Kaluza-Klein induced gravity inflation

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A D-dimensional induced gravity theory is studied carefully in a $4+(D-4)$ dimensional Friedmann-Robertson-Walker space-time. We try to extract information about the symmetry-breaking potential in search of an inflationary solution with a nonexpanding internal space. We find that the induced gravity model imposes strong constraints on the form of the symmetry-breaking potential in order to generate an acceptable inflationary universe. These constraints are analyzed carefully in this paper.

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I. INTRODUCTION

The scale-invariant model explains the origin of the scale parameters such as the gravitational constant, cosmological constant, as well as the masses for the fermion fields. Accordingly, all dimensionful parameters in the Einstein Lagrangian are replaced by an appropriate scalar measuring field with the proper power according to their conformal dimensions.

Scale invariance also appears to be very important in various branches of physics such as QCD [1] and many other inflationary models [2–4]. Local scale symmetry has also been suggested to be related to the missing Higgs problem in electroweak theory [5] as well as many other research interests [6,7]. It is also argued that scale-invariant effective theory has to do with physics near the fixed points of renormalization-group trajectory [6].

On the other hand, higher dimensional Kaluza-Klein theory [8–10] has been a focus of research interest for a long time. In addition, Kaluza-Klein theory should be related to the evolution of our early Universe if the compactification process is completed during the early stage of the Universe. Hence one is naturally led to the question of whether the scale-invariant effective action is manifest before the dimensional reduction process takes place. Therefore, we propose to study the effect of a D-dimensional induced gravity in the very early universe.

One notes that there have been studies based on a D-dimensional Friedmann-Robertson-Walker (DFRW) metric

$$ds^2 \equiv \hat{g}_{MN} dz^M dz^N = -dt^2 + a^2(t)\left[\frac{1}{1-kr^2}dr^2 + r^2 d\Omega_{D-1}\right], \tag{1.1}$$

in search of a physically acceptable low-energy effective theory [10–14]. Here $d\Omega_n$ denotes the n-dimensional solid angle with appropriate angular coordinates. Note, however, that the internal and external spaces will inflate all together if $a(t)$ undergoes an exponentially expanding process under the DFRW metric. Hence it would be interesting to see if a more general $4+d$ dimensional FRW (4DFRW) space is capable of inducing inflating external space and contracting internal space at the same time. Here $d \equiv D-4$ denotes the dimension of the internal space. This kind of generalization is in fact a dimensional reduction from $M^D \to R^1 \times F^3 \times F^d$ with F^n denoting the n-dimensional maximally symmetric space [15].

In this paper, we will show that an induced inflationary solution with expanding external space and contracting internal space will require a very special symmetry-breaking scalar potential. The explicit form of the symmetry-breaking potential required by the expanding solution with expanding external space and constant internal space will also be solved. In addition, we will also show that the most favorable solution appears to be the same as the result of DFRW space for the induced conventional-ϕ^4 model. Our result indicates that the compactification process must have been completed before the inflationary process unless the symmetry-breaking potential takes an unordinary form.

This paper will be organized as follows: (i) In Sec. II, we will introduce the D-dimensional induced gravity theory. The D-dimensional equations of motion will also be compactified into $4+d$ dimensional in this section. (ii) We will also solve the equations of motion for an inflationary solution in the limit of slow-rollover approach in Sec. III. It will be shown first that the existence of a solution with expanding external space and contracting internal space will impose a number of constraints on the coupled symmetry-breaking potential. In particular, a solution with expanding external space and constant internal space is also solved while searching for the possible candidate for the scalar potential. (iii) The conventional ϕ^4 model with a coupled spontaneous symmetry breaking (SSB) ϕ^4 potential is solved and analyzed in Sec. IV. (iv) Finally, conclusions are drawn in Sec. V.

II. INDUCED GRAVITY IN D DIMENSION

In this paper we will consider the following induced gravity action:

$$S = -\int d^D z \sqrt{\hat{g}}\left[\frac{1}{2}\epsilon\phi^2\hat{R} + \frac{1}{2}\partial^M\phi\partial_M\phi + V(\phi)\right]. \tag{2.1}$$

The scalar field ϕ in Eq. (2.1) is the measuring field designed to replace the dimensionful Newtonian constant as it ap-

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peared in the four-dimensional-induced gravity models. One should replace all dimensionful coupling constants with appropriate scalar fields according to their dimensions. After this replacement, one can show that, apart from a possible symmetry-breaking potential V, the action (2.1) is invariant under the following scale transformation: $g_{MN} \to \Lambda^2 g_{MN}$, $\phi \to \Lambda^{(1-D/2)}\phi$.

