $2m$. For if $E_\gamma,E_\nu>2m$, the dominant decay modes should become $\gamma\to e^+e^-$ and $\nu\to\nu e^+e^-$ respectively. For incoming energies below the pair-production threshold, it turns out that the photon momenta in both $\gamma\to\nu\bar{\nu}$ and $\nu\to\nu\gamma$ are spacelike [18]. Hence the former process is kinematically forbidden. The amplitude of the latter process can be written as

$$\mathcal{M}(\nu(p_1)\to\nu(p_2)\gamma(q))=-\frac{G_F}{\sqrt{2}e}Z\epsilon^\alpha\overline{u}(p_2)\gamma^\beta(1-\gamma_5)u(p_1)\times(g_V\Pi_{\alpha\beta}(-q)-g_A\Pi^5_{\alpha\beta}(-q)), \tag{69}$$

where $\Pi_{\alpha\beta}$ and $\Pi^5_{\alpha\beta}$ are vector-vector and vector-axial vector two-point functions given by

$$\Pi_{\alpha\beta}(q)=-e^2\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}[\gamma_\alpha\mathcal{G}(k-q)\gamma_\beta\mathcal{G}(k)],$$

$$\Pi^5_{\alpha\beta}(q)=-e^2\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}[\gamma_\alpha\mathcal{G}(k-q)\gamma_\beta\gamma_5\mathcal{G}(k)]. \tag{70}$$

The factor Z is the wave-function renormalization constant of the photon field, induced by the effect of external magnetic fields. Since the deviation of Z from the unity is rather small, proportional to the fine structure constant α, we shall set $Z=1$ in our subsequent discussions.

The structures of the two-point functions $\Pi_{\alpha\beta}$ and $\Pi^5_{\alpha\beta}$ were given in previous literature [6,12]

$$\Pi_{\alpha\beta}(q)=A(q_\parallel^2 g_{\parallel\alpha\beta}-q_{\parallel\alpha}q_{\parallel\beta})+B(-q_\perp^2 g_{\perp\alpha\beta}-q_{\perp\alpha}q_{\perp\beta})+C(q^2 g_{\alpha\beta}-q_\alpha q_\beta),$$

$$\Pi^5_{\alpha\beta}(q)=C_\parallel(q_\parallel^2\tilde{F}_{\alpha\beta}+q_{\parallel\alpha}(\tilde{F}q)_\beta+q_{\parallel\beta}(\tilde{F}q)_\alpha)+C_\perp(-q_\perp^2\tilde{F}_{\alpha\beta}+q_{\perp\alpha}(\tilde{F}q)_\beta+q_{\perp\beta}(\tilde{F}q)_\alpha). \tag{71}$$

We wish to remind the reader that $q_\perp^2=(q^1)^2+(q^2)^2$ for a magnetic field in the $+z$ direction. The calculations of $\Pi_{\alpha\beta}$ and $\Pi^5_{\alpha\beta}$ for $B<B_c$ are straightforward using the weak field expansion derived in Eqs. (48), (49), and (50). Due to charge-conjugation and gauge invariances, the magnetic-field effects to $\Pi_{\alpha\beta}$ and $\Pi^5_{\alpha\beta}$ begin at the order e^2B^2 and

e^3B^3 respectively. The subleading contributions are then of the order e^4B^4 and e^5B^5 respectively. The coefficient functions of $\Pi_{\alpha\beta}(q)$ are given by

$$A=\frac{i\alpha}{\pi}\left[-\frac{7}{45}\left(\frac{B}{B_c}\right)^2+\left(\frac{26}{315}-\frac{52}{945}\frac{\omega^2}{m^2}\sin^2\theta\right)\left(\frac{B}{B_c}\right)^4+\cdots\right],$$

$$B=\frac{i\alpha}{\pi}\left[\frac{4}{45}\left(\frac{B}{B_c}\right)^2+\left(-\frac{16}{105}+\frac{4}{135}\frac{\omega^2}{m^2}\sin^2\theta\right)\left(\frac{B}{B_c}\right)^4+\cdots\right],$$

$$C=\frac{i\alpha}{\pi}\left[\left(\frac{2}{45}-\frac{1}{105}\frac{\omega^2}{m^2}\sin^2\theta\right)\left(\frac{B}{B_c}\right)^2+\left(-\frac{4}{105}+\frac{44}{1575}\times\frac{\omega^2}{m^2}\sin^2\theta-\frac{10}{2079}\frac{\omega^4}{m^4}\sin^4\theta\right)\left(\frac{B}{B_c}\right)^4+\cdots\right], \tag{72}$$

where ω is the photon energy while θ is the angle between the the magnetic-field direction and the direction of photon propagation. For the coefficient functions of $\Pi^5_{\alpha\beta}(q)$, we find

