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Meng-Kiat Chuah

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By MENG-KIAT CHUAH

Abstract. Let K be a compact connected semi-simple Lie group, let G be its complexification, and let $G = KAN$ be an Iwasawa decomposition. Let B be the Borel subgroup containing A and N. Let P be a parabolic subgroup of G containing B, and (P,P) its commutator subgroup. In this paper, we perform geometric quantization and symplectic reduction to the pseudo-Kähler forms on the complex homogeneous space $G/(P,P)$. The reduced space is a disjoint union of copies of the flag manifold G/P, and this allows us to study the signatures of the K-invariant pseudo-Kähler forms on G/P via symplectic reduction. We also discuss the connectivity of the reduced space.

1. Introduction. Let K be a compact connected semi-simple Lie group, let G be its complexification, and let $G = KAN$ be an Iwasawa decomposition. Let T be the centralizer of A in K, so that $H = TA$ is a Cartan subgroup, and $B = HN$ is a Borel subgroup of G. Let P be a parabolic subgroup of G containing B, and (P,P) its commutator subgroup. Consider the complex homogeneous space $G/(P,P)$. The study of geometric quantization of Kähler forms on $G/(P,P)$ started with the special case where $P = B$ and hence $G/(P,P) = G/N$ [7], following a suggestion of A. S. Schwarz [15]. This was generalized to the general parabolic subgroups in [6], which constructs a model for K (in the sense of I. M. Gelfand and A. Zelevinski [8]) via L^2-holomorphic sections of the pre-quantum line bundle. The present paper is a continuation of [6]. It extends Kähler forms to pseudo-Kähler forms, and extends L^2-holomorphic sections to cohomology of the L^2-Dolbeault complex. Pseudo-Kähler forms are symplectic forms ω of type $(1,1)$. Namely, they are a generalization of Kähler forms in the sense that under the complex structure J, the symmetric form $\omega(\cdot, J\cdot)$ is required to be nondegenerate, but not necessarily positive definite. In this paper, we perform geometric quantization [14] and symplectic reduction [17] to the pseudo-Kähler forms on $G/(P,P)$ under certain compact Lie group actions. An idea which links these two processes is proposed by V. Guillemin and S. Sternberg [9], called "geometric quantization commutes with reduction." A survey of this idea can be found in [18]. We shall show that this principle works in our situation, and leads to unitary irreducible K-representations. The reduced space is a disjoint union of copies of the flag manifold G/P. We describe the signatures of the K-invariant pseudo-Kähler forms on G/P via the above unitary irreducible representations. Connectivity of the reduced space is also studied.

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Our projects are carried out in the following sections: We study the pseudo-Kähler forms on $G/(P,P)$ in §2 and quantize them in §3, in terms of cohomology of the L^2-Dolbeault complex. In §4, we perform symplectic reduction on $G/(P,P)$ and show that the reduced space is the flag manifold G/P up to connected components. As corollaries, we classify the K-invariant pseudo-Kähler forms on G/P and reproduce A. Borel's result on Fubini-Study forms. As an application, we study the signatures of these pseudo-Kähler forms in §5. Finally, connectivity of the reduced space is discussed in §6. We now describe these projects in more details.

We adopt the convention that the Lie algebra of a Lie group is denoted by the corresponding lower case German letter. For example, the Lie algebras of H, T are respectively $\mathfrak{h}, \mathfrak{t}$.

Let $\Delta \subset \mathfrak{h}^*$ be the roots. A positive system of Δ is determined by $\mathfrak{n}$, where $\mathfrak{n}$ corresponds to the negative root spaces. Let $\Delta_0 \subset \Delta$ be the simple positive roots. We shall say that $\sigma \subset \mathfrak{t}^*$ is a cell if there exists $S \subset \Delta_0$ such that

$$\sigma = \{\lambda \in \mathfrak{t}^*; (S, \lambda) > 0, (\Delta_0 \backslash S, \lambda) = 0\}. \tag{1.1}$$

Here the pairing $(,)$ on $\mathfrak{t}^*$ is given by the Killing form, and can be thought of as an inner product. This way, the dominant Weyl chamber is the disjoint union of all the cells. These cells have various dimensions. For instance, the origin and the open Weyl chamber are respectively cells with dimensions 0 and rank K. There is a bijective correspondence between the cells $\{\sigma\}$ and the parabolic subgroups P containing B, given by Langlands decomposition ([12], p. 132):

$$P = M_\sigma A_\sigma N_\sigma \ ; \ A_\sigma \subset A, N_\sigma \subset N.$$

Here $A_\sigma = A$ and $N_\sigma = N$ exactly when $P = B$. The subalgebra $\mathfrak{a}_\sigma \subset \mathfrak{a}$ is identified with $\mathfrak{t}_\sigma \subset \mathfrak{t}$ by the complex structure. Here $\mathfrak{t}_\sigma^*$ is the span of $\sigma \subset \mathfrak{t}^*$. The corresponding complex torus $H_\sigma = T_\sigma A_\sigma$ normalizes (P,P), so it acts on $G/(P,P)$ on the right.

From (1.1), every positive root α satisfies $(\alpha, \sigma) \geq 0$. Let $\overline{S}$ be the positive roots α in which $(\alpha, \sigma) > 0$. So $S = \overline{S} \cap \Delta_0$. We define

$$(\mathfrak{t}_\sigma^*)_{reg} = \{\lambda \in \mathfrak{t}_\sigma^*; (\alpha, \lambda) \neq 0 \text{ for all } \alpha \in \overline{S}\}. \tag{1.2}$$

It satisfies $\sigma \subset (\mathfrak{t}_\sigma^*)_{reg} \subset \mathfrak{t}_\sigma^*$. In fact $(\mathfrak{t}_\sigma^*)_{reg}$ is a disjoint union of open cones in $\mathfrak{t}_\sigma^*$, one of them being σ. The points in $(\mathfrak{t}_\sigma^*)_{reg}$ are called σ-regular. In particular if σ is the open Weyl chamber, a σ-regular point is simply called regular.

Let K^σ be the centralizer of σ in K, and $K_{ss}^\sigma \subset K^\sigma$ its commutator subgroup. In [6], we see that

$$G/(P,P) = (K/K_{ss}^\sigma)A_\sigma. \tag{1.3}$$

It identifies A_σ with the subset $\{ea\}$ of $G/(P,P)$, where $e \in K/K^\sigma_{ss}$ is the identity coset and $a \in A_\sigma$. The above description also identifies the K-invariant functions $C^\infty_K(G/(P,P))$ with functions on A_σ. Observe that A_σ and $\mathfrak{a}_\sigma$ are diffeomorphic by the exponential map. This leads to the following identification

$$(1.4) \qquad C^\infty_K(G/(P,P)) \cong C^\infty(A_\sigma) \cong C^\infty(\mathfrak{a}_\sigma),$$

which we adopt throughout this paper. By the complex structure, $\mathfrak{a}^*_\sigma \cong \mathfrak{t}^*_\sigma$. So the gradient of $F \in C^\infty(\mathfrak{a}_\sigma)$ is given by

$$(1.5) \qquad F'\colon\ \mathfrak{a}_\sigma \longrightarrow \mathfrak{t}^*_\sigma.$$

Let $U_F \subset \mathfrak{t}^*_\sigma$ denote the image of $\frac{1}{2}F'$.

We shall say that $F \in C^\infty(\mathfrak{a}_\sigma)$ is nonsingular if its Hessian matrix is nonsingular everywhere. Further, if the Hessian matrix is positive definite everywhere, we say that F is strictly convex.

In [6], we study the $K \times T_\sigma$-invariant Kähler forms on $G/(P,P)$. The following theorem is a simple extension to the pseudo-Kähler forms.

THEOREM 1. *Every $K \times T_\sigma$-invariant closed (1,1)-form on $G/(P,P)$ is given by $\omega = \sqrt{-1}\partial\bar\partial F$, where $F \in C^\infty(\mathfrak{a}_\sigma)$. There exists a unique moment map Φ: $G/(P,P) \longrightarrow \mathfrak{k}^*$ for the K-action given by $\Phi(a) = \frac{1}{2}F'(a) \in \mathfrak{t}^*_\sigma$ for all $a \in A_\sigma \subset G/(P,P)$, so $\Phi(A_\sigma) = U_F$. Here ω is pseudo-Kähler if and only if F is nonsingular and $U_F \subset (\mathfrak{t}^*_\sigma)_{reg}$. It is Kähler if and only if F is strictly convex and $U_F \subset \sigma$.*

This is proved in §2. Since Φ is K-equivariant, the theorem says that F' determines Φ.

Given a $K \times T_\sigma$-invariant pseudo-Kähler form ω, there exists a pre-quantum line bundle $\mathbf{L}$ [14] whose Chern class is $[\omega]$. Since ω is exact, $[\omega] = 0$, so $\mathbf{L}$ is a trivial bundle. It carries a connection whose curvature is ω, as well as an invariant Hermitian structure.

Let $\lambda \in \mathfrak{h}^*_\sigma$ be an integral weight. We shall always write $\chi = e^\lambda$ for its character. Namely, χ: $H_\sigma \longrightarrow \mathbf{C}^\times$ is the multiplicative homomorphism satisfying

$$(1.6) \qquad \chi(e^v) = \exp(\lambda, v),\ v \in \mathfrak{h}_\sigma.$$

Let α be an element of a right H_σ-module. We say that α *transforms by $\lambda \in \mathfrak{h}^*_\sigma$ under the right H_σ-action* (respectively *transforms by $\lambda \in \mathfrak{t}^*_\sigma$ under the right T_σ-action*) if $h \cdot \alpha = \chi(h)\alpha$ for all $h \in H_\sigma$ (respectively $h \in T_\sigma$).

Let $\mathbf{\Omega}^{0,q}(\mathbf{L})$ denote the Dolbeault $(0,q)$-forms on $G/(P,P)$ with coefficients in $\mathbf{L}$. They form a chain complex under the Dolbeault operator $\bar\partial_{\mathbf{L}}$. Using the Hermitian structure on $\mathbf{L}$, we shall construct (in (3.2)) a $K \times T_\sigma$-invariant L^2-structure on $\mathbf{\Omega}^{0,q}(\mathbf{L})$. We say that an element of $\mathbf{\Omega}^{0,q}(\mathbf{L})$ is *square-integrable* if it converges under this L^2-structure. Let $\lambda \in \mathfrak{t}_\sigma^*$ be an integral weight. Let $\Omega^{0,q}_{2,\lambda}(\mathbf{L})$ be the differential forms $\alpha \in \Omega^{0,q}(\mathbf{L})$ in which α and $\bar\partial_{\mathbf{L}}\alpha$ are both square-integrable, and that α transforms by $\lambda \in \mathfrak{t}_\sigma^*$ under the right T_σ-action. The cohomology of the complex $\{\Omega^{0,q}_{2,\lambda}(\mathbf{L}), \bar\partial_{\mathbf{L}}\}_q$ is denoted by $(H^q_\omega)_\lambda$.

We want to let the L^2-structure on the differential forms define a unitary structure on $(H^q_\omega)_\lambda$. This is done by selecting a unique representative for each cohomology class. From the imbedding $\mathfrak{h}_\sigma \hookrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/[\mathfrak{p},\mathfrak{p}]$, we can identify $\wedge^{0,1}\mathfrak{h}_\sigma$ with $K \times H_\sigma$-invariant anti-holomorphic vector fields whose values at the identity coset lie in $\wedge^{0,1}\mathfrak{h}_\sigma$. Given $\alpha \in \Omega^{0,q}(\mathbf{L})$, we say that α *annihilates* $\mathfrak{h}_\sigma$ if the interior product $\iota(v)\alpha \in \Omega^{0,q-1}(\mathbf{L})$ vanishes for all $v \in \wedge^{0,1}\mathfrak{h}_\sigma$.