We will denote $D=4+d$ from now on in this paper. Here $d=D-4$ is the dimension of the internal space in the Kaluza-Klein theory we are going to study in this paper. We will also use a hat notation, $\hat{R}_{MN}$, to represent the physical variables in D dimension and the nonhatted variable, R_{ab}, will represent the same physical variables evaluated solely in the four-dimensional physical external space. Barred notation, $\overline{R}_{mn}$, will also be employed to denote the same physical variables evaluated in d-dimensional internal space. Note that by *the same physical variables* we mean that they are defined by the same notation except the metric is replaced by the appropriate metric defined in its own space. Furthermore, we will use capital indices $M,N, \cdots (=0,1,2,\ldots,D-1)$ to denote the D-dimensional space-time indices. Also lower case Latin indices from the beginning $(a,b,c,\ldots)$ of the alphabet will denote the four-dimensional space time indices $(a,b,c=0,1,2,3)$. In addition, $i,j,k,l(=1,2,3)$ labels the spatial 3-manifold. Finally, we will use lower case Latin indices from the middle $(m,n,\ldots)$ of the alphabet to label the d-dimensional compactified internal space.

Note that the Kaluza-Klein dimensional reduction process we will adopt in this paper is the following $4+d$ dimensional Friedmann-Robertson-Walker metric (4DFRW) [15,16]

$$ds^2 \equiv \hat{g}_{MN} dz^M dz^N = -dt^2 + a^2(t) h_{ij}(x) dx^i dx^j + b^2(t) \overline{h}_{mn}(y) dy^m dy^n. \tag{2.2}$$

Here $h_{ij}dx^i dx^j \equiv (1-k_1 r^2)^{-1} dr^2 + r^2 d\Omega_3$ and $\overline{h}_{mn} dy^m dy^n \equiv (1-k_2 s^2)^{-1} ds^2 + s^2 d\Omega_d$ with k_1, $k_2 = 0, \pm 1$ denoting the signature of the external space and internal space, respectively.

Note that if we adopt the compactification ansatz $\phi(z) = \phi(x)\kappa^{d/2}$, the compactified four-dimensional effective Einstein action, except the SSB ϕ^4 potential term, will remain scale invariant under the four-dimensional scale transformation: $g_{ab}(x) \to \Lambda(x)^2 g_{ab}(x)$, $\phi(x) \to \Lambda(x)^{(2-D)/2}\phi(x)$. This shows that this is a consistent and scale-invariant way to carry out the compactification process. Note also that κ is a dimension-one constant parameter such that $\int d^d y \kappa^d$ is dimensionless and will be set as 1 for latter convenience. Note that we will also use the same ϕ notation for $\phi(z)$ and $\phi(x)$ for convenience.

The equations of motion can be obtained from varying the action (2.1) with respect to ϕ and $\hat{g}_{MN}$, respectively. As a result, one has

$$\epsilon \phi \hat{R} - D_M \partial^M \phi + \frac{\partial}{\partial \phi} V(\phi) = 0, \tag{2.3}$$

$$\epsilon \phi^2 \hat{G}_{MN} = \epsilon (D_M \partial_N - \hat{g}_{MN} D^P \partial_P) \phi^2 + \hat{T}^{\phi}_{MN}. \tag{2.4}$$

Here $\hat{G}_{MN} \equiv \frac{1}{2}\hat{R}\hat{g}_{MN} - \hat{R}_{MN}$ defines the Einstein tensor. Moreover, $\hat{T}^{MN}_{\phi} \equiv \partial^M \phi \partial^N \phi - \hat{g}^{MN}[\frac{1}{2}\partial^P \phi \partial_P \phi + V(\phi)]$ is the energy momentum tensor associated with ϕ. Furthermore, the curvature tensor $\hat{R}_{MNOP}$ is defined by $[D_M, D_N]\hat{A}_O = \hat{R}^P_{ONM}\hat{A}_P$. In addition, the Ricci tensor and scalar curvature are defined by $\hat{R}_{ON} \equiv \hat{R}^P_{ONP}$ and $\hat{R} \equiv \hat{R}_{ON}\hat{g}^{ON}$, respectively.

For latter convenience, we will define $\phi^2 \equiv e^{\varphi}$, $a \equiv e^{\alpha}$, $b \equiv e^{\beta}$, $V(\phi) \equiv U(\varphi)$ throughout this paper. Hence, one can bring Eqs. (2.3) and (2.4) into a more comprehensive form in terms of the new variables and parameters defined earlier. Indeed, Eqs. (2.3) and (2.4) can be written as

$$\hat{G}_{MN} = D_M \partial_N \varphi + \partial_M \varphi \partial_N \varphi - \hat{g}_{MN}(D^P \partial_P \varphi + \partial^P \varphi \partial_P \varphi) - \hat{T}^{\varphi}_{MN}, \tag{2.5}$$

$$\hat{R} = \frac{1}{4\epsilon}(\partial^M \varphi \partial_M \varphi + 2D^P \partial_P \varphi) - \frac{8}{\epsilon} e^{-\varphi} \frac{\partial U(\varphi)}{\partial \varphi}. \tag{2.6}$$

