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Sequential construction of a circular consecutive-2 system

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Abstract

Derman, Lieberman and Ross solved the problem of sequentially assigning n components with different reliabilities to a consecutive-2 linear system to maximize the system reliability. Furthermore, they show that the optimal assignment is invariant, i.e., it depends only on the ranking of the component reliabilities, but not their values. We study the same problem for the consecutive-2 circular system and prove that an invariant optimal assignment does not exist. But we reduce the number of candidates of an optimal assignment from $n!$ to $\lfloor n/2 \rfloor - 2$. We also find the first-order invariant optimal assignment. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

A linear (circular) consecutive-k system is a line (cycle) of n components such that the system has failed if and only if some consecutive k components have all failed. Let p_i denote the probability that component i is working and let $q_i = 1 - p_i$ denote the probability that component i has failed. Derman, Lieberman and Ross (DLR) [1] studied two problems of how to construct a consecutive-2 line with n given components to maximize its reliability. They assume that the state of a component becomes known once it is added to the system. The first problem they considered is to construct the line sequentially, adding the components one by one and taking full advantage of the knowledge of the states of components already added. The second problem is to construct the line nonsequentially by specifying the complete sequencing of components together.

Let

$$p = \{p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n\}$$

denote the set of component reliabilities to be assigned. An optimal assignment is called *invariant* if it depends only on the ranking of p_i, but not their actual values. DLR gave an invariant assignment to the first problem and conjectured an invariant solution for the second. The conjecture was later independently proved by Du and Hwang [2] and Malon [3]. The former actually proved the conjecture for the more general circular system. A natural expectation is that the remaining case out of the four possibilities, namely, the sequential cycle, will also be invariant. In this paper we show that this is not so; but we reduce the search of an optimal cycle to $\lfloor n/2 \rfloor - 2$ candidates instead of the original $n!$, and we give simple formulas to compare these candidates.

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2. Reduction of the candidate set

Since our goal is to find an invariant assignment, all algorithms considered here are assumed to be dependent on the p_i's only through their ranks. DLR gave the following greedy algorithm $G(n)$ for the sequential line and proved its invariance.

Step 1. Assign component 1 as the initial component.

Step 2. Suppose component j is the last component assigned. If component j is working, assign the worst component from the remaining lot next. If component j fails, assign the best component from the remaining lot next.

Note that the optimality of Step 2 is independent of Step 1. Namely, if component i is assigned first, the subsequent optimal strategy depends only on the state of that component, but not on its label i. Let $G(n;i)$ be an extension of $G(n)$ to the case that component i is assigned first. Then $G(n;i)$ is invariant among those algorithms where component i is assigned first.

In the cycle case, after the initial component is assigned to the system, both its neighbors can receive the next assignment, and this duality of receivership exists until the assignment of the last component. We now modify $G(n;i)$ for the sequential cycle:

Step 1. Assign component i as the initial component.

Step 2. If component i is working, use $G(n-1)$ on the set $p\backslash\{p_i\}$.

Step 3. If component i fails, assign the best component x_i from the remaining lot to either neighbor of component i next. If x_i is working, use $G(n-2;i)$ on the remaining $n-2$ components (starting by assigning the currently best component y_i to the other neighbor of i).

Note that whenever two consecutive components both failed, the system fails no matter how the remaining components are assigned. In particular, if either x_i or y_i fails, then the system fails and subsequent assignments are inconsequential.

Let P denote the probability function, and let $R(A)$ denote the reliability of a cycle constructed under algorithm A.

Lemma 1. *Let A denote an algorithm on $p=\{p_1,\ldots,p_n\}$ and let p' be obtained from p except changing one p_i to a larger p'_i. Then*

$$\frac{R(A\text{ on }p')}{R(A\text{ on }p)}\leqslant\frac{p'}{p}.$$

Proof. Let $S_{(i)}$ denote a sequence of states of components, excluding the state of component i, on a particular realization of A, and let s_i denote the state of component i. Define

$$W_i=\{S_{(i)}\colon\ S_{(i)}\cup s_i\text{ works if and only if } s_i\text{ is working}\},$$
$$W_{(i)}=\{S_{(i)}\colon\ S_{(i)}\cup s_i\text{ works regardless of what } s_i\text{ is}\}.$$

Then

$$\frac{R(A\text{ on }p')}{R(A\text{ on }p)}=\frac{p'_iP(W_i)+P(W_{(i)})}{p_iP(W_i)+P(W_{(i)})}\leqslant\frac{p'_i}{p_i}.\qquad\square$$

Theorem 2. *Among the class of sequential cycle algorithms which start with component i, $G(n;i)$ is invariant.*

Proof. Theorem 2 is trivially true for $n=1,2,3$. We prove the general $n\geqslant 4$ case.