Let $\bar\partial^*_{\mathbf{L}}$ be the formal adjoint of $\bar\partial_{\mathbf{L}}$ relative to the L^2-structure. If $\alpha \in \Omega^{0,q}_{2,\lambda}(\mathbf{L})$ satisfies $\bar\partial_{\mathbf{L}}\alpha = \bar\partial^*_{\mathbf{L}}\alpha = 0$, we say that α is *harmonic*.

Let W be the Weyl group corresponding to $\mathfrak{h} \subset \mathfrak{g}$, and let ρ denote half the sum of all positive roots. Consider the conditions

(1.7) (i) $\lambda \in U_F$;

(ii) there exists $\tau \in W$ of length q such that $\tau(\lambda + \rho) - \rho$ is dominant.

In §3, we prove:

THEOREM 2. *Let $\omega = \sqrt{-1}\partial\bar\partial F$ be a $K \times T_\sigma$-invariant pseudo-Kähler form, with F strictly convex. Then $(H^q_\omega)_\lambda$ vanishes unless (1.7) is valid. When this happens, $(H^q_\omega)_\lambda$ is an irreducible K-representation with highest weight $\tau(\lambda + \rho) - \rho$. Each cohomology class $\xi \in (H^q_\omega)_\lambda$ has a unique harmonic representative α which annihilates $\mathfrak{h}_\sigma$, so $(H^q_\omega)_\lambda$ becomes a unitary K-representation via $\|\xi\| = \|\alpha\|$.*

The special case $q = 0$ of this theorem is discussed in [6]. Our construction of unitary structure for $(H^q_\omega)_\lambda$ shall be further justified in Theorem 4, when we compare $(H^q_\omega)_\lambda$ with another unitary representation $H^q_{(\omega_\lambda)}$.

In §4, we perform symplectic reduction with respect to the right T_σ-action preserving $\omega = \sqrt{-1}\partial\bar\partial F$. We shall show (in Proposition 4.1) that the right T_σ-action preserving ω is Hamiltonian, with a canonical K-invariant moment map

$$\Phi_r\colon\ G/(P,P) \longrightarrow \mathfrak{t}_\sigma^*.$$

We call it the right moment map, to distinguish it from the K-moment map Φ. Let $\lambda \in \mathfrak{t}_\sigma^*$ be in the image of Φ_r. There is a free T_σ-action on $\Phi_r^{-1}(\lambda) \subset G/(P,P)$, and the quotient $R_\lambda = \Phi_r^{-1}(\lambda)/T_\sigma$ has a natural K-invariant pseudo-Kähler form ω_λ. It satisfies $\pi^*\omega_\lambda = \imath^*\omega$, where

$$\text{(1.8)} \qquad \pi\colon\ \Phi_r^{-1}(\lambda) \longrightarrow R_\lambda,\ \imath\colon\ \Phi_r^{-1}(\lambda) \longrightarrow G/(P,P)$$

are respectively the natural fibration and inclusion. The procedure

$$(G/(P,P),\omega,\lambda) \rightsquigarrow (R_\lambda,\omega_\lambda) \tag{1.9}$$

is called symplectic reduction [17]. Here R_λ is called the reduced space, and ω_λ the reduced form. We shall show (in Propositions 4.2 and 4.4) that R_λ is the flag manifold G/P up to connected components, and ω_λ is a K-invariant pseudo-Kähler form.

We consider the conditions for two reduced spaces to be isomorphic pseudo-Kähler manifolds. Suppose that for $i = 1,2$, λ_i is in the image of the right moment map of ω_i. Let $(\omega_i)_{\lambda_i}$ be the reduced form on R_{λ_i}. We introduce the notions of

$$\lambda_1 \sim \lambda_2, (\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}, (\omega_1)_{\lambda_1} \approx (\omega_2)_{\lambda_2} \tag{1.10}$$

as follows: Namely, $\lambda_1 \sim \lambda_2$ if they lie in the same coadjoint K-orbit. Also, $(\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}$ if there is a K-equivariant symplectomorphism between R_{λ_i}. In particular if this symplectomorphism can be made holomorphic, we write $(\omega_1)_{\lambda_1} \approx (\omega_2)_{\lambda_2}$.

THEOREM 3. *The right T_σ-action has a unique K-invariant moment map which agrees with Φ on A_σ. The reduced space is a disjoint union of copies of G/P; each copy with the same K-invariant pseudo-Kähler form. Conversely, every pseudo-Kähler form on G/P can be obtained via (1.9); and F can be chosen as strictly convex for $\omega = \sqrt{-1}\partial\bar\partial F$. Also, $(\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}$ if and only if $\lambda_1 \sim \lambda_2$; while $(\omega_1)_{\lambda_1} \approx (\omega_2)_{\lambda_2}$ if and only if $\lambda_1 = \lambda_2$.*

Theorem 3 will be proved in §4. Assume that the reduced space is connected. Then the theorem says that symplectic reduction (1.9) is independent of ω, and simplifies to

$$(G/(P,P),\lambda) \rightsquigarrow (G/P,\omega_\lambda). \tag{1.11}$$

Recall that $(\mathfrak{t}_\sigma^*)_{reg}$ are the σ-regular points, introduced in (1.2). We shall use Theorem 3 to classify the K-invariant pseudo-Kähler forms on G/P:

COROLLARY 3A. *The set of all K-invariant pseudo-Kähler forms on G/P is bijective to $(\mathfrak{t}_\sigma^*)_{reg}$.*

Consider the K-invariant pseudo-Kähler forms on G/P which are positive definite, namely the Kähler forms. They are commonly called the Fubini-Study forms. Our theorem recovers A. Borel's classical result:

COROLLARY 3B. [3] *The set of all Fubini-Study forms on G/P is bijective to σ.*

We now consider the signatures of the K-invariant pseudo-Kähler forms Ω on the flag manifold G/P. Let d be the dimension of G/P, and let J be the complex structure. We say that Ω has signature $(d-s, s)$ if $\Omega(\cdot, J\cdot)$ has exactly s mutually orthogonal negative eigenvectors. For instance, Kähler forms have signature $(d, 0)$. Observe that by Theorem 3, each Ω is necessarily given by $\Omega = \omega_\lambda$, where ω is a $K \times T_\sigma$-invariant pseudo-Kähler form on $G/(P, P)$. Therefore, we can use symplectic reduction to study the signature of Ω. This will be discussed in the next theorem.

Given a reduced form ω_λ on G/P, we perform geometric quantization to it, as in §3. Let $\mathbf{L}_\lambda$ be the pre-quantum line bundle corresponding to ω_λ. We then obtain the Dolbeault $(0, q)$-cohomology of G/P with coefficients in $\mathbf{L}_\lambda$, which are automatically square-integrable because G/P is compact. We denote them by $H^q_{(\omega_\lambda)}$. The K-action on G/P lifts to a unitary K-representation on $H^q_{(\omega_\lambda)}$. Suppose that in $\omega = \sqrt{-1}\partial\bar{\partial}F$, F is strictly convex (which is always possible, by Theorem 3). Recall that a point in $\mathfrak{t}^*$ is said to be regular if its pairing with every root is nonzero, and that ρ is half the sum of positive roots. The next theorem proves that geometric quantization commutes with reduction, and uses this principle to reveal the signature of ω_λ.

THEOREM 4. *$H^q_{(\omega_\lambda)} \cong (H^q_\omega)_\lambda$. They vanish for all q unless $\lambda \in U_F$ and $\lambda + \rho$ is regular. When this happens, they vanish for all q except when $(d-q, q)$ is the signature of ω_λ.*

This is proved in §5. An explicit unitary K-equivariant isomorphism for $H^q_{(\omega_\lambda)} \cong (H^q_\omega)_\lambda$ is given in (5.2). In the case where they do not vanish, they are irreducible with highest weight computed in Theorem 2.

For the special case where ω_λ or $-\omega_\lambda$ is Kähler, (i.e., ω_λ has signature $(d, 0)$ or $(0, d)$), Theorem 4 leads easily to the familiar vanishing theorems of K. Kodaira:

COROLLARY 4A. [13] *If ω_λ is Kähler, then $H^q_{(\omega_\lambda)}$ vanishes for $q \geq 1$. If $-\omega_\lambda$ is Kähler, then $H^q_{(\omega_\lambda)}$ vanishes for $q < \dim G/P$.*

We know from Theorem 3 that each connected component of the reduced space R_λ is a copy of the flag manifold G/P. It would be nice to know when R_λ is connected, so that R_λ is equivalent to G/P itself. Apart from obvious cases such as when ω or $-\omega$ is Kähler, satisfactory conditions for connectivity are still not known. In §6, we address this issue and formulate this problem in terms of basic calculus.

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2. Pseudo-Kähler structures. In this section, we apply results in [6] to classify $K\times T_\sigma$-invariant pseudo-Kähler forms on $G/(P,P)$ and prove Theorem 1. Let K^σ be the centralizer of T_σ in K, with semi-simple commutator subgroup $K^\sigma_{ss} = (K^\sigma, K^\sigma)$. Recall from (1.3) that $G/(P,P) = (K/K^\sigma_{ss})A_\sigma$. With this description, the right action of T_σ is clear because it commutes with K^σ_{ss}. Also, the right A_σ-action is simply group operation on the A_σ component.

PROPOSITION 2.1. *Every $K \times T_\sigma$-invariant closed (1,1)-form on $G/(P,P)$ is given by $\omega = \sqrt{-1}\partial\bar\partial F$, where $F \in C^\infty(\mathfrak{a}_\sigma)$. There exists a unique moment map Φ: $G/(P,P) \longrightarrow \mathfrak{k}^*$ for the K-action given by $\Phi(a) = \frac{1}{2}F'(a) \in \mathfrak{t}^*_\sigma$ for all $a \in A_\sigma \subset G/(P,P)$.*

Proof. Let ω be a closed $K \times T_\sigma$-invariant $(1,1)$-form on $G/(P,P)$. The arguments for Theorem 1 of [6] are valid even when ω is not Kähler, so we get $\omega = \sqrt{-1}\partial\bar\partial F$, where F is $K \times T_\sigma$-invariant.

Given $v \in \mathfrak{k}$, let v^l be the infinitesimal vector field on $G/(P,P)$ obtained from the K-action. Set $\beta = \frac{\sqrt{-1}}{2}(-\partial F + \bar\partial F)$. Then β is a $K \times T_\sigma$-invariant real 1-form satisfying $d\beta = \omega$. So a moment map is given by ([1], Theorem 4.2.10)

$$(\Phi(p), v) = -(\beta, v^l)_p \tag{2.1}$$

for all $p \in G/(P,P)$. By K-equivariance of Φ, it suffices to consider (2.1) for $p \in A_\sigma$. This is computed in ([6], §3) as $\frac{1}{2}F'$, with image in $\mathfrak{t}^*_\sigma$.

