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Cycle-symmetric matrices and convergent neural networks

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Abstract

This work investigates a class of neural networks with cycle-symmetric connection strength. We shall show that, by changing the coordinates, the convergence of dynamics by Fiedler and Gedeon [Physica D 111 (1998) 288] is equivalent to the classical results. This presentation also addresses the extension of the convergence theorem to other classes of signal functions with saturations. In particular, the result of Cohen and Grossberg [IEEE Trans. Syst. Man Cybernet. SMC-13 (1983) 815] is recast and extended with a more concise verification. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The notion of neural network, in addition to biological modeling, has been applied to various scientific areas such as circuit architecture and numerical computations. In designing a neural network, it is usually of prime importance to guarantee the convergence of the corresponding dynamical system, cf. [1–4,6,7,9]. The convergence of dynamics refers to every solution tending to a stationary solution as time goes to positive infinity. Such a convergence is often concluded by constructing a Lyapunov function and then applying LaSalle's invariance principle. Classical results on the construction of the Lyapunov function require the symmetry for the matrix of connection strength between neurons. For example, the works by Cohen and Grossberg [4] and by Chua and Yang [1] made this assumption. A significant progress has been made by Fiedler and Gedeon [5]. They successfully extended the Lyapunov function, hence the convergence of dynamics, to more general matrices of connection strength. The following preparation is needed to define such matrices.

For an undirected graph G without loops or multiple edges, a path is defined as a sequence of vertices $v_1 v_2 \cdots v_k$, $k \geq 1$, where $v_i v_{i+1}$ is an edge of G for each $i \in \{1, \dots, k-1\}$ and there are no repeated vertices except possibly the first and the last. By a cycle, we mean a closed path of length greater than or equal to three, that is, $v_1 = v_k$ and $k \geq 3$. Let B denote an $n \times n$ matrix with entries $\beta_{ij} \in \mathbf{R}$, $i, j \in \{1, 2, \dots, n\}$. If $\beta_{ij} \neq 0$ whenever $\beta_{ji} \neq 0$, then a

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graph with n vertices can be defined from B . Indeed, for $i \neq j$, an unordered pair (i, j) with $\beta_{ij} \neq 0$ is an edge of this graph. Consider the class of matrices with entries β_{ij} satisfying

$$(H_1) \quad \beta_{ij}\beta_{ji} > 0 \text{ if } \beta_{ij} \neq 0,$$

$$(H_2) \quad \prod_C \beta_{ik} = \prod_C \beta_{ki}, \text{ along every cycle } C,$$

where \prod denotes the product. Notably, condition (H_1) means that the entries of B are sign-symmetric. Such class of matrices, called *cycle-symmetric* herein, has been investigated in [11,12]. In fact, these matrices are characterized as matrices which are similar to symmetric matrices by real diagonal matrices. Restated, if B satisfies (H_1) and (H_2) , then there exists an invertible diagonal matrix P such that PBP^{-1} is a symmetric matrix. Notably, this characterization theorem was first obtained by Parter and Youngs [12]. Maybe [11] then considered a class of so-called combinatorially symmetric matrices ($B = [\beta_{ij}]$ with $\beta_{ji} \neq 0$ if $\beta_{ij} \neq 0$) and weakened the condition (H_1) . A matrix is called pseudosymmetric therein if it is similar to a symmetric matrix by a real diagonal matrix. However, (H_1) and (H_2) are the basic conditions for a matrix to be pseudosymmetric.

Fiedler and Gedeon [5] generalized the Lyapunov function for the neural network proposed in [4] to accommodate the network with the connection strength satisfying (H_1) and (H_2) . The condition (H_1) was further weakened in [6]. The studies in [5,6] thus concluded the convergence of dynamics for the system with a larger class of connection strength.

The first goal of this paper is to show that, with the characterization of the cycle-symmetric matrices, a change of coordinates can transform the system to a similar system, but with symmetric connection strength. Therefore, the convergence results in [5] are equivalent to the classical ones. This approach answers the question raised in [6] (also mentioned in [5]), which is whether the characterization theorems in [11,12] can be applied directly to prove the convergence theorem. The new treatment in this presentation is considered a more natural generalization of the classical results, since, for example, symmetric matrices are easier to handle in various related computations.