Hence one has

$$\hat{R} = \frac{1}{4\epsilon}(\partial_a \varphi \partial^a \varphi + D_a \partial^a \varphi) - \frac{2}{\epsilon} e^{-\varphi} \partial_{\varphi} U(\varphi), \tag{2.7}$$

$$\hat{G}_{ab} = (\partial_a \varphi \partial_b \varphi + D_a \partial_b \varphi) - g_{ab}(\partial^c \varphi \partial_c \varphi + D^P \partial_P \varphi) - T^{\varphi}_{ab}, \tag{2.8}$$

$$\hat{G}_{mn} = D_m \partial_n \varphi - \overline{g}_{mn}(\partial^c \varphi \partial_c \varphi + D^P \partial_P \varphi) - \overline{T}^{\varphi}_{mn}. \tag{2.9}$$

Here we have defined the generalized energy momentum tensor for φ and $\hat{T}^{\varphi}_{MN}$ as

$$\hat{T}^{\varphi}_{MN} = \frac{1}{4\epsilon}\left(\frac{1}{2}\hat{g}_{MN}\partial_P \varphi \partial^P \varphi - \partial_M \varphi \partial_N \varphi\right) + \frac{V}{\epsilon} e^{-\varphi} \hat{g}_{MN}. \tag{2.10}$$

Therefore, one has

$$T^{\varphi}_{ab} = \frac{1}{4\epsilon}\left(\frac{1}{2} g_{ab}\partial^c \varphi \partial_c \varphi - \partial_a \varphi \partial_b \varphi\right) + \frac{1}{\epsilon} e^{-\varphi} U(\varphi) g_{ab}, \tag{2.11}$$

$$\overline{T}^{\varphi}_{mn} = \frac{1}{2}\frac{1}{4\epsilon}\overline{g}_{mn}\partial^c \varphi \partial_c \varphi + \frac{1}{\epsilon} e^{-\varphi} U(\varphi) \overline{g}_{mn}, \tag{2.12}$$

respectively. In addition, with the compactified metric

$$ds^2 \equiv \hat{g}_{MN} dz^M dz^N = g_{ab}(z) dx^a dx^b + \overline{g}_{mn}(z) dy^m dy^n, \tag{2.13}$$

and by setting $\phi(z) = \phi(x)$, one can derive the following compactified identities for the curvature terms:

$$\hat{R}=R+2dD_a\partial^a\beta+d(d+1)\partial_a\beta\partial^a\beta-d(d-1)k_2e^{-2\beta}, \tag{2.14}$$

$$\hat{G}_{ab}=G_{ab}+t_{ab}, \tag{2.15}$$

$$\hat{G}_{mn}=\frac{1}{2}\bar{g}_{mn}[R+2(d-1)D_a\partial^a\beta+d(d-1)\partial_a\beta\partial^a\beta-(d-1)(d-2)k_2e^{-2\beta}]. \tag{2.16}$$

Here the generalized energy momentum tensor t_{ab} is given by

$$t_{ab}\equiv\frac{1}{2}g_{ab}[2dD_c\partial^c\beta+d(d+1)\partial_c\beta\partial^c\beta-d(d-1)k_2e^{-2\beta}]-d(D_a\partial_b\beta+\partial_a\beta\partial_b\beta). \tag{2.17}$$

Note that $\hat{g}^{MN}\hat{G}_{MN}=(D/2-1)\hat{R}$. Hence one can obtain the following equation of φ:

$$\partial_a\varphi\partial^a\varphi+D_a\partial^a\varphi+d\partial_a\beta\partial^a\varphi=\kappa_3\frac{e^{-\varphi}}{\epsilon}[(D-2)\partial_\varphi U-DU], \tag{2.18}$$

by eliminating the $\hat{R}$ term in the trace of $\hat{G}_{MN}$ Eq. (2.5) and the φ Eq. (2.7). Here one has set $\kappa_3=4\epsilon/[4(D-1)\epsilon+D-2]$. Finally, one can show that the trace of the $\hat{G}_{mn}$ Eq. (2.9), $\bar{g}^{mn}\hat{G}_{mn}$, and the trace of the $\hat{G}_{mn}$ Eq. (2.16) gives two constraint equations related to R. One can eliminate these R terms and obtain the following equation for $b(t)$:

$$d\partial_a\beta\partial^a\beta+D_a\partial^a\beta-(d-1)k_2e^{-2\beta}=\left(1+\frac{1}{4\epsilon}\right)[\partial_a\varphi\partial^a\varphi+D_a\partial^a\varphi+d\partial_a\beta\partial^a\varphi]-\partial_a\beta\partial^a\varphi+\frac{e^{-\varphi}}{\epsilon}[U-\partial_\varphi U]. \tag{2.19}$$