$$C_{\parallel}=\frac{i\alpha}{B\pi}\left[\frac{1}{70}\frac{\omega^2}{m^2}\sin^2\theta\left(\frac{B}{B_c}\right)^3+\left(-\frac{26}{945}\frac{\omega^2}{m^2}\sin^2\theta+\frac{10}{693}\frac{\omega^4}{m^4}\sin^4\theta\right)\left(\frac{B}{B_c}\right)^5+\cdots\right],$$

$$C_{\perp}=\frac{i\alpha}{B\pi}\left[\left(-\frac{1}{15}+\frac{1}{70}\frac{\omega^2}{m^2}\sin^2\theta\right)\left(\frac{B}{B_c}\right)^3+\left(\frac{8}{63}-\frac{86}{945}\frac{\omega^2}{m^2}\sin^2\theta+\frac{10}{693}\frac{\omega^4}{m^4}\sin^4\theta\right)\times\left(\frac{B}{B_c}\right)^5+\cdots\right]. \tag{73}$$

It should be noted that the validity of weak-field expansion in Eqs. (72) and (73) also depends on the ratio $r\equiv\omega^2\sin^2\theta B^2/m^2B_c^2$, in addition to the requirement $(B/B_c)^2\ll 1$. For a sufficiently large photon energy such that $r>1$, the expansion in Eqs. (72) and (73) may break down. However, since we have limited the photon energy to $\omega<2m$, the ratio r is automatically smaller than 1.

The computation of $\nu\to\nu\gamma$ width requires the knowledge of photon index of refraction $n\equiv|\vec{q}|/\omega$. The index of refraction can be calculated from the two-point function $\Pi_{\alpha\beta}(q)$. It is well known that n depends on the photon polarizations. For the magnetic field in the $+z$ direction, the polarization states with distinct index of refraction are $\epsilon^{\mu}_{\perp}=(0,0,1,0)$ and $\epsilon^{\mu}_{\parallel}=(0,-\cos\theta,0,\sin\theta)$. Here we have adopted the convention that $q^{\mu}=(\omega,\omega\sin\theta,0,\omega\cos\theta)$, i.e., photon propagates on the $x-z$ plane with an angle θ to the magnetic field direction. Hence $\vec{\epsilon}_{\perp}$ is the polarization vector perpendicular to the $x-z$ plane while $\vec{\epsilon}_{\parallel}$ lies on the $x-z$ plane. The photon dispersion relation is given by

$$q^2-i\Pi_a=0, \tag{74}$$

where $\Pi_a=\epsilon^{\alpha}_a\Pi_{\alpha\beta}\epsilon^{\beta}_a$. Here the index a stands for the polarization states, namely $a=\perp$ or $\parallel$. Combining Eqs. (72) and (74), we arrive at

$$[1+iB\sin^2\theta+iC]q^2=iB\omega^2\sin^2\theta,$$

$$[1+iC]q^2=-iA\omega^2\sin^2\theta, \tag{75}$$

for polarization states $a=\perp$ and $a=\parallel$ respectively. Since the electomagnetic coupling constant is rather small, the left hand side of the above equations may be approximated by q^2. Using the definition $q^2=\omega^2\cdot(1-n^2)$, we obtain

$$n_{\perp}=1+\frac{\alpha}{\pi}\left[\frac{2}{45}\left(\frac{B}{B_c}\right)^2+\left(-\frac{8}{105}+\frac{2}{135}\frac{\omega^2}{m^2}\sin^2\theta\right)\times\left(\frac{B}{B_c}\right)^4+\cdots\right]\sin^2\theta,$$

$$n_{\parallel}=1+\frac{\alpha}{\pi}\left[\frac{7}{90}\left(\frac{B}{B_c}\right)^2+\left(-\frac{13}{315}+\frac{26}{945}\frac{\omega^2}{m^2}\sin^2\theta\right)\left(\frac{B}{B_c}\right)^4+\cdots\right]\sin^2\theta. \tag{76}$$

It is seen that the leading contributions to $n_{\perp}$ and $n_{\parallel}$ agree with the results obtained by Adler [18]. The next-to-leading contribution to $n_{\perp,\parallel}$ depend on both the photon energy ω and the photon propagation direction [19].