Our second objective in this investigation is to extend the convergence theory to other signal functions, in particular, the signal functions with saturations. Such functions have been used as output functions in the cellular neural networks [1–3]. Similar signal functions have also been considered in [4], where the transition of zero slope to positive slope in the signal functions relates to the notion of inhibitory signal threshold. Based on our previous technique of changing coordinates, we shall extend the convergence of dynamics to the system with more general saturated signal functions. This work not only provides an explicit formulation of these signal functions but also develops a new concise treatment for the proof of convergence. Our first step is to partition the phase space as the configurations of the signal functions are respected. The convergence of dynamics is then established by constructing a global Lyapunov function as well as certain regional Lyapunov functions. The latter ones are naturally incorporated with the existence for the equilibrium of the system and the partitioning of phase space. This approach is more straightforward than the one in [4] and is more general than the one in [10]. This investigation further explores the intrinsic structures of the model equations discussed in this presentation. Indeed, for example, for an arbitrary dynamical system with a global Lyapunov function, the existence of a regional Lyapunov function on the set where the global Lyapunov function is constant is not automatically valid.

We shall present our results for strictly increasing and two-sided saturated signal functions in Section 2. Extension of the convergence theorem to more general saturated signal functions will be discussed in Section 3.

2. Main results

We consider the following system proposed by Cohen and Grossberg [4], and later investigated in [5,6],

$$\frac{dx_i}{dt} = a_i(\mathbf{x}) \left[\gamma_i(x_i) - \sum_{j=1}^n \beta_{ij} f_j(x_j) \right], \quad i = 1, 2, \dots, n, \quad (2.1)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$. Denote by $\mathcal{F}_i(\mathbf{x})$ the right-hand side of (2.1) and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n)$. The following assumptions have been made in [5,6] in addition to (H_1) , (H_2) :

(H_3) $a_i(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbf{R}^n$ and every $i = 1, 2, \dots, n$,

(H_4) $f'_i(\xi) > 0$ for all $\xi \in \mathbf{R}$ and every $i = 1, 2, \dots, n$.

There are extra conditions which guarantee the dissipativeness, hence the existence of the global attractor, for the system (2.1).

(H_5) All f_i are bounded; $a_i(\mathbf{x}) > 0$ for all sufficiently large $|\mathbf{x}|$; $\gamma_i(x_i)x_i \rightarrow -\infty$ as $|x_i| \rightarrow \infty$.

Let $B = [\beta_{ij}]$ be a cycle-symmetric matrix, that is, B satisfies (H_1) and (H_2) . By the theorem in [11,12], there exists an invertible diagonal matrix P such that $PBP^{-1} = A$ with $A = [\alpha_{ij}]$, a symmetric matrix. Denote the diagonal entries of P by p_1, p_2, \dots, p_n , where every p_i is nonzero. Set $\mathbf{y} = P\mathbf{x}$, that is, $y_i = p_i x_i$ for each i . Eq. (2.1) in new variables is given by the following form:

$$\begin{aligned} \frac{dy_i}{dt} &= p_i a_i(P^{-1}\mathbf{y}) \left[\gamma_i(p_i^{-1}y_i) - \sum_{j=1}^n \beta_{ij} f_j(p_j^{-1}y_j) \right] \\ &= a_i(P^{-1}\mathbf{y}) \left[p_i \gamma_i(p_i^{-1}y_i) - \sum_{j=1}^n p_i \beta_{ij} p_j^{-1} p_j f_j(p_j^{-1}y_j) \right] = \tilde{a}_i(\mathbf{y}) \left[\tilde{\gamma}_i(y_i) - \sum_{j=1}^n \alpha_{ij} \tilde{f}_j(y_j) \right], \end{aligned}$$

where $\tilde{a}_i(\mathbf{y}) = a_i(P^{-1}\mathbf{y})$, $\tilde{\gamma}_i(y_i) = p_i \gamma_i(p_i^{-1}y_i)$, and $\tilde{f}_i(y_i) = p_i f_i(p_i^{-1}y_i)$. Notice that \tilde{a}_i satisfies (H_3) , \tilde{f}_i satisfies (H_4) , and \tilde{a}_i , \tilde{f}_i , and $\tilde{\gamma}_i$ satisfy (H_5) . Therefore, the Lyapunov function

$$V(\mathbf{y}) = -\sum_{i=1}^n \left\{ \int^{y_i} \tilde{\gamma}_i(\xi) \tilde{f}'_i(\xi) d\xi - \frac{1}{2} \sum_{j=1}^n \alpha_{ij} \tilde{f}_i(y_i) \tilde{f}_j(y_j) \right\},$$

which was proposed in [4] for symmetric connection strength, still holds here. We thus obtain the main theorem in [5].