Suppose i is working. Then the n-cycle is reduced to the $(n-1)$-line for which $G(n-1)$ is invariant.

Let A_j be an algorithm which assigns component j after the initial component i has failed, let A'_j be obtained from A_j by switching component j with component x_i, and let $\overline{A}_j$ be the part of A_j after component j. Then by Lemma 1,

$$\frac{R(A_j\mid\text{component }i\text{ failed})}{R(A'_j\mid\text{component }i\text{ failed})}=\frac{p_jR(\overline{A}_j\text{ on }p\backslash\{p_i,p_j\})}{x_iR(\overline{A}_j\text{ on }p\backslash\{p_i,x_i\})}\leqslant\frac{p_j}{x_i}\left(\frac{x_i}{p_j}\right)=1.$$

Therefore we may assume that $p_j=x_i$. When x_i is working, the n-cycle is reduced to an $(n-1)$-line headed by a failing component i. The optimality of $G(n-2;i)$ has been established before. $\square$

Theorem 2 reduces the number of candidates of an optimal assignment from $n!$ to n. In the next section we show that more than half of the n candidates can be further eliminated.

3. Comparisons of $G(n;i)$

Let R_i denote the reliability of $G(n;i)$, and let $P_{j,k}(i)$ denote the probability that $G(n;i)$ works conditional on component j working and component k failed. Then for $i = 1, \ldots, n-1$.

Lemma 3. $(R_{i+1} - R_i)(P_{i,i+1}(i) - P_{i+1,i}(i)) \geqslant 0$.

Proof. Note that $G(i)$ and $G(i+1)$ differ only in two positions, the first position and the position $G(i)$ has component $i+1$ and $G(i+1)$ has component i (this is because $i+1$ has the same rank in $G(i)$ as i in $G(i+1)$ after ignoring the initial component). Hence $P_{i+1,i}(i+1) = P_{i,i+1}(i)$ since the states of the components in the two differing positions agree. Similarly, $P_{i,i+1}(i+1) = P_{i+1,i}(i)$. Thus

$$\begin{aligned}
R_{i+1} - R_i &= p_{i+1}q_i P_{i+1,i}(i+1) + q_{i+1}p_i P_{i,i+1}(i+1) \\
&\quad - p_{i+1}q_i P_{i+1,i}(i) - q_{i+1}p_i P_{i,i+1}(i) \\
&= p_{i+1}q_i P_{i,i+1}(i) + q_{i+1}p_i P_{i+1,i}(i) \\
&\quad - p_{i+1}q_i P_{i+1,i}(i) - q_{i+1}p_i P_{i,i+1}(i) \\
&= (p_{i+1}q_i - q_{i+1}p_i)\big(P_{i,i+1}(i) - P_{i+1,i}(i)\big) \\
&= (p_{i+1} - p_i)\big(P_{i,i+1}(i) - P_{i+1,i}(i)\big).
\end{aligned}$$

Therefore $R_{i+1} - R_i$ has the same sign as $P_{i,i+1}(i) - P_{i+1,i}(i)$. □

Next we give closed-form solutions of $P_{i,i+1}(i)$ and $P_{i+1,i}(i)$. Consider the set $N = \{1, \ldots, n\}$ and $S \subset N$. Let (I,J) be a partition of $N \backslash S$ such that $|I| = k$. Let $P_k(k,n)$ denote the sum of $\binom{n-|S|}{k}$ terms where each term can be represented by $\prod_{i\in I} q_i \prod_{j\in J} p_j$ with a distinct (I,J) (S can be omitted if empty). For example,

$$P(1,3) = q_1p_2p_3 + p_1q_2p_3 + p_1p_2q_3$$

and

$$P_3(2,4) = q_1q_2p_4 + q_1p_2q_4 + p_1q_2q_4.$$

We adopt the convention

$$\prod_{j=x}^{y} p_j = 1 \quad \text{if } y < x.$$

Lemma 4.

$$P_{i,i+1}(i) = \sum_{k=0}^{m} P_{\{i,i+1\}}(k, n-k-1) \prod_{\substack{j=n-k \\ j\neq i,i+1}}^{n} p_j,$$

where $m = \min\{n-i-1, \lfloor n/2\rfloor - 1\}$.