Given the above photon dispersion relation, we proceed to compute the $\nu\to\nu\gamma$ width in the subcritical background magnetic field. We note that the most recent calculation of $\nu\to\nu\gamma$ width is performed by Ioannisian and Raffelt [17]. Following their approach, we write the width of this process as

$$\Gamma=\frac{1}{16\pi E_1^2}\int_0^{\omega_{\max}}d\omega\sum_{\text{pols}}|\mathcal{M}|^2, \tag{77}$$

where $\mathcal{M}$ is the amplitude given by Eq. (69), E_1 is the neutrino energy, and $\omega_{\max}=\min(E_1,\omega_c)$ with ω_c the critical photon energy beyond which the photon four momentum becomes timelike and the Cherenkov condition no longer holds. We note that, in deriving the above width, one has taken the collinear approximation that the particles in the initial and final states are all parallel with one another. The correction to such an approximation is small, proportional to the fine structure constant α. From Eq. (69), we obtain

$$|\mathcal{M}|^2=\frac{4g_A^2G_F^2}{e^2}\epsilon^{\alpha}(a)\epsilon^{*\alpha'}(a)(p_1^{\beta}p_2^{\beta'}+p_1^{\beta'}p_2^{\beta})\Pi^5_{\alpha\beta}\Pi^5_{\alpha'\beta'}, \tag{78}$$

where $a=\parallel,\perp$. The contribution by the vector-vector two-point function $\Pi_{\alpha\beta}$ is negligible since both p_1 and p_2 are approximately parallel to the photon momentum q and $q^{\beta}\Pi_{\alpha\beta}=0$ due to the gauge invariance. The fact that both p_1

and p_2 are approximately parallel to q also has a consequence on the polarization dependencies of $|\mathcal{M}|^2$. This is easily seen with

$$q^\beta \Pi^5_{\alpha\beta} = 2(C_\parallel q_\parallel^2 - C_\perp q_\perp^2)(\tilde{F}q)_\alpha . \tag{79}$$

For a B field in the $+z$ direction, $(\tilde{F}q)_\alpha$ is nonvanishing only for $\alpha=0,3$. Given $\epsilon_\perp^\mu=(0,0,1,0)$ and $\epsilon_\parallel^\mu=(0,-\cos\theta,0,\sin\theta)$ as stated earlier, one immediately sees that $|\mathcal{M}|^2$ vanishes for a photon in a $\perp$ mode. Hence the photon radiated from the neutrino is polarized, with its polarization vector lying on the surface spanned by $\vec{q}$ and $\vec{B}$.

The width of $\nu\to\nu\gamma$ can be readily calculated using Eqs. (77), (78) and (71). We have

$$\Gamma = \frac{g_A^2 G_F^2 B^2}{2\pi^2 E_1 \alpha} \sin^6\theta \int_0^{\omega_{\max}} d\omega (E_1-\omega)\omega^4 |C_\parallel - C_\perp|^2. \tag{80}$$

Since $C_\parallel$ and $C_\perp$ are already given by Eq. (73), Γ can be easily determined once $\omega_{\max}$ is specified. Since $E_1<2m$, which implies $\omega<2m$, the photon refractive index is always greater than 1 as indicated by Eq. (76). Hence the critical energy ω_c for photon dispersion relation to cross the light cone is greater than $2m$. Thus $\omega_{max}\equiv\min(E_1,\omega_c)=E_1$. The width Γ is given as follows:

$$\Gamma = \frac{2G_F^2 \alpha E_1^5}{135(2\pi)^4}\sin^6\theta \left[\frac{1}{50}\left(\frac{B}{B_c}\right)^6 - \left(\frac{8}{105} - \frac{1}{49}\frac{E_1^2\sin^2\theta}{m^2}\right)\left(\frac{B}{B_c}\right)^8 + \cdots\right]. \tag{81}$$

Comparisons of our result with the earlier results of Refs. [12,17] are in order. First, we focus on the weak field region $B<B_c$ while Refs. [12,17] considers the general magnetic field and the corresponding coefficient functions $C_{\parallel,\perp}$ are expressed in double integrals. Second, due to a different convention, the coefficient functions obtained in Ref. [17], denoted as $C'_{\parallel,\perp}$, are related to ours via the relation $|C_\parallel - C_\perp| = (e^4/32\pi^2 m^2)|C'_\parallel - 2C'_\perp|$ where

$$C'_\parallel = im^2 \int_0^\infty ds \int_{-1}^1 dv\, e^{-is\phi_0}(1-v^2),$$

$$C'_\perp = im^2 \int_0^\infty ds \int_{-1}^1 dv\, e^{-is\phi_0} R, \tag{82}$$

with

$$\phi_0 = m^2 + \frac{1-v^2}{4} q_\parallel^2 + \frac{\cos(eBsv)-\cos(eBs)}{2eBs\sin(eBs)} q_\perp^2,$$

$$R = \frac{1 - v\sin(eBsv)\sin(eBs) - \cos(eBs)\cos(eBsv)}{\sin^2(eBs)}. \tag{83}$$