Therefore, we will take Eq. (2.19) as the independent β equation. This will soon be shown to be helpful in our analysis below. Finally, one can show that the G_{tt} component of Eq. (2.8), the φ equation (2.18), and the β equation (2.19) becomes

$$\alpha'^2+\frac{k_1}{a^2}+d\alpha'\beta'+\frac{d(d-1)}{6}(\beta'^2+k_2e^{-2\beta})+\alpha'\varphi'+\frac{d}{3}\beta'\varphi'=\frac{1}{24\epsilon}\varphi'^2+\frac{U}{3\epsilon}e^{-\varphi}, \tag{2.20}$$

$$\varphi''+3\alpha'\varphi'+d\beta'\varphi'+\varphi'^2=-\frac{\kappa_3}{\epsilon}e^{-\varphi}[(D-2)\partial_\varphi U-DU], \tag{2.21}$$

$$\beta''+3\alpha'\beta'+d\beta'^2+(d-1)k_2e^{-2\beta}+\beta'\varphi'=\frac{\kappa_3}{\epsilon}e^{-\varphi}\left[\partial_\varphi U+\left(1+\frac{1}{2\epsilon}\right)U\right]. \tag{2.22}$$

Note also that the G_{ij} component of Eq. (2.8) can be deduced from the four-dimensional Bianchi identity $D_aG^{ab}=0$ associated with the four-dimensional FRW metric. Hence it is in fact redundant. Therefore, Eqs. (2.20)–(2.22) are in fact the complete set of equations of motion one needs for solving α, β, and φ.

III. INFLATIONARY UNIVERSE

If one assumes the slow-rollover approximation, namely, $a'/a\gg|\varphi'|$, one can show that

$$\alpha'^2+d\alpha'\beta'+\frac{d(d-1)}{6}\beta'^2=\frac{U}{3\epsilon}e^{-\varphi}, \tag{3.1}$$

$$3\alpha'\beta'+d\beta'^2=\frac{\kappa_3}{\epsilon}e^{-\varphi}\left[\partial_\varphi U+\left(1+\frac{1}{2\epsilon}\right)U\right], \tag{3.2}$$

$$(3\alpha'+d\beta')\varphi'=\frac{\kappa_3}{\epsilon}e^{-\varphi}[(d+4)U-(d+2)\partial_\varphi U]. \tag{3.3}$$

Here we have set $k_1=k_2=0$ for simplicity. Note that the issue of the noncompact internal space has recently been the subject of renewed interest [9]. An exotic class of Kaluza-Klein models in which the internal space is neither compact nor even of finite volume was considered and gravity is used to trap particles near a four-dimensional submanifold of the higher dimensional spacetime.

Moreover, we have also assumed that $|\varphi''|\ll\alpha'|\varphi'|$ and $|\beta''|\ll\beta'^2$. We will show shortly that these assumptions can be met rather easily.

We will assume for the moment during the slow-rollover period that $\alpha=\alpha_0t$ and $\beta=-k\alpha_0t$ for some positive real number k and α_0. This kind of solution represents a brief moment of inflating scale factor a accompanied by a contracting internal scale factor b. This will be helpful in finding a possible constraint on the form of the symmetry-breaking potential one would require for a more realistic model. One can also assume that $U\sim U_0\equiv V(\phi=\phi_0)$, while $\phi\sim\phi_0$ during the inflationary phase. Therefore, one can show that Eqs. (3.1)–(3.3) can be brought to the following form:

$$d(d-1)k^2-6dk+6=\tilde{k}, \tag{3.4}$$

$$k(dk-3)=\frac{\tilde{k}\kappa_3}{4}(s-s_-), \tag{3.5}$$

$$dk-3=\frac{(d+2)\tilde{k}\kappa_3\alpha_0}{4\varphi_0'}(s-s_+). \tag{3.6}$$

Here $\tilde{k}\equiv 2U_0/\epsilon\alpha_0^2\phi_0^2$, $\varphi_0'\equiv\varphi'(t_0)$, $s_-\equiv-2-1/\epsilon$, and $s_+\equiv 2(d+4)/(d+2)$. In addition, we have also defined $s\equiv\phi_0(\partial_\phi U)_0/U_0$ as the scaling factor of U evaluated at $\phi=\phi_0$.

We will first study the case where $d>1$. Note that Eq. (3.4) indicates that $d(d-1)k^2-6dk+6\equiv d(d-1)(k-k_+)(k-k_-)>0$ under the assumption that U is positive everywhere. Here we have defined $k_\pm\equiv 3/(d-1)\pm\sqrt{3d(d+2)}/d(d-1)$ as the roots of the k equation. Therefore, one has either $k>k_+$ or $k<k_-$ from the k inequality. In addition, Eqs. (3.5) and (3.6) give

$$\alpha_0=\frac{s-s_-}{k(d+2)(s-s_+)}\varphi_0'. \tag{3.7}$$

This shows that $\varphi_0'(s-s_+)(s-s_-)>0$, since k is assumed to be positive.