To check that the moment map is unique, suppose Ψ: $G/(P,P) \longrightarrow \mathfrak{k}^*$ is another K-moment map. Define ψ, ϕ: $\mathfrak{k} \longrightarrow C^\infty(G/(P,P))$ by $\psi^v(p) = (\Psi(p), v)$ and $\phi^v(p) = (\Phi(p), v)$ for all $v \in \mathfrak{k}$ and $p \in G/(P,P)$. Then

$$d\psi^v = \iota(v^l)\omega = d\phi^v,$$

so there exists a constant $c = c(v)$ such that $\psi^v = \phi^v + c(v)$. Since c depends on v linearly, $c \in \mathfrak{k}^*$. Note that

$$(\Psi(p), v) = (\Phi(p), v) + c(v).$$

Therefore, since Ψ and Φ are K-equivariant, c is fixed by the coadjoint Ad^*_k for all $k \in K$. Equivalently, c annihilates $[\mathfrak{k}, \mathfrak{k}]$. Since $\mathfrak{k}$ is semi-simple, $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$, and so $c = 0$. Hence $\Psi = \Phi$, and the proposition follows. □

Consider $\mathfrak{k} = \mathfrak{t} + V$, where V is orthocomplement to $\mathfrak{t}$ via the Killing form. In [6], we decompose the real vector space V into two dimensional subspaces V_i, indexed by the positive roots α_i. In fact if we identify $\mathfrak{k} \cong \mathfrak{k}^*$ by the Killing form, then

$$[V_i, V_i] \cong \mathbf{R}(\alpha_i). \tag{2.2}$$

If the positive root α_i satisfies $(\alpha_i, v) = 0$ for all $v \in \sigma$, we write $(\alpha_i, \sigma) = 0$. Otherwise we write $(\alpha_i, \sigma) > 0$. Equivalently, $(\alpha_i, \sigma) > 0$ exactly when $\alpha_i \in \overline{S}$, where $\overline{S}$ is the subset of positive roots given in (1.2). Let $\mathfrak{t}_\sigma^\perp \subset \mathfrak{t}$ be the complement of $\mathfrak{t}_\sigma$ in $\mathfrak{t}$. Observe that

$$\mathfrak{k} = \mathfrak{k}_{ss}^\sigma + \mathfrak{t}_\sigma + \left(\sum_{(\alpha_i,\sigma)>0} V_i \right), \mathfrak{k}_{ss}^\sigma = \mathfrak{t}_\sigma^\perp + \left(\sum_{(\alpha_i,\sigma)=0} V_i \right). \tag{2.3}$$

For $v \in \mathfrak{k}$, recall that v^l is the infinitesimal vector field on $G/(P,P)$ generated by the left K-action. Consider $a \in A_\sigma \subset G/(P,P)$. By (3.4) of [6], the tangent space of $G/(P,P)$ at a is given by

$$(\mathfrak{t}_\sigma^l)_a + (\mathfrak{a}_\sigma^l)_a + \sum_{(\alpha_i,\sigma)>0} (V_i^l)_a. \tag{2.4}$$

Since ω is K-invariant, to check if it is pseudo-Kähler or Kähler, it suffices to consider ω at the tangent space (2.4) for all $a \in A_\sigma$. Recall from [6] that

$$\omega(\mathfrak{t}^l + \mathfrak{a}^l, V^l)_a = 0.$$

So to check if ω is pseudo-Kähler or Kähler, we may consider ω at $(\mathfrak{t}_\sigma^l + \mathfrak{a}_\sigma^l)_a$ and $\sum_{(\alpha_i,\sigma)>0} (V_i^l)_a$ separately.

PROPOSITION 2.2. *Restrict ω to $(\mathfrak{t}_\sigma^l + \mathfrak{a}_\sigma^l)_a$. It is nondegenerate if and only if F is nonsingular, and it is positive definite if and only if F is strictly convex.*

Proof. Since F is K-invariant, by (1.4), $F \in C^\infty(\mathfrak{a}_\sigma)$. In [6], we see that the restriction of ω to $(\mathfrak{t}_\sigma^l + \mathfrak{a}_\sigma^l)_a$ is essentially given by the Hessian matrix $(\frac{\partial^2 F}{\partial x_i \partial x_j}(a))$. Therefore, it is nondegenerate exactly when F is nonsingular, and it is positive definite exactly when F is strictly convex. □

Recall that $(\mathfrak{t}_\sigma^*)_{reg}$ denotes the σ-regular points, defined in (1.2).

PROPOSITION 2.3. *Restrict ω to $\sum_{(\alpha_i,\sigma)>0} (V_i^l)_a$. It is nondegenerate if and only if $\Phi(a) \in (\mathfrak{t}_\sigma^*)_{reg}$, and it is positive definite if and only if $\Phi(a) \in \sigma$.*

Proof. For $i \neq j$, $[V_i, V_j] \subset V$. By Proposition 2.1 $\Phi(a) \in \mathfrak{t}_\sigma^*$, so

$$\omega(V_i^l, V_j^l)_a = (\Phi(a), [V_i, V_j]) \subset (\mathfrak{t}_\sigma^*, V) = 0.$$

So $\{(V_i^l)_a\}_{(\alpha_i,\sigma)>0}$ are mutually orthogonal with respect to ω, and we may evaluate ω on each of them separately.

Let $\zeta_i, \gamma_i \in V_i$ be the basis as in (2.6) of [6]. The complex structure of $G/(P,P)$ sends ζ_i^l to γ_i^l, and γ_i^l to $-\zeta_i^l$. Further, $[\zeta_i, \gamma_i] = \alpha_i$ under (2.2). Therefore,

$$\omega(\zeta_i^l, \gamma_i^l)_a = (\Phi(a), [\zeta_i, \gamma_i]) = (\Phi(a), \alpha_i). \tag{2.5}$$

Let i vary among $(\alpha_i, \sigma) > 0$. Then (2.5) is nonzero for all i if and only if $\Phi(a) \in (\mathfrak{t}_\sigma^*)_{reg}$. Also, (2.5) is positive for all i if and only if $\Phi(a) \in \sigma$. This completes the proof. □

Proof of Theorem 1. The first two statements of Theorem 1 follow from Proposition 2.1, while the rest follows from Propositions 2.2 and 2.3. □

3. Geometric quantization. In this section, we consider the problem of geometric quantization associated to a $K \times T_\sigma$-invariant pseudo-Kähler form ω on $G/(P, P)$. Namely, we construct unitary K-representations out of ω.

As mentioned in §1, ω leads to a pre-quantum line bundle $\mathbf{L}$ [14], which carries a connection and an invariant Hermitian structure. Since ω is exact, the Chern class of $\mathbf{L}$ is $[\omega] = 0$, so $\mathbf{L}$ is a trivial bundle. In [6], we show that the choice of a nonvanishing section of $\mathbf{L}$ can be made to have the following nice properties.

PROPOSITION 3.1. [6] *There exists a unique nonvanishing $K \times T_\sigma$-invariant holomorphic section s satisfying $\langle s, s\rangle_{ka} = e^{-F(a)}$ for all $ka \in (K/K_{ss}^\sigma)A_\sigma$.*

From now on, we shall always let s denote this section. Let $\mathbf{\Omega}^{0,q}(G/(P, P))$ and $\mathbf{\Omega}^{0,q}(\mathbf{L})$ respectively denote the Dolbeault $(0, q)$-forms on $G/(P, P)$ with coefficients in $\mathbf{C}$ and $\mathbf{L}$. Using the section s, we can express a typical element α of $\mathbf{\Omega}^{0,q}(\mathbf{L})$ as $\alpha_0 \otimes s$, where $\alpha_0 \in \mathbf{\Omega}^{0,q}(G/(P, P))$. We construct an Hermitian structure $\langle \cdot, \cdot\rangle^{\mathbf{L}}$ on $\mathbf{\Omega}^{0,q}(\mathbf{L})$ as follows.

Choose a $K \times H_\sigma$-invariant Hermitian structure on the anti-holomorphic bundle $\wedge^{0,1}(\mathfrak{g}/[\mathfrak{p}, \mathfrak{p}])$ over $G/(P, P)$. One such choice is available via the Killing form of $\mathfrak{g}$, by observing the fact that H_σ normalizes (P, P). It leads to a $K \times H_\sigma$-invariant Hermitian structure $\langle \cdot, \cdot\rangle$ on $\mathbf{\Omega}^{0,q}(G/(P, P))$. In other words if $\alpha_0, \beta_0 \in \mathbf{\Omega}^{0,q}(G/(P, P))$ and $p \in G/(P, P)$, then $\langle \alpha_0, \beta_0\rangle_p \in \mathbf{C}$. Also, under the left action L and right action R, $\langle \alpha_0, \beta_0\rangle_{kph} = \langle L_k^* R_h^* \alpha_0, L_k^* R_h^* \beta_0\rangle_p$ for all $k \in K$ and $h \in H_\sigma$. From the Hermitian structure $\langle \cdot, \cdot\rangle$ on $\mathbf{L}$ and the section s of Proposition 3.1, we obtain a Hermitian structure on $\mathbf{\Omega}^{0,q}(\mathbf{L})$ by

$$\langle \alpha_0 \otimes s, \beta_0 \otimes s\rangle^{\mathbf{L}} = \langle \alpha_0, \beta_0\rangle \langle s, s\rangle = \langle \alpha_0, \beta_0\rangle e^{-F} \tag{3.1}$$

for all $\alpha_0 \otimes s, \beta_0 \otimes s \in \mathbf{\Omega}^{0,q}(\mathbf{L})$.

Let μ be the $K \times A_\sigma$-invariant measure on $G/(P, P)$, which is unique up to positive constant. In fact by (1.3), μ is the product of the K-invariant measure dk on K/K_{ss}^σ (exists because K and K_{ss}^σ are unimodular [11], p. 89) and the Haar measure da on A_σ. We construct an L^2-structure on $\mathbf{\Omega}^{0,q}(\mathbf{L})$ via

$$\|\alpha\|^2 = \int_{p \in G/(P,P)} \langle \alpha, \alpha\rangle_p^{\mathbf{L}} \mu \tag{3.2}$$

for all $\alpha \in \Omega^{0,q}(\mathbf{L})$. We are interested in knowing when (3.2) converges. When it converges, we say that α is square-integrable.

The integral weight $\lambda \in \mathfrak{t}_\sigma^*$ extends by complex linearity to $\lambda \otimes \mathbf{C}\colon\ \mathfrak{h}_\sigma \longrightarrow \mathbf{C}$. The integral condition means that there is a multiplicative homomorphism $\chi\colon\ H_\sigma \longrightarrow \mathbf{C}^\times$ satisfying (1.6). The following lemma will be helpful in deciding convergence of (3.2). Recall from (1.5) that U_F is the image of $\frac{1}{2}F'$.