Theorem 2.1. *Assume (H_1) – (H_5) . The dynamics of (2.1) are convergent if every equilibrium is isolated.*

The second goal of this presentation is to extend the convergence of dynamics for (2.1) to other signal functions f_i . In particular, we consider sigmoidal f_i with some saturations. In this case, the slope of f_i becomes only nonnegative (compare with (H_4)). These functions are described as follows.

(H'_4) Let $\{b_i\}_1^n, \{c_i\}_1^n, \{u_i\}_1^n, \{v_i\}_1^n$ be sequences of real numbers with $b_i < c_i, u_i < v_i$ for each i . For $i = 1, 2, \dots, n$, let $f_i = f_i(\xi_i)$ be a function which is continuous on \mathbf{R} , increasing on $[b_i, c_i]$, $f_i(\xi_i) = v_i$ for all $\xi_i \geq c_i$, and $f_i(\xi_i) = u_i$ for all $\xi_i \leq b_i$.

A typical function f_i satisfying (H'_4) is depicted in Fig. 1. The phase space \mathbf{R}^n for the dynamical system generated by (2.1) can be decomposed into 3^n regions, corresponding to the partitioning of the domains in definition of these sigmoidal functions f_i . The following labeling and notations are used to describe these regions. Denote by \mathbf{N}_n the set of positive integers from 1 to n , and by $\mathcal{A}^{\mathbf{N}_n}$ the set of all functions $\sigma : \mathbf{N}_n \rightarrow \mathcal{A}$, where $\mathcal{A} := \{-1, 0, 1\}$. It follows that

$$\bigcup_{\sigma \in \mathcal{A}^{\mathbf{N}_n}} \Omega_\sigma = \mathbf{R}^n,$$

where

$$\Omega_\sigma := \{\mathbf{x} = \{x_i\} \in \mathbf{R}^n \mid x_i \geq c_i \text{ if } \sigma_i = 1; x_i \leq b_i \text{ if } \sigma_i = -1; b_i < x_i < c_i \text{ if } \sigma_i = 0\}.$$

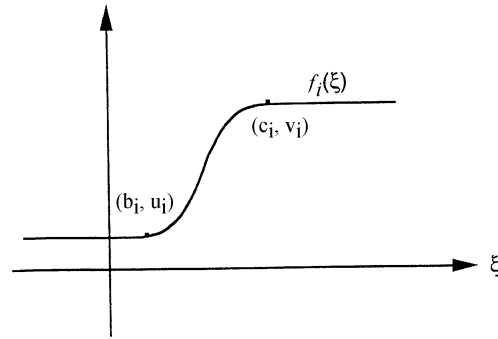


Fig. 1. Graph of signal function f_i in (H'_4) .

An illustration of the decomposition for $n = 2$ is provided in Fig. 2. Let $\Lambda_e = \{\{\sigma_i\} \in \mathcal{A}^{\mathbf{N}_n} \mid \sigma_i = -1 \text{ or } 1\}$, $\Lambda_m = \{\{\sigma_i\} \in \mathcal{A}^{\mathbf{N}_n} \mid \sigma_i = 0 \text{ for some } i \in \mathbf{N}_n \text{ and } |\sigma_j| = 1 \text{ for some } j \in \mathbf{N}_n\}$. These 3^n regions can then be classified into three categories: Ω_σ is called an *exterior region* if $\sigma \in \Lambda_e$, a *mixed region* if $\sigma \in \Lambda_m$ and an *interior region* if $\sigma_i = 0$ for all $i \in \mathbf{N}_n$. Accordingly, there is only one interior region and it will be denoted by Ω_0 .

As a consequence, the equilibria for (2.1) can be classified into three types, according to their locations. An equilibrium $\bar{\mathbf{x}} = \{\bar{x}_i\}_1^n$ is called *exterior* if $\bar{\mathbf{x}}$ lies in an exterior region, *mixed* if $\bar{\mathbf{x}}$ lies in a mixed region, and *interior* if $\bar{\mathbf{x}}$ lies in the interior region.