Moreover, one also assumes that $dU(\phi)/dt<0$ such that the scalar field is rolling down from some initial value ϕ_0 to the minimum potential energy state ϕ_m. This means that $s\varphi_0'<0$. Therefore, one finds that there are only two kinds of combination capable of supporting this process. The first one is (1) $\varphi_0'<0$, $0<s<s_+$ and the second one is (2) $\varphi_0'>0$, $s>s_-$. One can further rule out case (2) from the assumption that $\alpha_0\gg|\varphi_0'|$. Indeed, Eq. (3.7) indicates that the slow-rollover assumption is equivalent to $|s-s_-|\gg k(d+2)|s-s_+|$. Hence case (1) can be shown to give a constraint

$$s\gg\frac{k(d+2)s_+ + s_-}{k(d+2)+1}. \tag{3.8}$$

This can easily be achieved provided that $s_+\gg s_-$. Note that this is true if $\epsilon\ll 1$ [4]. Similarly, case (2) will give a contradictory result $s_-\gg s_+$. Therefore, case (2) is ruled out.

In addition, case (1) and Eq. (3.6) shows that $dk-3>0$. Hence the constraint on k obtained earlier is further restricted to the case where $k>k_+$. This is because $3/d<k<k_-$ leads to a contradiction $3>d(d+2)$.

In short, the induced Kaluza-Klein compactification admits chaotic inflation only if the symmetry-breaking potential obeys a number of constraints listed earlier. They are

$$(a)\quad s_+\gg s_-, \tag{3.9}$$

$$(b)\quad s_+>s\gg[k(d+2)s_+ + s_-]/[k(d+2)+1], \tag{3.10}$$

$$(c)\quad \varphi_0'<0, \tag{3.11}$$

$$(d)\quad k>k_+. \tag{3.12}$$

For example, one would have (a) $8/5\gg s_-$, (b) $8/5>s\gg(5k+s_-)/(8k+1)$, and (c) $k>1$ as the constraint on k and s for the case where $d=6$ or equivalently $D=10$.

One can easily construct an effective symmetry-breaking potential by expanding the potential around the initial point ϕ_0. Explicitly, it will take the form $U=U_0+sU_0(\phi-\phi_0)+\cdots$ around the initial point. For example, one can show that the conventional ϕ^4 model with $U=(\lambda/8)(\phi^2-v^2)^2$ does not satisfy the constraint obtained earlier. Indeed, one can show that Eq. (3.6) gives

$$\Lambda>\frac{\lambda}{8(d+4)}(\phi_0^2-v^2)[d\phi_0^2+(d+4)v^2] \tag{3.13}$$

for the conventional ϕ^4 model with an additional positive definite cosmological constant term Λ. This clearly shows that the chaotic inflation condition $\phi_0^2>v^2$ is inconsistent with the case where $\Lambda=0$. Note that the no-hair conjecture states that cosmologies with a positive cosmological constant would approach the de Sitter solution asymptotically [17]. Even some counter examples are found, and are shown to hold for very general conditions [18]. Our result appears to favor the above conjecture with the inclusion of the higher dimensional space. Therefore, the conventional ϕ^4 model with vanishing cosmological constant cannot support an inflationary solution with expanding external space and contracting internal space. We will solve the conventional ϕ^4 model later in Sec. IV.

For the case where $d=1$, the situation is rather different. Equation (3.4) implies that $k<1$ for positive U_0. In addition, Eq. (3.5) gives $s<s_-(<0)$, while Eq. (3.6) implies $\varphi_0'>0$ (new inflationary solution). In addition, the slow-rollover assumption indicates that $s_--s\gg k(10-3s)$. Therefore, one obtains $(3k-1)s\gg 10k-s_-(>0)$. This implies that $k<1/3$. In summary, one has (a) $s<s_-(<0)$, (b) $k<1/3$, and (c) $\varphi_0'>0$. Therefore, the five-dimensional Kaluza-Klein new inflationary solution with expanding external space and contracting internal space can also be arranged if the field parameters are chosen appropriately.

One can also study the case where the internal-space scale factor remains constant, i.e., $b=b_0$ [14] or equivalently $k=0$ in the early universe. In this case, the equations will become

$$\alpha'^2+\frac{k_1}{a^2}+\frac{d(d-1)}{6}\frac{k_2}{b_0^2}+\alpha'\varphi' =\frac{1}{24\epsilon}\varphi'^2+\frac{U}{3\epsilon}e^{-\varphi}, \tag{3.14}$$

$$\varphi''+3\alpha'\varphi'+\varphi'^2=-\frac{\kappa_3}{\epsilon}e^{-\varphi}[(D-2)\partial_\varphi U-DU], \tag{3.15}$$

$$(d-1)\frac{k_2}{b_0^2}=\frac{\kappa_3}{\epsilon}e^{-\varphi}\left[\partial_\varphi U+\left(\frac{1}{2\epsilon}+1\right)U\right]. \tag{3.16}$$