LEMMA 3.2. [7] *$\int_{a\in A_\sigma} \chi(a)^2 e^{-F(a)}\, da < \infty$ if and only if $\lambda \in U_F$.*

Proof. Consider the exponential map

$$\exp\colon\ \mathfrak{a}_\sigma \longrightarrow A_\sigma, \exp(v) = a.$$

Using these variables v and a, (1.6) says that $\chi(a)^2 = \exp(2\lambda, v)$. Let dV be the Lebesgue measure on $\mathfrak{a}_\sigma$. Recall that we identify $F(a)$ with $F(v)$ by (1.4). Thus

$$\int_{a\in A_\sigma} \chi(a)^2 e^{-F(a)}\, da = \int_{v\in\mathfrak{a}_\sigma} \exp((2\lambda, v) - F(v))\, dV. \tag{3.3}$$

In [7] Appendix, we show that RHS of (3.3) converges if and only if $\lambda \in U_F$. This proves the lemma. □

PROPOSITION 3.3. *Let $0 \neq \alpha = \alpha_0 \otimes s \in \Omega^{0,q}(\mathbf{L})$, and suppose that $R_a^*\alpha_0 = \chi(a)\alpha_0$ for all $a \in A_\sigma$. Then α is square-integrable if and only if $\lambda \in U_F$.*

Proof. Define $\phi\colon\ \Omega^{0,q}(\mathbf{L}) \longrightarrow C^\infty(A_\sigma)$ by

$$(\phi(\alpha))_a = \int_{k\in K/K_{ss}^\sigma} \langle \alpha_0, \alpha_0\rangle_{ka}\, dk \tag{3.4}$$

for all $\alpha = \alpha_0 \otimes s \in \Omega^{0,q}(\mathbf{L})$ and $a \in A_\sigma$. Suppose that $\alpha \neq 0$, and $R_a^*\alpha_0 = \chi(a)\alpha_0$ for all $a \in A_\sigma$. We claim that there exists a constant $c > 0$ satisfying

$$(\phi(\alpha))_a = c\chi(a)^2. \tag{3.5}$$

Observe that χ maps A_σ into $\mathbf{R}^+$. The Hermitian structure on $\Omega^{0,q}(G/(P,P))$ is right A_σ-invariant, so

$$\begin{aligned} R_a^*\langle \alpha_0, \alpha_0\rangle &= \langle R_a^*\alpha_0, R_a^*\alpha_0\rangle \\ &= \langle \chi(a)\alpha_0, \chi(a)\alpha_0\rangle \\ &= \chi(a)^2\langle \alpha_0, \alpha_0\rangle. \end{aligned} \tag{3.6}$$

Since $G/(P,P) = (K/K_{ss}^\sigma)A_\sigma$, we extend $\chi\colon\ A_\sigma \longrightarrow \mathbf{R}^+$ to a K-invariant function

$$E_\chi\colon\ G/(P,P) \longrightarrow \mathbf{R}^+. \tag{3.7}$$

Consider the positive function $0 < f = \langle\alpha_0, \alpha_0\rangle E_\chi^{-2} \in C^\infty(G/(P,P))$. By (3.6), f is is right A_σ-invariant. Namely, $f(ka) = f(k)$ for all $k \in K/K_{ss}^\sigma$ and $a \in A_\sigma$. So (3.4) becomes

$$\begin{aligned}(\phi(\alpha))_a &= \int_{k\in K/K_{ss}^\sigma} (\langle\alpha_0, \alpha_0\rangle E_\chi^{-2})_{ka} E_\chi^2(ka)\, dk \\ &= \int_{k\in K/K_{ss}^\sigma} f(k)\, dk \chi(a)^2.\end{aligned}$$

The integral in the last expression converges because K/K_{ss}^σ is compact, and its value is positive because $f > 0$. This proves (3.5).

We now consider the square-integrability of α:

$$\begin{aligned}\|\alpha\|^2 &= \int_{p\in G/(P,P)} \langle\alpha, \alpha\rangle_p^{\mathbf{L}} \mu && \text{by (3.2)} \\ &= \int_{ka\in(K/K_{ss}^\sigma)A_\sigma} \langle\alpha_0, \alpha_0\rangle_{ka} e^{-F(a)}\, dk\, da && \text{by (3.1)} \\ &= \int_{a\in A_\sigma} (\phi(\alpha))_a e^{-F(a)}\, da && \text{by (3.4)} \\ &= c\int_{a\in A_\sigma} \chi(a)^2 e^{-F(a)}\, da. && \text{by (3.5).}\end{aligned}$$

By Lemma 3.2, the final expression converges if and only if $\lambda \in U_F$. This proves the proposition. ☐

Let $\bar\partial_{\mathbf{L}}$: $\mathbf{\Omega}^{0,q}(\mathbf{L}) \longrightarrow \mathbf{\Omega}^{0,q+1}(\mathbf{L})$ be the Dolbeault operator. Using the holomorphic section s of Proposition 3.1, we get

$$\bar\partial_{\mathbf{L}}(\alpha_0 \otimes s) = (\bar\partial\alpha_0) \otimes s, \tag{3.8}$$

where $\alpha_0 \in \mathbf{\Omega}^{0,q}(G/(P,P))$. Let $\lambda \in \mathfrak{t}_\sigma^*$ be an integral weight. Define

$$\mathbf{\Omega}_{2,\lambda}^{0,q}(\mathbf{L}) = \{\alpha \in \mathbf{\Omega}^{0,q}(\mathbf{L});\, \|\alpha\| < \infty, \|\bar\partial_{\mathbf{L}}\alpha\| < \infty, R_t^*\alpha = \chi(t)\alpha \text{ for all } t \in T_\sigma\}.$$

We get the subcomplex

$$\ldots \longrightarrow \mathbf{\Omega}_{2,\lambda}^{0,q-1}(\mathbf{L}) \longrightarrow \mathbf{\Omega}_{2,\lambda}^{0,q}(\mathbf{L}) \longrightarrow \mathbf{\Omega}_{2,\lambda}^{0,q+1}(\mathbf{L}) \longrightarrow \ldots .$$

Its resulting cohomology is denoted by $(H_\omega^q)_\lambda$.

We also let $\mathbf{\Omega}_\lambda^{0,q}(G/(P,P))$ denote the elements of $\mathbf{\Omega}^{0,q}(G/(P,P))$ which transform by $\lambda \in \mathfrak{t}_\sigma^*$ under the right T_σ-action. Its corresponding cohomology is denoted by $H_\lambda^{0,q}(G/(P,P))$. Since s is right T_σ-invariant and holomorphic, from the natural map $\mathbf{\Omega}^{0,q}_{2,\lambda}(\mathbf{L}) \longrightarrow \mathbf{\Omega}^{0,q}_{\lambda}(G/(P,P))$ where $\alpha_0 \otimes s \mapsto \alpha_0$, we obtain

$$\kappa\colon\ (H^q_\omega)_\lambda \longrightarrow H^{0,q}_\lambda(G/(P,P)).$$

PROPOSITION 3.4. *The map κ intertwines with the K-action. If $\lambda \in U_F$, then κ is an isomorphism. If $\lambda \notin U_F$, then $(H^q_\omega)_\lambda = 0$.*

Proof. By K-invariance of the section s, it follows that κ intertwines with the K-action.

We first assume that $\lambda \in U_F$. For injection, suppose that $\kappa[\alpha_0 \otimes s] = [\alpha_0] = 0 \in H^{0,q}_\lambda(G/(P,P))$ for some $(0,q)$-form α_0. Since $\bar\partial\alpha_0 = 0$ and α_0 transforms by $\lambda \in \mathfrak{t}^*_\sigma$ under the right T_σ-action, it also transforms by the complexified weight $\lambda \otimes \mathbf{C} \in \mathfrak{h}^*_\sigma$ under the right H_σ-action. By ([5], Theorem 2(i)), $H^{0,q}_\lambda(G/(P,P)) \cong H^{0,q}_{\lambda\otimes\mathbf{C}}(G/(P,P))$. So since $[\alpha_0] = 0 \in H^{0,q}_{\lambda\otimes\mathbf{C}}(G/(P,P))$, there exists β_0 satisfying $\bar\partial\beta_0 = \alpha_0$ and $R^*_h\beta_0 = \chi(h)\beta_0$ for all $h \in H_\sigma$. By Proposition 3.3, $\beta = \beta_0 \otimes s$ is square-integrable because $\lambda \in U_F$. It follows that $\beta \in \mathbf{\Omega}^{0,q-1}_{2,\lambda}(\mathbf{L})$. By (3.8), $\bar\partial_{\mathbf{L}}\beta = \alpha_0 \otimes s$. So $[\alpha_0 \otimes s] = 0$. This proves that κ is injective.

For surjection, pick $[\alpha_0] \in H^{0,q}_\lambda(G/(P,P))$. Since $\bar\partial\alpha_0 = 0$ and $R^*_t\alpha_0 = \chi(t)\alpha_0$ for all $t \in T_\sigma$, it follows that $R^*_a\alpha_0 = \chi(a)\alpha_0$ for all $a \in A_\sigma$. By Proposition 3.3, $\alpha = \alpha_0 \otimes s$ is square-integrable because $\lambda \in U_F$. Hence $[\alpha] \in (H^q_\omega)_\lambda$ and $\kappa[\alpha] = [\alpha_0]$. This shows that κ is surjective. We have proved the proposition for the case $\lambda \in U_F$.

Next assume that $\lambda \notin U_F$. Pick a $\bar\partial_{\mathbf{L}}$-closed element $\alpha = \alpha_0 \otimes s$ of $\mathbf{\Omega}^{0,q}_{2,\lambda}(\mathbf{L})$. By (3.8) we get $\bar\partial\alpha_0 = 0$; and $R^*_t\alpha_0 = \chi(t)\alpha_0$ because s is right T_σ-invariant. Therefore, α_0 transforms by the complexified weight $\lambda \otimes \mathbf{C} \in \mathfrak{h}^*_\sigma$ under the right H_σ-action. By Proposition 3.3, $\alpha = 0$ because $\lambda \notin U_F$. We conclude that the only $\bar\partial_{\mathbf{L}}$-closed element of $\mathbf{\Omega}^{0,q}_{2,\lambda}(\mathbf{L})$ is 0. So $(H^q_\omega)_\lambda = 0$ whenever $\lambda \notin U_F$. This proves the proposition. ☐

Proposition 3.4 leads to most of Theorem 2, except for the unitary structure of the K-representation $(H^q_\omega)_\lambda$. The rest of this section aims to select a canonical representative for each cohomology class in $(H^q_\omega)_\lambda$. This way, the L^2-norms of these representatives make $(H^q_\omega)_\lambda$ a unitary K-representation.

The integral weight $\lambda \otimes \mathbf{C} \in \mathfrak{h}^*_\sigma$ leads to a homogeneous line bundle

$$G \times_{\lambda\otimes\mathbf{C}} \mathbf{C} \longrightarrow G/P \tag{3.9}$$

over the flag manifold G/P, denoted by $\mathbf{L}_\lambda$. The natural fibration $\pi\colon\ G/(P,P) \longrightarrow G/P$ leads to an injection

$$\pi^*\colon\ \mathbf{\Omega}^{0,q}(G/P, \mathbf{L}_\lambda) \longrightarrow \mathbf{\Omega}^{0,q}_\lambda(G/(P,P)), \tag{3.10}$$

whose image is denoted by I^q_λ. In [5], we study the Dolbeault complex

$\Omega^{0,q}_\lambda(G/(P,P))$ as the product of two subcomplexes,

$$\Omega^{0,q}_\lambda(G/(P,P)) = \oplus_{r+s=q} I^r_\lambda \otimes (C^\infty(A_\sigma) \otimes \wedge^{0,s}\mathfrak{h}^*_\sigma). \tag{3.11}$$

The next lemma shows that the various components of the bigrading (r,s) are everywhere pairwise orthogonal under the $K \times H_\sigma$-invariant Hermitian structure.

LEMMA 3.5. *If* $(r,s) \neq (t,u)$, *then*

$$\langle I^r_\lambda \otimes C^\infty(A_\sigma) \otimes \wedge^{0,s}\mathfrak{h}^*_\sigma, I^t_\lambda \otimes C^\infty(A_\sigma) \otimes \wedge^{0,u}\mathfrak{h}^*_\sigma\rangle = 0.$$

Proof. Our Hermitian structure is obtained from one on the anti-holomorphic bundle. So it suffices to show that

$$\langle C^\infty(A_\sigma) \otimes \wedge^{0,1}\mathfrak{h}^*_\sigma, I^1_\lambda\rangle = 0. \tag{3.12}$$

Suppose otherwise; namely, there exist $\alpha \in C^\infty(A_\sigma) \otimes \wedge^{0,1}\mathfrak{h}^*_\sigma$ and $\beta \in I^1_\lambda$ such that $0 \neq \langle\alpha,\beta\rangle \in C^\infty(G/(P,P))$. By the $K \times H_\sigma$-action, we may assume that

$$\langle\alpha,\beta\rangle_e \neq 0, \tag{3.13}$$

where $e \in G/(P,P)$ is the identity coset. Recall the following results from (2.3): The complex vector space $\mathfrak{g}/[\mathfrak{p},\mathfrak{p}]$ decomposes into complex subspaces $\{\mathfrak{h}_\sigma, V_i\}_i$, where i is indexed over the simple roots α_i satisfying $(\alpha_i,\sigma) > 0$. If we imbed $V^* \subset (\mathfrak{g}/[\mathfrak{p},\mathfrak{p}])^*$ by the Killing form, then V^* annihilates $\mathfrak{h}_\sigma$. Since β is in the image of π^*, β_e annihilates $\wedge^{0,1}\mathfrak{h}_\sigma$. Therefore, $\beta_e \in \wedge^{0,1}V^*$.