With this classification, we elaborate on the existence for each type of the equilibria in the following. If substituting $\{x_i\}_1^n$ by $\{\bar{x}_i\}_1^n$ into the right-hand side of (2.1) yields zero and $b_i < \bar{x}_i < c_i$ for each $i \in \mathbf{N}_n$, then $\{\bar{x}_i\}_1^n$ is an interior equilibrium.

(2.1) restricted to an exterior region Ω_σ , $\sigma \in \Lambda_e$ takes the following form:

$$\frac{dx_i}{dt} = a_i(\mathbf{x}) \left[\gamma_i(x_i) - \sum_{j=1}^n \beta_{ij} \omega_j \right], \tag{2.2}$$

where

$$\omega_j = v_j \quad \text{if } \sigma_j = 1, \quad \omega_j = u_j \quad \text{if } \sigma_j = -1. \tag{2.3}$$

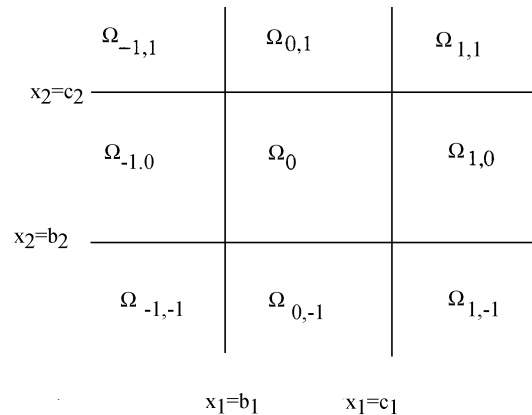


Fig. 2. Decomposition of phase space for $n = 2$.

Thus, $\bar{\mathbf{x}} = \{\bar{x}_i\}_1^n$ is an exterior equilibrium of (2.1) in Ω_σ if it satisfies (2.2) as well as $\bar{x}_i \geq c_i$ for i with $\sigma_i = 1$ and $\bar{x}_i \leq b_i$ for i with $\sigma_i = -1$.

Consider a mixed region Ω_σ , $\sigma \in \Lambda_m$. Let $J_0 = \{i \in \mathbf{N}_n : \sigma_i = 0\}$ and $J_1 = \mathbf{N}_n \setminus J_0$. For $i \in J_0$, the i th component of the vector field $\mathcal{F}(\mathbf{x})$ in (2.1) restricted to Ω_σ becomes

$$\mathcal{F}(\mathbf{x})_i = a_i(\mathbf{x}) \left[\gamma_i(x_i) - \sum_{j \in J_0} \beta_{ij} f_j(x_j) - \sum_{j \in J_1} \beta_{ij} \omega_j \right], \tag{2.4}$$

where

$$\omega_j = v_j \quad \text{if } \sigma_j = 1, \quad \omega_j = u_j \quad \text{if } \sigma_j = -1. \tag{2.5}$$

Assume that $a_i(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbf{R}^n$. Suppose there exist real numbers $\{\bar{x}_i\}_1^n$ with $\bar{\mathbf{x}} = \{\bar{x}_i\}_1^n$ such that substituting $\{x_i\}_1^n$ by $\{\bar{x}_i\}_1^n$ into (2.4) yields zero. Then, (2.4) also vanishes for $\mathbf{x} = \{x_i\}_1^n$ with $x_i = \bar{x}_i$ if $i \in J_0$, and any $x_i \leq b_i$ if $\sigma_i = -1$, as well as any $x_i \geq c_i$ if $\sigma_i = 1$. Therefore, we have the following subsets of the phase space, which possesses certain invariant property. Namely,

$$I_\sigma = \{\mathbf{x} \in \mathbf{R}^n \mid x_i = \bar{x}_i \text{ if } i \in J_0, x_i \leq b_i \text{ if } \sigma_i = -1, x_i \geq c_i \text{ if } \sigma_i = 1\}. \tag{2.6}$$

An orbit starting on I_σ remains on I_σ before it enters the other regions $\Omega_{\sigma'}$ neighboring Ω_σ . Note that an equilibrium in Ω_σ , $\sigma \in \Lambda_m$, must lie on such a subset I_σ . Indeed, $\bar{\mathbf{x}} = \{\bar{x}_i\}_1^n$ is a mixed equilibrium in Ω_σ if the vector field in (2.1) vanishes at $\bar{\mathbf{x}}$ (the i th component of the vector field is as (2.4) for $i \in J_0$), moreover, $b_i < \bar{x}_i < c_i$ for $i \in J_0$, and $\bar{x}_i \geq c_i$ for $i \in J_1$ with $\sigma_i = 1$ and $\bar{x}_i \leq b_i$ for $i \in J_1$ with $\sigma_i = -1$.