To compare the two sets of results, it is useful to realize that one can rotate the integration contour, $s\to -is$, in the above integrals, provided $q_0\equiv\omega<2m$. In this way, the phase $e^{-is\Phi_0}$ turns into $e^{-s\Phi_0}$ and becomes highly suppressed for a large s. For $B<B_c$, such a behavior permits one to simultaneously perform the weak-field and low-energy expansions with respect to $C_{\parallel,\perp}$. The results of expanding $|C'_\parallel - 2C'_\perp|$ may be organized into the sum of the following series $\Sigma_{n=0} a_n(\omega^2\sin^2\theta/m^2)^n (B^2/B_c^2)^n$, $\Sigma_{n=0} b_n(\omega^2\sin^2\theta/m^2)^n (B^2/B_c^2)^{n+1}$, $\Sigma_{n=0} c_n(\omega^2\sin^2\theta/m^2)^n (B^2/B_c^2)^{n+2}\ldots$. One observes that the coefficients a_1 and a_2 correspond to the $O(\omega^2B^2)$ and $O(\omega^4B^4)$ terms in our $|C_\parallel - C_\perp|$ respectively. We found that all the coefficients a_i's vanish. This is indeed reflected in our calculations where the $O(\omega^2B^2)$ and $O(\omega^4B^4)$ terms in $|C_\parallel - C_\perp|$ vanish as well. We also found agreements between the coefficients $b_{0,1}$ and the corresponding $O(B^2)$, $O(\omega^2B^4)$ terms in $|C_\parallel - C_\perp|$. Although we did not compare the coefficient c_0 with the $O(B^4)$ term in $C_\parallel$, due to the growing complexity in computing the general coefficients $c_i's$, the agreements we just found with respect to the first two series seems rather compelling. Because of these agreements, we also confirm the statement made in Ref. [17] that the earlier calculation on $\Pi^5_{\mu\nu}$ is incorrect.

From the above comparisons, we have seen that our approach, in spite of less general, is convenient for obtaining the analytic amplitudes of physical processes in a subcritical background magnetic field. In such a magnetic field, it suffices to know the leading and subleading terms in the weak-field expansion. Our approach produces those terms directly from Feynman diagrams.

The work on extending the present analysis to the more complicated processes, such as the photon splitting $\gamma\to\gamma\gamma$ and the pair production $\gamma\to e^+e^-$ is currently being pursued. For the latter process, we have exploited the analytical properties of the vacuum polarization function $\Pi^{\mu\nu}$ in the background magnetic field. For a subcritical magnetic field, it is possible to obtain a simple expression for the absorption coefficient(the pair-production width) for arbitrary photon energies [20]. This is an improvement to the previous work where a simplified expression is possible only for $\omega\gg m$ [21]. For the former process, $\gamma\to\gamma\gamma$, our result shall serve as an additional check to the previous results [18,22].

IV. CONCLUSION

In this paper, we have developed the weak-field expansion technique for processes occurring in a background magnetic field. This expansion is performed with respect to internal electron propagators which are affected by the background magnetic field. In some processes, our approach is valid for general external momenta even if they are much greater than the electron mass m. For external momenta much greater than m, the effective-Lagrangian approach is no longer appropriate. To illustrate this point, we calculated the amplitude of $\gamma\gamma\to\nu\bar{\nu}$ under a background magnetic field, and consequently determined the stellar energy-loss rate Q due to this process. It is interesting to find that the effective-Lagrangian approach is inappropriate for computing the stellar energy-loss rate due to $\gamma\gamma\to\nu\bar{\nu}$, unless the star temperature is less than $0.01\,m$. This result reflects clearly the importance of our approach. In fact, our approach can be applied to many other processes. In this regard, we also discussed the processes $\gamma\to\nu\bar{\nu}$ and $\nu\to\nu\gamma$ under a strong background magnetic field. We found that the validity of weak-field expansion with respect to the above processes are also determined by the parameter $r\equiv\omega^2\sin^2\theta B^2/m^2B_c^2$, besides the requirement $B<B_c$. For energy below pair production threshold, the parameter r is less than 1, which causes no trouble to the weak-field expansion. We found that $\gamma\to\nu\bar{\nu}$ is kinematically forbidden while $\nu\to\nu\gamma$ is permitted by the phase space. Our predictions on the latter process agree with previous works [17]. It has also been pointed out that our approach, although less general, is convenient for obtaining the analytic amplitudes of physical processes in a subcritical background magnetic field.

We are currently extending the weak-field expansion technique to the photon splitting process $\gamma\to\gamma\gamma$ [18,22] and the pair production process $\gamma\to e^+e^-$ [18,21]. Both processes are of great interests in the physics of pulsars on which the background magnetic fields are close to the critical value B_c.

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