If we further identify $\wedge^{0,1}(\mathfrak{g}/[\mathfrak{p},\mathfrak{p}])^*$ with $K \times A_\sigma$-invariant differential forms on $G/(P,P)$, then $C^\infty(A_\sigma)\otimes\wedge^{0,1}\mathfrak{h}^*_\sigma$ are right T_σ-invariant, while $\wedge^{0,1}V^*_i$ transform by α_i under the right T_σ-action. In other words, if $\chi_i\colon T_\sigma \longrightarrow S^1$ are the characters associated to the simple roots α_i, then

$$R^*_t u = u, R^*_t v_i = \chi_i(t)v_i; u \in C^\infty(A_\sigma) \otimes \wedge^{0,1}\mathfrak{h}^*_\sigma, v_i \in \wedge^{0,1}V^*_i. \tag{3.14}$$

Replace β with a $K \times A_\sigma$-invariant (0,1)-form $\gamma \in \wedge^{0,1}V^* \subset \Omega^{0,1}(G/(P,P))$ satisfying $\gamma_e = \beta_e$. This way, $\langle\alpha,\gamma\rangle_e \neq 0$ by (3.13). Write $\gamma = \sum_i c_i\gamma_i$, with $\gamma_i \in \wedge^{0,1}V^*_i$. Then $\langle\alpha, c_i\gamma_i\rangle_e \neq 0$ for some i. For such i, pick $t \in T_\sigma$ such that $\chi_i(t) \neq 1$. Then

$$\begin{aligned} 0 \neq \langle\alpha, c_i\gamma_i\rangle_e &= L^*_t R^*_t \langle\alpha, c_i\gamma_i\rangle_e \\ &= \langle R^*_t\alpha, c_i R^*_t\gamma_i\rangle_e \quad \text{by left invariance} \\ &= \langle\alpha, \chi_i(t)c_i\gamma_i\rangle_e. \quad \text{by (3.14).} \end{aligned}$$

This is a contradiction. We have proved (3.12), and the lemma follows. □

This lemma allows us to look at each component of the bigrading (r, s) of (3.11) separately. It turns out that the component $(r, s) = (q, 0)$ is the most significant, as it is related to some data from G/P. To obtain useful information from G/P, we now construct an Hermitian structure on $\mathbf{\Omega}^{0,q}(G/P, \mathbf{L}_\lambda)$.

Recall from (3.10) that $\mathbf{\Omega}^{0,q}(G/P, \mathbf{L}_\lambda)$ injects into $\mathbf{\Omega}^{0,q}_\lambda(G/(P, P))$ via π^*. Also, recall from (3.7) that E_χ is a K-invariant function on $G/(P, P)$. The Hermitian structure on $\mathbf{\Omega}^{0,q}_\lambda(G/(P, P))$ leads to one on $\mathbf{\Omega}^{0,q}(G/P, \mathbf{L}_\lambda)$, still denoted by $\langle \cdot, \cdot \rangle$, by

$$\langle \alpha, \beta \rangle_{\pi(p)} = \langle \pi^*\alpha, \pi^*\beta \rangle_p E_\chi^{-2}(p) \tag{3.15}$$

for all $\pi(p) = G/P$ and $\alpha, \beta \in \mathbf{\Omega}^{0,q}(G/P, \mathbf{L}_\lambda)$. The next lemma shows that this is well defined.

LEMMA 3.6. *The value of (3.15) is independent of the choice of p in the fiber of π, so the Hermitian structure on $\mathbf{\Omega}^{0,q}(G/P, \mathbf{L}_\lambda)$ is well defined.*

Proof. The fiber of π is $H_\sigma/(H_\sigma \cap (P, P))$. We need to show that the value of (3.15) stays the same if we replace p by ph, $h \in H_\sigma$. Equivalently, we need to show that the function $\langle \pi^*\alpha, \pi^*\beta \rangle E_\chi^{-2} \in C^\infty(G/(P, P))$ is invariant under the right H_σ-action.

Write $h = ta \in T_\sigma A_\sigma$, and consider the right actions R_t, R_a seperately. For R_t, note that E_χ^{-2} is automatically right T_σ-invariant because it is K-invariant. So we only need to consider $\langle \pi^*\alpha, \pi^*\beta \rangle$. In this case,

$$\begin{aligned} R_t^* \langle \pi^*\alpha, \pi^*\beta \rangle &= \langle R_t^*\pi^*\alpha, R_t^*\pi^*\beta \rangle \\ &= \langle \chi(t)\pi^*\alpha, \chi(t)\pi^*\beta \rangle \\ &= \chi(t)\overline{\chi(t)} \langle \pi^*\alpha, \pi^*\beta \rangle \\ &= \langle \pi^*\alpha, \pi^*\beta \rangle, \end{aligned} \tag{3.16}$$

as $\chi(t) \in S^1$. So $\langle \pi^*\alpha, \pi^*\beta \rangle E_\chi^{-2}$ is right T_σ-invariant.

For R_a, observe that $\chi(a) \in \mathbf{R}^+$. So by repeating the argument of (3.16), we get $R_a^*\langle \pi^*\alpha, \pi^*\beta \rangle = \chi(a)^2 \langle \pi^*\alpha, \pi^*\beta \rangle$. Together with the factor E_χ^{-2}, we see that $\langle \pi^*\alpha, \pi^*\beta \rangle E_\chi^{-2}$ is right A_σ-invariant. This proves the lemma. □

From (3.15), it follows that

$$\pi^*\langle \alpha, \beta \rangle = \langle \pi^*\alpha, \pi^*\beta \rangle E_\chi^{-2} \in C^\infty(G/(P, P)) \tag{3.17}$$

for all $\alpha, \beta \in \mathbf{\Omega}^{0,q}(G/P, \mathbf{L}_\lambda)$.

It is well known that the flag manifold $K/K^\sigma = G/P$ has K-invariant symplectic forms; a fact which will also be discussed in the next section. Therefore, the flag manifold has K-invariant volume form dk_0. The fibration π restricts to $K/K^\sigma_{ss} \longrightarrow K/K^\sigma$, with compact fiber. So by normalizing dk_0 suitably ([11], p. 95),

$$\int_{k\in K/K^\sigma_{ss}} (\pi^*\varphi)(k)\,dk = \int_{\pi(k)\in K/K^\sigma} \varphi(\pi(k))\,dk_0 \tag{3.18}$$

for all $\varphi \in C^\infty(K/K^\sigma)$. This identity will be useful later.

By the previous lemma, both $\Omega^{0,q}_\lambda(\mathbf{L})$ and $\Omega^{0,q}(G/P, \mathbf{L}_\lambda)$ have Hermitian structures. We respectively let $\bar\partial^*_{\mathbf{L}}$ and $\bar\partial^*$ denote the formal adjoints of $\bar\partial_{\mathbf{L}}$ and $\bar\partial$ relative to the corresponding L^2-structures. A square-integrable differential form annihilated by $\bar\partial_{\mathbf{L}}$ and $\bar\partial^*_{\mathbf{L}}$ (or $\bar\partial$ and $\bar\partial^*$) is called harmonic. Recall from §1 that $\alpha \in \Omega^{0,q}(\mathbf{L})$ is said to annihilate $\mathfrak{h}_\sigma$ if $\iota(v)\alpha = 0$ for all $v \in \wedge^{0,1}\mathfrak{h}_\sigma$.

PROPOSITION 3.7. *Each cohomology class in $(H^q_\omega)_\lambda$ has a unique harmonic representative which annihilates $\mathfrak{h}_\sigma$.*

Proof. We may assume that $\lambda \in U_F$, for otherwise the arguments of Proposition 3.4 show that 0 is the only $\bar\partial_{\mathbf{L}}$-closed element of $\Omega^{0,q}_{2,\lambda}(\mathbf{L})$, and there is nothing to prove.

Let $\beta \in \Omega^{0,q}(G/P, \mathbf{L}_\lambda)$. Since $\lambda \in U_F$, Proposition 3.3 says that $\pi^*\beta \otimes s$ is square-integrable. We claim that

$$\bar\partial^*_{\mathbf{L}}(\pi^*\beta \otimes s) = \pi^*\bar\partial^*\beta \otimes s. \tag{3.19}$$

In view of Lemma 3.5, to prove (3.19), it suffices to show that

$$\begin{aligned}\int_{p\in G/(P,P)} &\langle(\pi^*\alpha)f \otimes s, \bar\partial^*_{\mathbf{L}}(\pi^*\beta\otimes s)\rangle^{\mathbf{L}}_p \mu \\ &= \int_{p\in G/(P,P)} \langle(\pi^*\alpha)f \otimes s, \pi^*\bar\partial^*\beta\otimes s\rangle^{\mathbf{L}}_p \mu\end{aligned} \tag{3.20}$$

for all square-integrable $(\pi^*\alpha)f \otimes s \in I^{q-1}_\lambda \otimes C^\infty(A_\sigma)\otimes \mathbf{L} \subset \Omega^{0,q-1}(\mathbf{L})$. The first integral of (3.20) is

$$\begin{aligned}&\int_{p\in G/(P,P)} \langle(\pi^*\alpha)f \otimes s, \bar\partial^*_{\mathbf{L}}(\pi^*\beta\otimes s)\rangle^{\mathbf{L}}_p \mu \\ &= \int_{p\in G/(P,P)} \langle\bar\partial((\pi^*\alpha)f), \pi^*\beta\rangle_p e^{-F(p)}\mu && \text{by (3.1)} \\ &= \int_{p\in G/(P,P)} \langle(\bar\partial\pi^*\alpha)f + (-1)^{q-1}\pi^*\alpha\wedge\bar\partial f, \pi^*\beta\rangle_p e^{-F(p)}\mu \\ &= \int_{p\in G/(P,P)} f\langle\pi^*\bar\partial\alpha, \pi^*\beta\rangle_p e^{-F(p)}\mu && \text{by Lemma 3.5} \\ &= \int_{k\in K/K^\sigma_{ss}} \pi^*\langle\bar\partial\alpha,\beta\rangle_k\,dk \int_{a\in A_\sigma} f(a)\chi(a)^2 e^{-F(a)}\,da && \text{by (3.17)}\end{aligned} \tag{3.21}$$

$$= \int_{\pi(k)\in K/K^\sigma} \langle \bar{\partial}\alpha, \beta\rangle_{\pi(k)}\, dk_0 \int_{a\in A_\sigma} f(a)\chi(a)^2 e^{-F(a)}\, da. \qquad \text{by (3.18).}$$

The second integral of (3.20) is

$$\begin{aligned}
(3.22)\quad & \int_{p\in G/(P,P)} \langle (\pi^*\alpha) f \otimes s, \pi^*\bar{\partial}^*\beta \otimes s\rangle_p^{\mathbf{L}} \mu \\
&= \int_{p\in G/(P,P)} f\langle \pi^*\alpha, \pi^*\bar{\partial}^*\beta\rangle_p e^{-F(p)}\mu && \text{by (3.1)} \\
&= \int_{k\in K/K_{ss}^\sigma} \pi^*\langle \alpha, \bar{\partial}^*\beta\rangle_k\, dk \int_{a\in A_\sigma} f(a)\chi(a)^2 e^{-F(a)}\, da && \text{by (3.17)} \\
&= \int_{\pi(k)\in K/K^\sigma} \langle \alpha, \bar{\partial}^*\beta\rangle_{\pi(k)}\, dk_0 \int_{a\in A_\sigma} f(a)\chi(a)^2 e^{-F(a)}\, da && \text{by (3.18).} \\
&= \int_{\pi(k)\in K/K^\sigma} \langle \bar{\partial}\alpha, \beta\rangle_{\pi(k)}\, dk_0 \int_{a\in A_\sigma} f(a)\chi(a)^2 e^{-F(a)}\, da.
\end{aligned}$$

So (3.21) and (3.22) imply (3.20), and hence (3.19).