Now we consider (2.1) with symmetric connection strength B and signal functions f_i satisfying (H'_4) . First, let us construct a global Lyapunov function:

$$V(\mathbf{x}) = -\sum_{i=1}^n \left\{ \int^{f_i(x_i)} \gamma_i(g_i(\xi)) d\xi - \frac{1}{2} \sum_{j=1}^n \beta_{ij} f_i(x_i) f_j(x_j) \right\}, \tag{2.7}$$

where $g_i : [u_i, v_i] \rightarrow [b_i, c_i]$ is defined by $g_i(\xi) = (f_i|_{[b_i, c_i]})^{-1}(\xi)$, and $(f_i|_{[b_i, c_i]})^{-1}$ is the inverse function of f_i restricted to $[b_i, c_i]$. If each f_i is differentiable on \mathbf{R} , then the derivative of V along an orbit of (2.1) is

$$\dot{V}(\mathbf{x}) = -\sum_{i=1}^n \dot{x}_i f'_i(x_i) \left[\gamma_i(x_i) - \sum_{j=1}^n \beta_{ij} f_j(x_j) \right] \tag{2.8}$$

$$= -\sum_{i=1}^n f'_i(x_i) a_i(\mathbf{x}) \left[\gamma_i(x_i) - \sum_{j=1}^n \beta_{ij} f_j(x_j) \right]^2. \tag{2.9}$$

The equality in (2.8) follows from the symmetry of $B = [\beta_{ij}]$ and the following observation. In the computation, (2.8) should only hold for the term $\gamma_i(x_i)$ with $x_i \in [b_i, c_i]$ according to the definition of g_i . However, for $x_i \geq c_i$ or $x_i \leq b_i$, $f'_i(x_i) = 0$. Thus, for x_i in these ranges, the i -term in the summation $\sum_{i=1}^n$ vanishes no matter what the terms in the bracket are. Since $f'_i(x_i) \geq 0$ for any x_i , $\dot{V}(\mathbf{x})$ in (2.9) is less than or equal to zero.

If some f_i is not differentiable, an alternative computation yields the same result. Namely, consider

$$\dot{V}(\mathbf{x}) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(\mathbf{x} + h\mathcal{F}(\mathbf{x})) - V(\mathbf{x})],$$

where $\mathcal{F}(\mathbf{x})$ is the vector field in (2.1), cf. [8]. The detailed computation is similar to the one in [10]. Let \mathcal{S} be the set on which V remains constant along an orbit of (2.1), that is,

$$\mathcal{S} = \{\mathbf{x} \in \mathbf{R}^n : \dot{V}(\mathbf{x}) = 0\}.$$

Then, the closure of \mathcal{S} can be represented by

$$\bar{\mathcal{S}} = (\cup_{\sigma \in \Lambda_e} \Omega_\sigma) \cup (\cup I_\sigma) \cup \mathcal{E}_0. \quad (2.10)$$

Herein, $\cup_{\sigma \in \Lambda_e} \Omega_\sigma$ is the union of all exterior regions, \mathcal{E}_0 is the set of equilibria in the interior region, and $\cup I_\sigma$ is the union of the subsets in mixed regions, as discussed in (2.6), whenever they exist. We shall call each point (an equilibrium) of \mathcal{E}_0 , each of the exterior regions Ω_σ , and each of these I_σ , a *component* of \mathcal{S} .

Next, we introduce the regional Lyapunov function V_σ for (2.1) restricted to each exterior region Ω_σ or each I_σ in a mixed region. Consider an exterior region Ω_σ , $\sigma \in \Lambda_e$. Let

$$V_\sigma(\mathbf{x}) = -\sum_{i=1}^n \left\{ \int^{x_i} \gamma_i(\xi) d\xi - x_i \sum_{j=1}^n \beta_{ij} \omega_j \right\}, \quad (2.11)$$

where ω_j is as defined in (2.3). The derivative of this function along a solution of (2.1) lying in Ω_σ is

$$\dot{V}_\sigma(\mathbf{x}) = -\sum_{i=1}^n \dot{x}_i \left[\gamma_i(x_i) - \sum_{j=1}^n \beta_{ij} \omega_j \right] = -\sum_{i=1}^n a_i(\mathbf{x}) \left[\gamma_i(x_i) - \sum_{j=1}^n \beta_{ij} \omega_j \right]^2 \leq 0.$$