Proof. Suppose that k more components other than $i+1$ fail. Then the $k+1$ largest components other than i (which is already assigned) must be working since each of them follows a failing component, except when $i+1 = n-k$, then only the k largest need be working since the failing $i+1$ is followed by the working i. If $i+1$ is among the k largest, then the system must fail. Therefore $k \leqslant n-i-1$. On the other hand, no system can work with a majority of failing components. Hence $k \leqslant \lfloor n/2\rfloor - 1$ (not counting $i+1$). The k failing components can be chosen arbitrarily except that the $k+1$ largest as well as i and $i+1$ are not candidates. □

Lemma 5.

$$P_{i+1,i}(i) = \sum_{k=0}^{\lfloor n/2\rfloor -1} P_{\{i,i+1\}}(k, n-2-k) \prod_{\substack{j=n-1-k \\ j\notin\{i,i+1\}}}^{n} p_j.$$

Proof. i is first assigned and fails. The currently two largest components must both be working, since one follows i and the other is reserved for last (to precede i). The rest of the argument is similar to the proof of Lemma 4 except that the restriction $k \leqslant n-i-1$ is not needed since $i+1$ is working. □

Theorem 6. *For* $n \geqslant 6$ *an optimal assignment must be among* $G(n;i)$ *for* $i \in \{\lceil n/2\rceil + 1, \lceil n/2\rceil + 2, \ldots, n-2\}$.

Proof. By Lemmas 4 and 5, for $i \leqslant \lceil n/2\rceil$,

$$\begin{aligned}
&P_{i,i+1}(i) - P_{i+1,i}(i) \\
&\quad = \sum_{k=0}^{\lfloor n/2\rfloor -1}\Bigg[P_{\{i,i+1\}}(k, n-k-1) \prod_{\substack{j=n-k \\ j\neq i,i+1}}^{n} p_j \\
&\qquad - P_{\{i,i+1\}}(k, n-2-k) \prod_{\substack{j=n-k-1 \\ j\neq i,i+1}}^{n} p_j\Bigg] \\
&\quad \geqslant 0,
\end{aligned}$$

since

$$P_{\{i,i+1\}}(k, n-k-1) \geqslant P_{\{i,i+1\}}(k, n-2-k)$$

and

$$\prod_{\substack{j=n-k\\ j\neq i,i+1}}^{n} p_j \geqslant \prod_{\substack{j=n-k-1\\ j\neq i,i+1}}^{n} p_j.$$

Furthermore,

$$\begin{aligned} &P_{n-1,n-2}(n-1)\\ &\quad \geqslant P_{\{n-2,n-1\}}(0, n-2)p_n + P_{\{n-2,n-1\}}(1, n-3)p_n\\ &\quad = P_{\{n-2,n-1\}}(0, n-1)p_n + P_{\{n-2,n-1\}}(1, n-2)p_n\\ &\quad = P_{n-2,n-1}(n-1).\\ &P_{n,n-1}(n-1)\\ &\quad \geqslant P_{\{n-1,n\}}(0, n-2)\\ &\quad = P_{\{n-1,n\}}(0, n-1)\\ &\quad = P_{n-1,n}(n-1). \end{aligned}$$

By Lemma 3, we have $R_{\lceil n/2\rceil+1} \geqslant R_{\lceil n/2\rceil} \geqslant \cdots \geqslant R_1$ and $R_{n-2} \geqslant R_{n-1} \geqslant R_n$. □

Note that we can use Lemmas 4 and 5 to compute $R_{i+1} - R_i$ for $i = \lceil n/2\rceil + 1, \ldots, n-2$, and determine the optimal $G(n; i)$ from these $\lfloor n/2\rfloor - 2$ values.

4. The issue of invariance

For $n = 1, 2, 3$, all assignments are equivalent. For $n = 4, 5$, it is easily verified that both $G(n; n-1)$ and $G(n; n-2)$ are invariant. For $n = 6, 7$, Theorem 3 tells us that $G(n; n-2)$ is invariant. For $n = 8$, Theorem 3 says that $G(8; 5)$ and $G(8; 6)$ are the only two candidates for optimality. We now compare R_6 with R_5, or equivalently, $P_{5,6}(5)$ with $P_{6,5}(5)$.

$$\begin{aligned} &P_{5,6}(5) - P_{6,5}(5)\\ &\quad = \big[P_{\{5,6\}}(0,7)p_8 + P_{\{5,6\}}(1,6)p_8p_7\\ &\qquad + P_{\{5,6\}}(2,5)p_8p_7\big]\\ &\qquad - \big[P_{\{5,6\}}(0,6)p_8p_7 + P_{\{5,6\}}(1,5)p_8p_7\\ &\qquad\quad + P_{\{5,6\}}(2,4)p_8p_7 + P_{\{5,6\}}(3,3)p_8p_7p_4\big]\\ &\quad = \big[P_{\{5,6\}}(0,7)p_8 - P_{\{5,6\}}(0,6)p_8p_7\big]\\ &\qquad - P_{\{5,6\}}(3,3)p_8p_7p_4\\ &\quad = p_8q_7p_4p_3p_2p_1 - p_8p_7p_4q_3q_2q_1\\ &\quad = p_8p_4(q_7p_3p_2p_1 - p_7q_3q_2q_1). \end{aligned}$$