Since $\lambda \in U_F$, Proposition 3.3 says that we have the mapping

$$(3.23)\qquad \theta\colon\ \Omega^{0,q}(G/P, \mathbf{L}_\lambda) \longrightarrow \Omega^{0,q}_{2,\lambda}(\mathbf{L});\ \theta(\alpha) = (\pi^*\alpha)\otimes s.$$

Since G/P is compact, the standard Hodge theory says that every cohomology class in $H^{0,q}(G/P, \mathbf{L}_\lambda)$ has a unique harmonic representative. Clearly θ is injective and $\bar{\partial}_{\mathbf{L}}\theta = \theta\bar{\partial}$. Further, (3.19) says that $\bar{\partial}^*_{\mathbf{L}}\theta = \theta\bar{\partial}^*$. So the image $I^q_\lambda \otimes s$ of θ also has the property that every cohomology class in $H^*(I^q_\lambda \otimes s, \bar{\partial})$ has a unique harmonic representative. We apply the Kunneth theorem to (3.11) and observe that the subcomplex $C^\infty(A_\sigma)\otimes \wedge^{0,q}\mathfrak{h}^*_\sigma$ has trivial cohomology. Hence $(H^q_\omega)_\lambda \cong H^*(I^q_\lambda \otimes s, \bar{\partial})$. The elements of $I^q_\lambda \otimes s$ are exactly those which annihilate $\mathfrak{h}_\sigma$. We conclude that every cohomology class in $(H^q_\omega)_\lambda$ has a unique harmonic representative which annihilates $\mathfrak{h}_\sigma$. The proposition follows. □

Proof of Theorem 2. Suppose that conditions (i) and (ii) in (1.7) are valid. Then condition (i) and Proposition 3.4 say that $(H^q_\omega)_\lambda \cong H^{0,q}_\lambda(G/(P,P))$. By ([5] Theorem 2(ii)), condition (ii) says that $H^{0,q}_\lambda(G/(P,P))$ is an irreducible K-representation with highest weight $\tau(\lambda+\rho)-\rho$.

Conversely, suppose that either condition in (1.7) fails. If (i) fails, then Proposition 3.4 says that $(H^q_\omega)_\lambda = 0$. If (i) is valid but (ii) fails, then

$$\begin{aligned}
(H^q_\omega)_\lambda &\cong H^{0,q}_\lambda(G/(P,P)) && \text{by (i) and Proposition 3.4} \\
&= 0. && \text{by failure of (ii) and [5] Theorem 2(ii).}
\end{aligned}$$

We conclude that $(H^q_\omega)_\lambda$ vanishes if either condition of (1.7) fails.

By Proposition 3.7, each cohomology class $\xi \in (H^q_\omega)_\lambda$ is given by $\xi = [\alpha]$, where α is the unique harmonic representative which annihilates $\mathfrak{h}_\sigma$. These representatives form a unitary K-representation under the L^2-structure (3.2). By setting $\|\xi\| = \|\alpha\|$, it makes $(H^q_\omega)_\lambda$ a unitary K-representation. Theorem 2 follows. □

4. Symplectic reduction. Let $\omega = \sqrt{-1}\partial\bar{\partial}F$ be a $K \times T_\sigma$-invariant pseudo-Kähler form on $G/(P,P)$. In this section, we show that the right T_σ-action is Hamiltonian, and perform symplectic reduction to it. Recall that $\Phi\colon G/(P,P) \longrightarrow \mathfrak{k}^*$ is the K-moment map of ω.

PROPOSITION 4.1. *The right T_σ-action has a K-invariant moment map $\Phi_r\colon G/(P,P) \longrightarrow \mathfrak{t}_\sigma^*$, which is unique up to addition by $\mathfrak{t}_\sigma^*$. If we further require that Φ_r agrees with Φ on A_σ, then Φ_r is unique and is given by $\Phi_r(ka) = \frac{1}{2}F'(a) \in (\mathfrak{t}_\sigma^*)_{reg}$ for all $ka \in (K/K^\sigma_{ss})A_\sigma = G/(P,P)$.*

Proof. Given $v \in \mathfrak{t}_\sigma$, let v^l and v^r respectively denote the infinitesimal vector fields generated by the left and right T_σ-actions. The real 1-form $\beta = \frac{\sqrt{-1}}{2}(-\partial F + \bar{\partial}F)$ is $K \times T_\sigma$-invariant and satisfies $d\beta = \omega$. By ([1], Theorem 4.2.10), a moment map is given by

$$(\Phi_r(p), v) = -(\beta, v^r)_p \tag{4.1}$$

for all $p \in G/(P,P)$ and $v \in \mathfrak{t}_\sigma$. Both β and v^r are K-invariant, so Φ_r is K-invariant. This shows the existence of a K-invariant moment map.

Since T_σ is abelian, any $c \in \mathfrak{t}_\sigma^*$ defines another K-invariant moment map Ψ_r by

$$(\Psi_r(p), v) = (\Phi_r(p), v) + c(v). \tag{4.2}$$

To prove the first statement of this proposition, it remains to show that conversely, any K-invariant right moment map Ψ_r has the form of (4.2). Now let Ψ_r be a K-invariant right moment map. Define $\phi, \psi\colon \mathfrak{t}_\sigma \longrightarrow C^\infty(G/(P,P))$ by $\phi^v(p) = (\Phi_r(p), v)$ and $\psi^v(p) = (\Psi_r(p), v)$, for all $p \in G/(P,P)$ and $v \in \mathfrak{t}_\sigma$. Then $d\phi^v = \iota(v^r)\omega = d\psi^v$. Therefore, each v determines a constant $c = c(v)$ in which $\psi^v - \phi^v = c(v)$. The constant c varies linearly with v, so in fact $c \in \mathfrak{t}_\sigma^*$. This shows that Ψ_r has the form (4.2), which proves the first statement of this proposition.

By (4.1) and (4.2), an arbitrary K-invariant right moment map is given by

$$(\Psi_r(ka), v) = -(\beta, v^r)_a + c(v). \tag{4.3}$$

Since $T_\sigma A_\sigma$ is abelian, $v^l_a = v^r_a$ for all $a \in A_\sigma$. Therefore, (4.3) becomes

$$\begin{aligned}(\Psi_r(ka), v) &= -(\beta, v^l)_a + c(v)\\ &= (\Phi(a), v) + c(v),\end{aligned}$$

by (2.1). So Ψ_r and Φ coincides on A_σ exactly when $c = 0$. In this case, Theorem 1 says that $\Phi_r(ka) = \frac{1}{2}F'(a) \in (\mathfrak{t}_\sigma^*)_{reg}$. This proves the proposition. □

From now on, Φ_r denotes the canonical K-invariant right moment map of Proposition 4.1. Let $\lambda \in (\mathfrak{t}_\sigma^*)_{reg}$ be in the image of Φ_r. We consider the reduced space $R_\lambda = \Phi_r^{-1}(\lambda)/T_\sigma$.

PROPOSITION 4.2. *Each connected component of R_λ is a copy of the flag manifold G/P.*

Proof. Since ω is pseudo-Kähler, Theorem 1 says that F is nonsingular. By the inverse function theorem, F' is a local diffeomorphism. So there exists a discrete set $\Gamma \subset A_\sigma$ such that $(\frac{1}{2}F')^{-1}(\lambda) = \Gamma$. By Proposition 4.1, $\Phi_r^{-1}(\lambda) = (K/K_{ss}^\sigma)\Gamma \subset (K/K_{ss}^\sigma)A_\sigma$. Consequently,

$$\Phi_r^{-1}(\lambda)/T_\sigma = (K/(K_{ss}^\sigma T_\sigma))\Gamma = (K/K^\sigma)\Gamma = (G/P)\Gamma, \tag{4.4}$$

and a typical connected component is $(G/P)a$, $a \in \Gamma$. □

Consider the inclusion $\imath$ and the fibration π, given in (1.8). The reduced form ω_λ is defined to be the unique symplectic form on R_λ such that $\pi^*\omega_\lambda = \imath^*\omega$. Since $\imath$ and π commute with the K-action, it is clear that ω_λ is K-invariant. Since K is semi-simple, the Whitehead lemma ([10], §52) says that its Lie algebra cohomology satisfies $H^1(\mathfrak{k}) = H^2(\mathfrak{k}) = 0$. Consequently ([10], Theorem 26.1), the K-action preserving ω_λ has a unique moment map

$$\psi\colon\ R_\lambda \longrightarrow \mathfrak{k}^*.$$

By (4.4), write a typical element of R_λ as ka, where $k \in K/K^\sigma$ and $a \in \Gamma$. If k is the identity coset, we write $a = ka$ for simplicity.

PROPOSITION 4.3. $\psi(a) = \lambda \in (\mathfrak{t}_\sigma^*)_{reg}$.

Proof. Pick $x \in \mathfrak{k}$. By abuse of notation, let x^l be the infinitesimal vector field for the K-action on $G/(P,P), \Phi_r^{-1}(\lambda)$ or R_λ, depending on the context. Also, let a denote the appropriate element in any of these three spaces. Since $\imath$ and π commute with the K-action,

$$\imath(a) = a, \pi(a) = a, \imath_*(x_a^l) = x_a^l, \pi_*(x_a^l) = x_a^l.$$

Since $\mathfrak{k}$ is semi-simple, up to linear combination, $x = [u, v]$. Then

$$\begin{aligned}(\psi(a), x) &= (\psi(a), [u,v]) = \omega_\lambda(u^l, v^l)_a = \pi^*\omega_\lambda(u^l, v^l)_a = \imath^*\omega(u^l, v^l)_a \\ &= \omega(u^l, v^l)_a = (\Phi(a), [u,v]) = (\lambda, [u,v]) = (\lambda, x).\end{aligned} \tag{4.5}$$

So $\psi(a) = \lambda$, and the proposition follows. □

The standard complex structure of G/P is given by the complex Lie group G under the quotient of its complex subgroup P. Since R_λ consists of copies of G/P, it becomes a complex manifold under the standard complex structure of G/P. This allows us to consider whether ω_λ is pseudo-Kähler or Kähler.

PROPOSITION 4.4. *The reduced form ω_λ is a K-invariant pseudo-Kähler form on R_λ. In particular it is Kähler if and only if $\lambda \in \sigma$.*

Proof. The K-invariance of ω_λ follows from the above discussions. So it remains to check its pseudo-Kähler and Kähler properties.