The equality holds if and only if $a_i(\mathbf{x})[\gamma_i(x_i) - \sum_{j=1}^n \beta_{ij} \omega_j] = 0$ for every $i \in \mathbf{N}_n$. That is, $\dot{V}_\sigma(\mathbf{x})$ only vanishes at an exterior equilibrium \mathbf{x} in Ω_σ .

Suppose I_σ lies in a mixed region Ω_σ , $\sigma \in \Lambda_m$. Recall that $J_0 = \{i \in \mathbf{N}_n : \sigma_i = 0\}$ and $J_1 = \mathbf{N}_n \setminus J_0$ and the notations in (2.6). Let

$$V_\sigma(\mathbf{x}) = -\sum_{i \in J_1} \left\{ \int^{x_i} \gamma_i(\xi) d\xi - x_i \sum_{j \in J_1} \beta_{ij} \omega_j - x_i \sum_{j \in J_0} \beta_{ij} f_j(\tilde{x}_j) \right\}, \quad (2.12)$$

where ω_j is as described in (2.5). It can be verified that $\dot{V}_\sigma(\mathbf{x})$, the derivative of V_σ along a solution of (2.1) lying in I_σ , vanishes only at a mixed equilibrium in I_σ .

With the global Lyapunov function V and these regional Lyapunov functions V_σ , we can then derive the following result. It extends Theorem 2.1 to the class of signal functions f_i satisfying (H'_4) .

Theorem 2.2. *Assume (H_1) , (H_2) , (H'_4) , (H_5) and that $a_i(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbf{R}^n$. (2.1) is convergent if every equilibrium is isolated.*

Proof. By changing the coordinates, it suffices to consider (2.1) with symmetric $B = [\beta_{ij}]$. Notably, if there is an equilibrium in a mixed region Ω_σ , then a subset I_σ described in (2.6) exists and this equilibrium lies on I_σ . With the assumption that every equilibrium is isolated, the components of \mathcal{S} are pairwise disjoint. Indeed, any two distinct exterior regions are disjoint. In addition, any two components belonging to two different regions Ω_σ , $\Omega_{\sigma'}$, $\sigma \neq \sigma'$, are disjoint, since there is a $j \in \mathbf{N}_n$ such that $x_j \neq \tilde{x}_j$ for any $\mathbf{x} = (x_1, \dots, x_n) \in \Omega_\sigma$ and any $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n) \in \Omega_{\sigma'}$. Furthermore, the same argument justifies that any two components of \mathcal{S} belonging to the same Ω_σ are disjoint. Consider an orbit $\phi(t, \mathbf{x}_0)$ and its ω -limit set, $\omega(\phi(t, \mathbf{x}_0))$. It follows from the existence of global Lyapunov function V that $\omega(\phi(t, \mathbf{x}_0)) \in \mathcal{S}$. Moreover, by the connectedness of the ω -limit set, $\omega(\phi(t, \mathbf{x}_0))$

lies in one component of \mathcal{S} . Let $\mathbf{x}^* \in \omega(\phi(t, \mathbf{x}_0))$. If $\mathbf{x}^* \in \mathcal{E}_0$, then \mathbf{x}^* is already an equilibrium. Suppose $\mathbf{x}^* \in \Omega_\sigma$, $\sigma \in \Lambda_e$. Then $\phi(t, \mathbf{x}^*) \in \Omega_\sigma$ for all t since $V(\phi(t, \mathbf{x}^*)) = V(\mathbf{x}^*)$ for all t and $V(\phi(t, \mathbf{x}^*))$ decreases as $\phi(t, \mathbf{x}^*)$ leaves Ω_σ . By the existence of regional Lyapunov functions $V_\sigma(\mathbf{x})$, (2.11), it follows that \mathbf{x}^* has to be an exterior equilibrium. The same argument holds for $I_\sigma \subset \Omega_\sigma$, $\sigma \in \Lambda_m$. That is, if $\mathbf{x}^* \in I_\sigma$, then \mathbf{x}^* has to be a mixed equilibrium in Ω_σ . It is also obvious that the ω -limit set of $\phi(t, \mathbf{x}_0)$ consists of a single equilibrium. This completes the proof. \square

Remark. It can be shown by Sard’s theorem that the equilibrium points of (2.1) are isolated for almost every matrix of connection strength B , with a mild assumption on the values of the signal functions at the inhibitory thresholds. The verification is similar to the one in [4].