Therefore, one finds that there is a strong constraint (3.16) left over for the b equation. This equation says that $\partial_\varphi U+(1+1/2\epsilon)U=0$ for a flat internal space (i.e., $k_2=0$). One can then show that either (i) the potential U has to be a special fractional polynomial functional of ϕ, namely, $U=k_0\phi^{-(2+1/\epsilon)}$ with a proportional constant k_0, or (ii) the dynamics of the scalar field has to be frozen, namely, the scalar field becomes a constant $\phi=\phi_0$. One can show that the first case would imply that $\alpha'\sim\varphi'/2$ under the constraint $\epsilon\ll 1$. This contradicts the slow rollover approximation. On the other hand, the case (ii) implies that $U(\varphi_0)=\partial_\varphi U(\varphi_0)=0$. Hence one has $a'^2=-k_1$ because of Eq. (3.14). Therefore, one needs $k_1=-1$ in order to admit a power-law inflation. One can hence tune the field parameters to induce enough inflation with an expanding external space and constant internal space. But this model cannot tell us when the inflationary phase should come to an end. One would have to expect that this induced gravity model remains valid only during the inflationary period and leave the problem to other resolutions.

On the other hand, one can show that the constraint (3.16)

$$(D-5)\frac{k_2\epsilon}{b_0^2\kappa_3}\phi^2=\partial_\varphi U+\left(1+\frac{1}{2\epsilon}\right)U \tag{3.17}$$

implies $\phi=\phi_0$ for the case $k_2\neq 0$ unless

$$U=k_0\phi^{-2-1/4\epsilon}+\frac{2(D-5)k_2\epsilon^2}{(1+4\epsilon)b_0^2\kappa_3}\phi^2. \tag{3.18}$$

If $\phi=\phi_0$, Eq. (3.15) implies that

$$[(D-2)\partial_\varphi U-DU]_{\phi_0}=0. \tag{3.19}$$

Equations (3.17) and (3.19) mean that all field parameters and initial conditions are constrained by these equations. In addition, Eq. (3.14) tells us that

$$\alpha'^2\sim(D-5)\frac{k_2}{3b_0^2} \tag{3.20}$$

independent of the form of potential U. Of course, the initial value of the scalar field ϕ_0 is determined by the form of potential and the two constraints just derived. This solution is an inflationary solution with expanding external space and constant internal space as long as $k_2(D-5)>0$ and $b_0\ll 1$. One can certainly tune b_0 to induce enough inflation with expanding external space and constant internal space. But this solution cannot tell us how to exit the inflationary phase at this point either. One would then have to expect again that this kind of induced gravity model would not remain effective as soon as the inflationary process is completed.

On the other hand, Eqs. (3.14) and (3.15) imply that

$$\alpha'^2=(1+6\epsilon)A+B, \tag{3.21}$$

$$\alpha'\varphi'=4\epsilon A+2B, \tag{3.22}$$

under the slow-rollover approximation if U is given by Eq. (3.18). Here

$$A=\frac{(D-5)k_2}{3b_0^2(1+4\epsilon)}, \tag{3.23}$$

$$B=\frac{k_0}{3\epsilon\phi_0^{4+1/4\epsilon}} \tag{3.24}$$

for $\phi\sim\phi_0$ in this inflationary phase. Therefore, one can choose $\epsilon\ll 1$ and $B\ll A$ in order to be consistent with the assumption that $\alpha'\gg|\varphi'|$. In addition, one can choose $A>0$ since $\alpha'^2>0$. This implies that $k_2=1$ for $D\geq 5$. In addition, $A\gg B$ implies that $k_0b_0^2\ll(D-5)\epsilon\phi_0^{4+1/4\epsilon}$ which can be achieved by tuning the field parameters appropriately. Moreover, one still needs to make sure that the potential U given by Eq. (3.18) has at least a local minimum ϕ_m far away from the initial data ϕ_0 such that inflation can exit in due time.

Fortunately, a local minimum always exists for a large class of parameters. Indeed, one can show that

$$k_0b_0^2\sim(D-5)(D-2)\epsilon^2\phi_m^{4+1/4\epsilon} \tag{3.25}$$

from $\partial_\phi U|_{\phi=\phi_m}=0$. Hence, one only needs

$$\epsilon\phi_m^{4+1/4\epsilon}\ll\frac{2}{D-2}\phi_0^{4+1/4\epsilon}. \tag{3.26}$$

In addition, the requirement $U''|_{\phi_m}>0$ can be made valid very easily.

Therefore, the inflationary process can properly work with the assumption $b=b_0$ for the case where $F^d=S^{d+1}$, and this has to come along with the potential of the form given by Eq. (3.18).