Recall $\zeta_i, \gamma_i \in V_i \subset V$ from Proposition 2.3, where $[\zeta_i, \gamma_i] \in \mathfrak{t}$ is identified with α_i by the Killing form. Here $\{\zeta_i, \gamma_i\}_{(\alpha_i, \mathfrak{t}_\sigma)>0}$ can be regarded as a basis of $\mathfrak{k}/\mathfrak{k}^\sigma$. The complex structure of G/P sends ζ_i^l to γ_i^l, and γ_i^l to $-\zeta_i^l$. Substitute $u = \zeta_i$ and $v = \gamma_i$ in (4.5), we get

$$\omega_\lambda(\zeta_i^l, \gamma_i^l)_a = \omega(\zeta_i^l, \gamma_i^l)_a = (\lambda, [\zeta_i, \gamma_i]) = (\lambda, \alpha_i). \tag{4.6}$$

Since ω is pseudo-Kähler, it follows from (4.6) that ω_λ is pseudo-Kähler too. In fact ω_λ is Kähler if and only if (4.6) is positive for all $(\alpha_i, \mathfrak{t}_\sigma) > 0$, or equivalently $\lambda \in \sigma$. Hence the proposition. ☐

For $i = 1, 2$, consider the reduced spaces $(R_{\lambda_i}, (\omega_i)_{\lambda_i})$, with moment maps $\psi_i \colon R_{\lambda_i} \longrightarrow \mathfrak{k}^*$. By the previous proposition, these reduced spaces are pseudo-Kähler. So we can compare them under the notions of $\sim$ and $\approx$ introduced in (1.10). If $\approx$ does not hold, we write $\not\approx$.

PROPOSITION 4.5. *Suppose that R_{λ_i} have the same number of connected components. Then $(\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}$ if and only if $\lambda_1 \sim \lambda_2$, and $(\omega_1)_{\lambda_1} \approx (\omega_2)_{\lambda_2}$ if and only if $\lambda_1 = \lambda_2$.*

Proof. Suppose that this proposition has been proved for all connected reduced spaces. Let R_λ be a reduced space, possibly nonconnected. For $i = 1, 2$, let $(G/P)a_i$ be connected components of R_λ. By Proposition 4.3, their moment maps satisfy $\psi_i(a_i) = \lambda$. So by the present proposition for connected reduced spaces, $(G/P)a_1$ and $(G/P)a_2$ are isomorphic pseudo-Kähler manifolds. We conclude that all connected components of R_λ are isomorphic to one another, and so the present proposition holds for nonconnected reduced spaces too.

From this observation, we only have to prove the proposition for connected reduced spaces. So assume that R_{λ_i} are connected for $i = 1, 2$. Write $R_{\lambda_i} = (K/K^\sigma)a_i$, for some $a_i \in A_\sigma$.

Suppose that $\lambda_1 \sim \lambda_2$. Then there is a coadjoint orbit $\mathcal{O} \subset \mathfrak{k}^*$ which contains λ_1 and λ_2. By Proposition 4.3, $\psi_i(a_i) = \lambda_i$. By Proposition 4.1, $\lambda_i \in (\mathfrak{t}_\sigma^*)_{reg} \subset \mathfrak{t}^*$, so the isotropy subgroup of λ_i in K is K^σ. Hence $\mathcal{O} = K/K^\sigma$. So ψ_i is a diffeomorphism from $(K/K^\sigma)a_i$ onto $\mathcal{O}$. In fact ψ_i is K-equivariant, so it identifies $(\omega_i)_{\lambda_i}$ with the Kirillov-Kostant-Souriau symplectic form ω_{KKS} on $\mathcal{O}$. We conclude that $(\omega_1)_{\lambda_1} \sim \omega_{KKS} \sim (\omega_2)_{\lambda_2}$.

Conversely, if $(\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}$, then ψ_i have the same image $\mathcal{O}$. By Proposition 4.3, $\psi_i(a_i) = \lambda_i \in \mathcal{O}$, so $\lambda_1 \sim \lambda_2$.

We next prove the last part of this proposition, where $\sim$ is replaced with $\approx$. Suppose that $\lambda_1 = \lambda_2$. By (4.5), for all $u, v \in \mathfrak{k}$,

$$(\omega_1)_{\lambda_1}(u^l, v^l)_{a_1} = (\lambda_i, [u, v]) = (\omega_2)_{\lambda_2}(u^l, v^l)_{a_2}. \tag{4.7}$$

Consider the K-equivariant biholomorphic map

$$\kappa\colon\ (G/P)a_1 \longrightarrow (G/P)a_2, \kappa(ga_1) = ga_2. \tag{4.8}$$

By (4.7), $\kappa^*(\omega_2)_{\lambda_2}$ and $(\omega_1)_{\lambda_1}$ agree on a_1. By K-invariance, they agree everywhere. So κ preserves the pseudo-Kähler structures, and $(\omega_1)_{\lambda_1} \approx (\omega_2)_{\lambda_2}$.

Conversely, suppose that $\lambda_1 \neq \lambda_2$. If λ_i are in different coadjoint K-orbits, then the first part of the proposition says that $(\omega_i)_{\lambda_i}$ are not symplectomorphic, so in particular $(\omega_1)_{\lambda_1} \not\approx (\omega_2)_{\lambda_2}$. Hence we may assume that λ_i are in the same orbit. Each connected component of $(\mathfrak{t}^*_\sigma)_{reg} \subset \mathfrak{k}^*$ intersects a K-orbit at most once. From $\lambda_i \in (\mathfrak{t}^*_\sigma)_{reg}$, $\lambda_1 \neq \lambda_2$ and $\lambda_1 \sim \lambda_2$, we conclude that λ_i are in different connected components of $(\mathfrak{t}^*_\sigma)_{reg}$. The holomorphic map (4.8) fails to preserve the pseudo-Kähler structures, because (4.6) and (4.7) show that there is sign problem. Other symplectomorphisms between $(\omega_i)_{\lambda_i}$ have to permute the connected components of $(\mathfrak{t}^*_\sigma)_{reg}$, so they cannot be holomorphic. We conclude that $(\omega_1)_{\lambda_1} \not\approx (\omega_2)_{\lambda_2}$. This proves the proposition. □

By this proposition, the reduced form ω_λ depends only on λ, and not on ω. So whenever the reduced space is connected, the reduction process is given by (1.11). Finally, we show that every pseudo-Kähler form on G/P can be obtained by (1.11).

PROPOSITION 4.6. *Every K-invariant pseudo-Kähler form on G/P can be obtained by symplectic reduction from $\omega = \sqrt{-1}\partial\bar\partial F$ on $G/(P, P)$, with F strictly convex.*

Proof. Let Ω be a K-invariant pseudo-Kähler form on G/P. Since K is semisimple, the K-action preserving Ω has a unique moment map ([10], §52 and Theorem 26.1) $\phi\colon\ G/P \longrightarrow \mathfrak{k}^*$. Let $e \in G/P = K/K^\sigma$ be the identity coset. The left action of K^σ fixes e, hence

$$x \in \mathfrak{k}^\sigma \Longrightarrow x^l_e = 0. \tag{4.9}$$

We claim that $\phi(e) \in \mathfrak{t}^*_\sigma$. Write $\mathfrak{k} = \mathfrak{t}_\sigma + \mathfrak{t}^\perp_\sigma + V$ as in (2.3). Up to linear combination, a typical element of V can be written as $[x, y] \in V$, where $x \in \mathfrak{t}$ and $y \in V$. Then

$$(\phi(e), [x, y]) = \Omega(x^l, y^l)_e = 0, \tag{4.10}$$

because it follows from $\mathfrak{t} \subset \mathfrak{k}^\sigma$ and (4.9) that $x_e^l = 0$. Consequently,

$$(\phi(e), V) = 0. \tag{4.11}$$

Note that $\mathfrak{t}_\sigma^\perp \subset \mathfrak{k}_{ss}^\sigma$ and $\mathfrak{k}_{ss}^\sigma$ is semi-simple. So up to linear combination, a typical element of $\mathfrak{t}_\sigma^\perp$ can be written as $[x, y] \in \mathfrak{t}_\sigma^\perp$, where $x, y \in \mathfrak{k}^\sigma$. By (4.9), $x_e^l = y_e^l = 0$. Duplicating the arguments in (4.10) gives $(\phi(e), [x, y]) = 0$ and

$$(\phi(e), \mathfrak{t}_\sigma^\perp) = 0. \tag{4.12}$$

By (4.11) and (4.12), $\phi(e) \in \mathfrak{t}_\sigma^*$ as claimed.

For $(\alpha_i, \sigma) > 0$, consider the basis $\zeta_i, \gamma_i \in V_i$ from Proposition 2.3. Since Ω is nondegenerate,

$$(\phi(e), \alpha_i) = \Omega(\zeta_i^l, \gamma_i^l)_e \neq 0.$$

Therefore, $\phi(e) \in (\mathfrak{t}_\sigma^*)_{reg}$.

Write $\phi(e) = \lambda \in (\mathfrak{t}_\sigma^*)_{reg}$. Let C_λ be the connected component of $(\mathfrak{t}_\sigma^*)_{reg}$ containing λ. So C_λ is an open cone in $(\mathfrak{t}_\sigma^*)_{reg}$, and we let n be its dimension. There exist $\lambda_1, \ldots, \lambda_n \in \mathfrak{t}_\sigma^*$ such that $\lambda = \sum_1^n \lambda_i$ and

$$C_\lambda = \left\{ \sum_1^n c_i \lambda_i ; c_i > 0 \right\}. \tag{4.13}$$

Identify $\mathfrak{t}_\sigma^* \cong \mathfrak{a}_\sigma^*$, and define $F \in C^\infty(\mathfrak{a}_\sigma)$ by

$$F(y) = 2 \sum_1^n \exp(\lambda_i, y). \tag{4.14}$$

By (1.4), we get $F \in C_K^\infty(G/(P, P))$ and $\omega = \sqrt{-1}\partial\bar{\partial}F$. We claim that ω is pseudo-Kähler:

Identify $\mathfrak{a}_\sigma$ with $\mathbf{R}^n$ by $\{\lambda_i\}_1^n$. Then the gradient function is $F'(y) = (2 \exp y_i)_i$, so $U_F = (\mathbf{R}^+)^n = C_\lambda$. The Hessian matrix of F is $(2 \frac{\partial^2}{\partial y_i \partial y_j} \sum_1^n \exp y_k)_{ij}$, which is the diagonal matrix with entries $(2 \exp y_1, \ldots, 2 \exp y_n)$. This is a positive definite matrix, so F is strictly convex. By Theorem 1, ω is pseudo-Kähler as claimed.

The K-moment map of ω is given by $\frac{1}{2}F'$. So at $e = \exp(0)$,

$$\Phi(e) = \frac{1}{2}F'(0) = (\exp 0)_i = (1, \ldots, 1) = \lambda.$$

Let ψ be the moment map of the reduced form ω_λ. By Propositions 4.1 and 4.3, $\psi(e) = \lambda$. We conclude that the two moment maps satisfy $\phi(e) = \psi(e)$. By an argument similar to Proposition 4.5, it follows that $\Omega = \omega_\lambda$. □

Proof of Theorem 3, Corollaries 3A and 3B. Theorem 3 follows directly from Propositions 4.1 through 4.6. We now prove the corollaries.

Consider the situation where the reduced space is connected, thus $R_\lambda = G/P$. Theorem 3 says that symplectic reduction (1.9) simplifies to (1.11). Let $I \subset \mathfrak{t}_\sigma^*$ be the set of all $\lambda \in \mathfrak{t}_\sigma^*$ that are in the image of some right moment map Φ_r. According to Theorem 3, (1.11) sets up a bijective correspondence between I and all the K-invariant pseudo-Kähler forms on G/P. So the proof of Corollary 3A amounts to showing that

$$I = (\mathfrak{t}_\sigma^*)_{reg}. \tag{4.15}$$

By Theorem 1, it is clear that $I \subset (\mathfrak{t}_\sigma^*)_{reg}$. So it remains to show the opposite. Given $\lambda \in (\mathfrak{t}_\sigma^*)_{reg}$, we construct F by (4.14) so that $\omega = \sqrt{-1}\partial\bar{\partial}F$ is pseudo-Kähler. As shown in Proposition 4.6, the moment map of ω sends e to λ, so $\lambda \in I$. This means that $(\mathfrak{t}_\sigma^*)_{reg} \subset I$, which completes the arguments for (4.15). Corollary 3A follows.