3. More generalizations

Theorem 2.2 is valid for other signal functions with saturations. For example, similar arguments as the proof of Theorem 2.2 confirm the convergence of (2.1) with one-sided signal functions f_i (as in Fig. 3). This class of signal functions fits the setting of suprathreshold and subthreshold variables in [4].

Our result can further be extended to stairway-like multi-saturated signal functions. Let $m > 1$ be an integer. For $i \in 1, 2, \dots, n$, let each of $\{b_1^i, c_1^i, b_2^i, c_2^i, \dots, b_m^i, c_m^i\}$ and $\{u_0^i, u_1^i, u_2^i, \dots, u_m^i\}$ be a partition of \mathbf{R} with $b_1^i < c_1^i < b_2^i < c_2^i < \dots < b_m^i < c_m^i$ and $u_0^i < u_1^i < u_2^i < \dots < u_m^i$. For each $i = 1, 2, \dots, n$, let f_i be a continuous function defined by

$$f_i(\xi) = \begin{cases} u_0^i & \text{if } -\infty < \xi \leq b_1^i, \\ u_j^i & \text{if } c_j^i \leq \xi \leq b_{j+1}^i, j = 1, \dots, m - 1, \\ \text{increasing} & \text{if } b_j^i \leq \xi \leq c_j^i, j = 1, \dots, m, \\ u_m^i & \text{if } c_m^i \leq \xi < \infty. \end{cases}$$

Such a signal function is demonstrated in Fig. 4. For each $i = 1, 2, \dots, n$, let $g_i : [u_0^i, u_m^i] \rightarrow \cup_{j=1}^m [b_j^i, c_j^i] \cup \{c_m^i\}$ be a function defined by $g_i(\xi) = (f_i|_{[b_j^i, c_j^i]})^{-1}(\xi)$ if $\xi \in [u_j^i, u_{j+1}^i]$ for $j = 1, 2, \dots, m - 1$ and $g_i(u_m^i) = c_m^i$, where $(f_i|_{[b_j^i, c_j^i]})^{-1}$ is the inverse function of f_i restricted to $[b_j^i, c_j^i]$. Then the function V in (2.7) is a global Lyapunov function for (2.1). The computations in (2.8) and (2.9) remain valid by similar arguments following (2.9). Thus, the convergence theorem for (2.1) with such signal functions can be analogously concluded by establishing the associated regional Lyapunov functions.

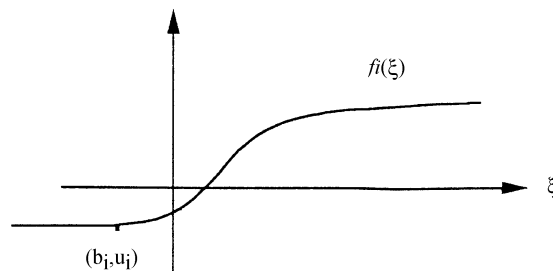


Fig. 3. One-sided saturated signal function.

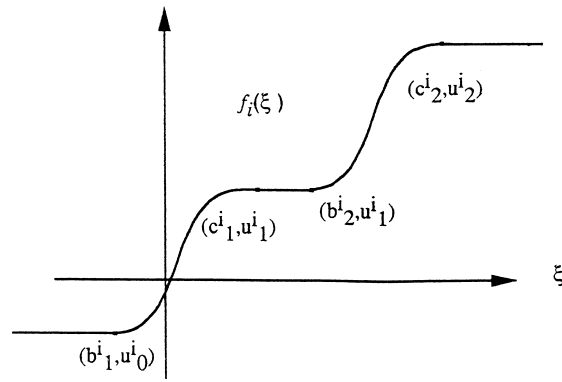


Fig. 4. Multi-saturated signal function.

Finally, we note that it is not necessary for signal functions f_i in (2.1) to have the same number of saturations to conclude the convergence of dynamics. Restated, the number of saturations can range from 0 to any positive integer $m + 1$ and m can vary with i .

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