IV. CONVENTIONAL ϕ^4 MODEL

One can also work on the model with a spontaneous symmetry breaking (SSB) ϕ^4 potential $U=(\lambda/8)(e^\varphi-v^2)^2$. This sort of potential will be referred to as the conventional ϕ^4 model in this paper. It is straightforward to show that $\partial_\varphi U=(\lambda/4)(e^\varphi-v^2)e^\varphi$. Hence Eq. (3.3) becomes

$$(3\alpha'+d\beta')\varphi'=-\frac{\kappa_3\lambda}{8\epsilon}e^{-\varphi}(e^\varphi-v^2)[de^\varphi+(d+4)v^2]. \tag{4.1}$$

This indicates that $3\alpha+d\beta$ is always an increasing function as long as the φ field is rolling down to its true vacuum $e^\varphi=v^2$. It also indicates that ϕ cannot go far away from its local minimum, hence it should oscillate around $\phi=v$ after the inflation is over. We will come back to this point shortly. Moreover, Eq. (3.2) becomes

$$3\alpha'\beta'+d\beta'^2=\frac{2U}{(d+2)\epsilon}e^{-\varphi} \tag{4.2}$$

if $\epsilon\ll 1$ and $|\phi^2-v^2|/\phi^2\gg 4\epsilon$. These assumptions can be adjusted rather easily. Together with Eq. (3.1), one finds that

$$2\alpha'^2+(d-2)\alpha'\beta'-d\beta'^2=(\alpha'-\beta')(2\alpha'+d\beta')\sim 0. \tag{4.3}$$

This means that $\alpha'=\beta'$ because the equation $\alpha'=-(d/2)\beta'$ contradicts Eq. (4.2). Hence, $b(t)$ increases along with the expanding $a(t)$ in the inflationary era under the slow-rollover approximation. Therefore, one has shown that the conventional ϕ^4 model supports DFRW space instead of the 4DFRW space. Hence the solution with expanding external space and contracting internal space cannot be found under the slow-rollover approximation. We will still, however, study the DFRW solution in detail in this section for completeness. Note that the presence of a nonvanishing cosmological constant in the conventional ϕ^4 model will not affect Eq. (4.3) under the same slow-rollover approximation.

Note that Eq. (4.2) gives us

$$\alpha' \sim \sqrt{\frac{\lambda v^4}{8(d+2)(d+3)}}\,\frac{v}{\phi_0}, \tag{4.4}$$

$$a \sim a_0 \exp\left(\sqrt{\frac{\lambda v^4}{8(d+2)(d+3)}}\,\frac{v}{\phi_0}\right). \tag{4.5}$$

Here one has set $\frac{1}{2}\epsilon v^2=1$ such that the gravitational constant measured today is set as 1 in the Planck unit. Moreover, ϕ_0, set to be positive, denotes the initial value of the ϕ field. Moreover, Eq. (4.1) gives

$$\phi' \sim \sqrt{\frac{\lambda(d+4)^2}{2(d+2)(d+3)}}v, \tag{4.6}$$

$$\phi \sim \phi_0 + \sqrt{\frac{\lambda(d+4)^2}{2(d+2)(d+3)}}vt. \tag{4.7}$$

Here we can see that the assumptions $|\varphi''| \ll \alpha'|\varphi'|$ and $|\beta''| \ll \beta'^2$ are both satisfied without imposing any further constraints.

One can further derive a few inequalities from the slow-rollover assumption $a'/a \gg |\varphi'|$. First of all, they give

$$v^2 \geqslant 4(d+4). \tag{4.8}$$

Note that the cosmological constant term $-\frac{1}{8}\lambda v^4$ at initial time should be less than 1, in the Planck unit, in order that the quantum effect can be neglected. We will be using the Planck unit from now on. In addition, if the scale factor $a(t)$ is capable of expanding some 60 e-fold in a time interval of roughly $\Delta T \sim 10^8$ Planck units, one should have the following inequality:

$$\frac{\lambda v^4}{8} \geqslant (d+2)(d+3)\frac{\phi_0^2}{v^2} \times 3.6 \times 10^{-13}. \tag{4.9}$$

Inequality (4.9) can be made valid rather easily. Indeed, these inequalities can be easily satisfied by choosing large v^2 (hence small ϵ) and a λ around the order of 10^{-17} as in Ref. [4]. Hence one shows that the slow-rollover approximation is indeed a good approach to this expanding solution.

Note that we can also extract information about α, β, and φ when $\varphi \to v$ near the end of the expansion. This can be done by analyzing Eqs. (2.20)–(2.22) by assuming $e^{\varphi}=\xi+v^2$ with $\xi \ll v^2$. Moreover, one can show that Eqs. (2.21)–(2.22) become

$$\partial_t(e^{3\alpha+d\beta+\varphi}\varphi') = -\frac{\kappa_3\lambda}{8\epsilon}e^{3\alpha+d\beta}(e^{\varphi}-v^2) \times [de^{\varphi}+(d+4)v^2], \tag{4.10}$$

$$\partial_t(e^{3\alpha+d\beta+\varphi}\beta') = \frac{\kappa_3\lambda}{8\epsilon}e^{3\alpha+d\beta}(e^{\varphi}-v^2) \times\left[\left(3+\frac{1}{2\epsilon}\right)e^{\varphi}-\left(1+\frac{1}{2\epsilon}\right)v^2\right]. \tag{4.11}$$

Hence Eqs. (4.10),(4.11) become

$$\partial_t(e^{3\alpha+d\beta}\xi') = -m_v(d+2)e^{3\alpha+d\beta}\xi, \tag{4.12}$$

$$\partial_t[e^{3\alpha+d\beta}\beta'(\xi+v^2)] = m_v e^{3\alpha+d\beta}\xi, \tag{4.13}$$

as one takes the limit $e^{\varphi}=\xi+v^2$. Here $m_v \equiv \kappa_3(\lambda v^4/8)$.