The arguments for Corollary 3B is very similar to Corollary 3A, with the following minor modification: Here I is defined for Kähler forms instead of pseudo-Kähler forms, and (4.15) is replaced by $I = \sigma$. Also, in (4.13), the open cone containing λ is $C_\lambda = \sigma$. By duplicating the above arguments, Corollary 3B follows. □

5. Signatures of pseudo-Kähler forms. In this section, we perform geometric quantization to the flag manifold G/P. We shall study the relation between quantization and reduction, compute the signature of the reduced form, and prove Theorem 4.

Let $\lambda \in \mathfrak{t}_\sigma^*$ be an integral weight in the image of the right moment map of some $K \times T_\sigma$-invariant pseudo-Kähler form ω on $G/(P,P)$. Suppose that the reduced space is connected, namely $(G/P, \omega_\lambda)$. Let $\mathbf{L}_\lambda$ be the pre-quantum line bundle [14] over G/P corresponding to ω_λ. It is equipped with a connection whose curvature is ω_λ. We again use this connection to define the $(0,q)$-Dolbeault complex with coefficients in $\mathbf{L}_\lambda$. They are automatically square-integrable because G/P is compact, and we denote their cohomology by $H^q_{(\omega_\lambda)}$. Let ψ be the moment map of the K-action preserving ω_λ. Proposition 4.3 says that λ is in the image of ψ. Hence the image of ψ is the coadjoint orbit containing λ. Therefore [2], $\mathbf{L}_\lambda$ is just the homogeneous line bundle described in (3.9). Then $H^q_{(\omega_\lambda)}$ can be computed by the Borel-Weil-Bott theorem [4]. Namely, $H^q_{(\omega_\lambda)}$ is an irreducible K-representation with highest weight α if there exists $\tau \in W$ of length q such that $\tau(\lambda+\rho)-\rho = \alpha$ is dominant, and it vanishes otherwise. Together with Theorem 2, $(H^q_\omega)_\lambda \cong H^q_{(\omega_\lambda)}$. On the other hand, if λ is not integral or does not lie in the image of the moment map, then clearly both $(H^q_\omega)_\lambda$ and $H^q_{(\omega_\lambda)}$ vanish. We conclude that geometric quantization commutes with reduction,

$$H^q_{(\omega_\lambda)} \cong (H^q_\omega)_\lambda. \tag{5.1}$$

Recall from Theorem 2 that $(H^q_\omega)_\lambda$ is a unitary K-representation, by using the L^2-norm of the unique harmonic $\mathfrak{h}_\sigma$-annihilating representative of each cohomology class. Similarly, since G/P is compact, each element of $H^q_{(\omega_\lambda)}$ has a unique harmonic representative. Similarly, the L^2-norm makes $H^q_{(\omega_\lambda)}$ a unitary K-representation. We now construct an explicit unitary K-intertwining isomorphism for (5.1). We assume that $\lambda \in U_F$, for otherwise everything vanishes. In this case, Lemma 3.2 says that $\int_{a\in A_\sigma} \chi^2(a)e^{-F(a)}\,da$ converges. For simplicity, write $N = \int_{a\in A_\sigma} \chi^2(a)e^{-F(a)}\,da$; so $0 < N < \infty$. Recall the section s of Proposition 3.1 and the injection π^* of (3.10). Define

$$\Theta\colon\ H^q_{(\omega_\lambda)} \longrightarrow (H^q_\omega)_\lambda, \Theta([\alpha]) = \frac{1}{\sqrt{N}}[(\pi^*\alpha)\otimes s]. \tag{5.2}$$

PROPOSITION 5.1. *The map Θ is a unitary K-equivariant isomorphism.*

Proof. Since π commutes with the K-action and s is K-invariant, it is clear that Θ also commutes with the K-action. If we compare Θ with θ of (3.23), then it follows from arguments of Proposition 3.7 that Θ is a bijection. It only remains to show that Θ is unitary. Switching to another cohomologous representative if necessary, let $\alpha \in \Omega^{0,q}(G/P, \mathbf{L}_\lambda)$ be the unique harmonic representative of $[\alpha]$. By Proposition 3.7, $(\pi^*\alpha)\otimes s$ is the unique harmonic representative of $[(\pi^*\alpha)\otimes s]$ which annihilates $\mathfrak{h}_\sigma$. So the norm of $[\alpha]$ is defined as $\|\alpha\|$. Then

$$\begin{aligned}
&\|\Theta^*[\alpha]\|^2\\
&= \frac{1}{N}\|(\pi^*\alpha)\otimes s\|^2\\
&= \frac{1}{N}\int_{p\in G/(P,P)} \langle \pi^*\alpha, \pi^*\alpha\rangle_p e^{-F(p)}\mu && \text{by (3.1) and (3.2)}\\
&= \frac{1}{N}\int_{p\in G/(P,P)} \pi^*\langle \alpha, \alpha\rangle_p E^2_\chi(p) e^{-F(p)}\mu && \text{by (3.17)}\\
&= \int_{k\in K/K^\sigma_{ss}} \pi^*\langle \alpha, \alpha\rangle_k\, dk\\
&= \int_{\pi(k)\in K/K^\sigma} \langle \alpha, \alpha\rangle_{\pi(k)}\, dk_0 && \text{by (3.18)}\\
&= \|[\alpha]\|^2.
\end{aligned}$$

Therefore, Θ is unitary. ◻

To compute the signature of ω_λ, we need the next proposition.

PROPOSITION 5.2. *Let α be a positive root. Suppose that $\tau(\lambda+\rho)-\rho$ is dominant for $\tau \in W$ and integral weight λ. Then $\tau(\alpha)$ is negative if and only if $(\lambda, \alpha) < 0$.*

Proof. Suppose that $\tau(\alpha)$ is negative. Then

$$\begin{aligned}(\tau(\lambda+\rho)-\rho, \tau\alpha) \leq 0 &\implies (\lambda+\rho, \alpha) \leq (\rho, \tau\alpha) \\ &\implies (\lambda+\rho, \alpha) < 0 \\ &\implies (\lambda, \alpha) < 0.\end{aligned}$$

Conversely, suppose that $\tau(\alpha)$ is positive. Then

$$\begin{aligned}(\tau(\lambda+\rho)-\rho, \tau\alpha) \geq 0 &\implies (\lambda+\rho, \alpha) \geq (\rho, \tau\alpha) \\ &\implies (\lambda+\rho, \alpha) > 0.\end{aligned} \tag{5.3}$$

Since the weight λ is integral, the last inequality in (5.3) implies that $(\lambda, \alpha) \geq 0$. This completes the proof of Proposition 5.2. □

Proof of Theorem 4 and Corollary 4A. The isomorphism $H^q_{(\omega_\lambda)} \cong (H^q_\omega)_\lambda$ follows from (5.1). According to Proposition 5.1, an explicit unitary K-equivariant isomorphism between them is given by (5.2). By Theorem 2, the spaces $(H^q_\omega)_\lambda \cong H^q_{(\omega_\lambda)}$ either are unitary irreducible K-representations or vanish. Theorem 2 also says that they are irreducible exactly when (1.7) holds. Condition (ii) of (1.7) implies that $\lambda + \rho$ is regular. Therefore to prove Theorem 4, it remains only to show that in this case, the signature of ω_λ is $(d-q, q)$. Here d is the dimension of G/P.

For τ in (1.7), there exist exactly q positive roots α_i such that $\tau(\alpha_i)$ are negative. By Proposition 5.2, these are all the positive roots α_i which satisfy $(\lambda, \alpha_i) < 0$.

The basis $\zeta_i, \gamma_i \in V_i$ in Proposition 4.4 satisfies $\omega_\lambda(\zeta_i^l, \gamma_i^l)_a = (\lambda, \alpha_i)$. The spaces $\{(V_i^l)_a\}_{(\alpha_i,\sigma)>0}$ are mutually orthogonal with respect to ω_λ. Since there are exactly q positive roots α_i in which $(\lambda, \alpha_i) < 0$, there correspond exactly q pairs of $\{\zeta_i, \gamma_i\}$ in which $\omega_\lambda(\zeta_i^l, \gamma_i^l)_a < 0$. So the pseudo-Kähler form ω_λ has signature $(d-q, q)$. This proves Theorem 4.

For the signature $(d-q, q)$ of ω_λ, set $q = 0$ and $q = d$ respectively for the cases where ω_λ and $-\omega_\lambda$ are Kähler. Then Corollary 4A follows immediately. □

6. Connectivity of reduced space. In this section, we consider the issue of whether the reduced space is connected. By Proposition 4.1, connectivity of the reduced space is equivalent to injectivity of the gradient function F'.

If ω or $-\omega$ is Kähler, then F or $-F$ is strictly convex, and so F' is injective. Consider however when ω is merely pseudo-Kähler, in particular when neither

F nor $-F$ is strictly convex. Then Theorem 1 says that F is nonsingular, and $U_F \subset (\mathfrak{t}_\sigma^*)_{reg}$. Note that $(\mathfrak{t}_\sigma^*)_{reg}$ consists of a disjoint union of connected cones. We can identify $\mathfrak{t}_\sigma^*$ with $\mathbf{R}^n$ by some suitable linear coordinates, such that the cone which contains U_F is identified with the first quadrant $\{x \in \mathbf{R}^n \; ; \; x_i > 0\} \subset \mathbf{R}^n$. The gradient function becomes

$$F' = \left(\frac{\partial F}{\partial x_i}\right)_i : \mathbf{R}^n \longrightarrow \mathbf{R}^n.$$

The connectivity problem of the reduced space can now be formulated in terms of basic calculus:

PROBLEM 6.1. *Suppose that the Hessian matrix of $F \in C^\infty(\mathbf{R}^n)$ is everywhere nonsingular, and $\frac{\partial F}{\partial x_i} > 0$ for all i. When is F' injective?*

The obvious sufficient condition for injectivity is where F or $-F$ is strictly convex, such as $F(x) = \sum_i \exp x_i$. However, other sufficient conditions are still not known. In the general setting where a compact Lie group acts in Hamiltonian fashion, a well known sufficient condition for the reduced space to be connected is properness of the moment map ([16], Theorem 1.1). Unfortunately, this never holds in our case:

PROPOSITION 6.2. *The right moment map Φ_r, or equivalently F', cannot be proper.*

Proof. Assume that F' is proper. By Proposition 4.1, this is equivalent to Φ_r being proper. Consequently ([16], Theorem 1.1), each $\Phi_r^{-1}(\lambda)$ is connected. This means that F' is injective. Since the Hessian of F is everywhere nonsingular, the inverse function theorem says that F' is a local diffeomorphism. Then F', being an injective local diffeomorphism, is a diffeomorphism onto an open set $U \subset \mathbf{R}^n$. The set U is not the entire $\mathbf{R}^n$, due to the condition $\frac{\partial F}{\partial x_i} > 0$. Pick a boundary point λ of U. It does not lie in U because U is open. Let C be a compact set which contains λ as an interior point. Then the inverse image $(F')^{-1}(C)$ is unbounded and hence not compact. This contradicts properness of F'. □

A better understanding of conditions for F' to be injective, and answers to Problem 6.1, would both be nice.

DEPARTMENT of APPLIED MATHEMATICS, NATIONAL CHIAO TUNG UNIVERSITY, 1001 TA HSUEH ROAD, HSINCHU, TAIWAN
Electronic mail: CHUAH@MATH.NCTU.EDU.TW

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