Note that the difference is positive if $p_1 \to 1$ (p_1 tending to 1), and is negative if $p_4 > p_3 = 0$. Therefore we conclude

Theorem 7. *No invariant assignment exists in general for the sequential cycle problem.*

A *k-cutset* is a set of k components whose failures bring down the system. Let $C_k(A)$ denote the set of k-cutsets for assignment A. Santha and Zhang [4] called assignment A^* to be *first-order invariant* if lexicographically,

$$\begin{aligned} &\big(|C_2(A^*)|, |C_3(A^*)|, \ldots, |C_n(A^*)|\big)\\ &\quad \leqslant \big(|C_2(A)|, |C_3(A)|, \ldots, |C_n(A)|\big) \quad \text{for all } A. \end{aligned}$$

This definition makes sense when $p_i \to 1$ for all i. For example, if an assignment is less likely to contain a 2-cutset than other assignments, then it is first-order invariant, since the sum of probabilities of all other cutsets tends to zero much faster than those of 2-cutsets. Clearly, an invariant assignment must be first-order invariant. In this section we prove that $G(n; n-1)$ and $G(n; n-2)$ are first-order invariant.

Let S_2 denote the set of all 2-cutsets and let $S_2(A)$ denote the subset of 2-cutsets which fail the algorithm A. Finally,

Lemma 8.

$$S_2\big(G(n; i)\big) = \bigcup_{j\neq x_i} \{\{x_i, j\}\} \cup \{\{i, y_i\}\}.$$

Proof. As soon as a failing component is assigned, $G(n; i)$ assigns x_i next to it. If x_i is also failing, then $G(n; i)$ does not succeed.

Also note that if i is failing, then $G(n; i)$ assigns x_i and y_i to its two sides. So $G(n; i)$ does not succeed if i and y_i both are failing.

The above also exhausts all possible failure cases. □

Lemma 9. *For any algorithm A, $|S_2(A)| = n$.*

Proof. Regardless of the sequence in assigning, there are n adjacent pairs of components, out of $\binom{n}{2}$ possible pairs, whose mutual failures cause the construction to fail. □

Let P be a partial order of ordered pairs (x, y), $x \geqslant y$, such that $(x, y) \geqslant (w, z)$ if and only if $x \geqslant w$ and $y \geqslant z$. A set Q of q pairs is said to *dominate* another set Q' of q pairs under P if and only if there exists a permutation of pairs in Q' such that the ith pair in Q is larger or equal to the ith pair of Q' for $i = 1, \ldots, q$. Define $\rho_i = q_i / p_i$. While a 2-cutset was previously written as an unordered pair, it can be readily converted to an ordered pair as prescribed in the beginning of this paragraph.

Lemma 10. $\mathrm{Prob}(S_2(A)) \geqslant \mathrm{Prob}(S_2(B))$ *if* $S_2(B)$ *dominates* $S_2(A)$.

Proof. Suppose $(x, y) \geqslant (w, z)$ under D. Let $K = \prod_{i=1}^{n} p_i$,

$$\begin{aligned}\mathrm{Prob}(x, y) &= K\rho_x\rho_y \leqslant K\rho_w\rho_z \\ &= \mathrm{Prob}(w, z). \qquad \square\end{aligned}$$

Lemma 11. *If $I_C(n)$ exists, then it is either $G(n; n-1)$ or $G(n; n-2)$.*

Proof. By Lemma 4,

$$S_2\big(G(n;n)\big) = \bigcup_{j \neq n-1} \{(n-1, j)\} \cup \{(n, n-2)\}. \qquad \square$$

Theorem 12. *$G(n; n-1)$ and $G(n; n-2)$ are first-order invariant.*

Proof. By Lemma 8,

$$\begin{aligned} S_2\big(G(n;n)\big) &= \Big(\bigcup_{j \neq n-1} (n-1, j)\Big) \cup (n, n-2), \\ S_2\big(G(n;n-1)\big) &= S_2\big(G(n;n-2)\big) \\ &= \Big(\bigcup_{j \neq n-1} (n, j)\Big) \cup (n-1, n-2), \end{aligned}$$

and

$$S_2\big(G(n;i)\big) = \Big(\bigcup_{j \neq n} (n, j)\Big) \cup (i, n-1)$$

for $1 \leqslant i \leqslant n-3$.

Thus $S_2(G(n; n-1)) = S_2(G(n; n-2))$ dominates $S_2(G(n; i))$ for all i, Theorem 12 now follows from Lemma 10. $\square$

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