Moreover, Eqs. (2.20)–(2.22) can be interpreted as a set of equations that allow one to express $\alpha(t)$ and $\beta(t)$ as functions of $\xi(t)$. Therefore, one can expand α and β as polynomials of ξ, i.e., one can write

$$\alpha(t) = \alpha_0(t) + \alpha_1(t)\xi(t) + \alpha_2(t)\xi^2(t) + \cdots \tag{4.14}$$

$$\beta(t) = \beta_0(t) + \beta_1(t)\xi(t) + \beta_2(t)\xi^2(t) + \cdots. \tag{4.15}$$

Therefore, the lowest (first) order in ξ of Eq. (4.12) is

$$\partial_t(e^{3\alpha_0+d\beta_0}\xi') = -m_v(d+2)e^{3\alpha_0+d\beta_0}\xi. \tag{4.16}$$

Moreover, the zeroth order in ξ of Eq. (4.13) can be shown to be

$$\partial_t[e^{3\alpha_0+d\beta_0}\beta_0'] = 0. \tag{4.17}$$

This means that $\partial_t[e^{d\beta_0}] = \text{const} \times e^{-3\alpha_0}$. Therefore, one has $\partial_t[e^{d\beta_0}] \sim 0$ since $e^{-3\alpha_0}$ is very close to 0 in the post-expansion era. Therefore, one can assume that $e^{d\beta_0}$ is changing very slowly as $\phi^2 \to v^2$. In addition, the zeroth order in ξ of Eq. (2.20) is

$$\alpha_0'^2 + d\alpha_0'\beta_0' + \frac{d^2-d}{6}\beta_0'^2 \sim 0. \tag{4.18}$$

This gives

$$\alpha_0' = -\frac{\sqrt{3}d \pm \sqrt{d^2+2d}}{2\sqrt{3}}\beta_0' \sim 0 \tag{4.19}$$

to this order of the limit. Hence, Eq. (4.16) becomes an equation for a simple harmonic oscillator

$$\xi'' = -m_v(d+2)\xi. \tag{4.20}$$

Note that the left-hand side of Eq. (4.20) approaches $-\lambda v^2 \xi$ in the limit $\epsilon \ll 1$. In short, one finds that ϕ field indeed oscillates about the local minimum of the symmetry-breaking potential U. Furthermore, Eqs. (4.19) indicate that $\alpha_0' \beta_0' < 0$ as ϕ approaches the local minimum of U. Therefore, $b(t)$ in fact starts decreasing if $a(t)$ remains increasing at later time. Note that above analysis is only a rough estimate, but it gives us a rough picture of what is going on when the ξ field approaches zero.

V. CONCLUSIONS

In summary, a D-dimensional induced gravity model in 4DFRW space is studied carefully. We present a careful and detailed analysis for the compactification process. This model is then solved for the inflationary solution in the slow-rollover approach. A number of constraints on the symmetry-breaking potential are found. These constraints are derived from the search for a inflationary solution with expanding external space and contracting, compactified internal space.

The result indicates that a possible form of the symmetry-breaking potential, prescribed by s, is constrained by Eqs. (3.4)–(3.6) due to the field equations. Here, $s \equiv \phi_0 (\partial_\phi U)_0 / U_0$ signifies the scaling factor of U evaluated at $\phi = \phi_0$. The cases where $d > 1$ and $d = 1$ are analyzed separately. Explicitly, constraints to the coupled potential are listed in Eqs. (3.9)–(3.12) for the case where $d > 1$. In particular, one shows that these constraints read (a) $8/5 \gg s_-$, (b) $8/5 > s \gg (5k + s_-)/(8k+1)$, and (c) $k > 1$ in the limit where $d = 6$. It was then shown that the conventional ϕ^4 model with an additional cosmological constant term fails to satisfy the above constraints. On the other hand, one shows that (a) $s < s_- (< 0)$, (b) $k < 1/3$, and (c) $\varphi_0' > 0$ for the case where $d = 1$. In addition, we also solve the case where the internal scale factor b remains constant during the inflationary phase.

An expanding solution is also found and analyzed for the conventional ϕ^4 model. In order to generate a solution with expanding external-space inflation in the very early universe, one finds that the internal space is expanding too under the slow-rollover approximation. Therefore, this indicates that dimensional reduction has to be completed before expanding external space starts to expand. With properly chosen free parameters and boundary conditions of the scalar field, one shows that enough expansion can be easily achieved regardless of the negative impact of the expanding internal space in the conventional ϕ^4 model.

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