On asymptotic stabilization of driftless systems

Yew-Wen Liang, Der-Cherng Liaw [*]

Department of Electrical and Control Engineering, National Chiao Tung University,
1001 Ta Hsueh Road, Hsinchu, Taiwan 30039, ROC

Abstract

A simple approach is proposed to investigate the stabilization of driftless systems. By taking the control input as a function of system states, we can transform the stabilization design into a stability problem. The known stability results on the nonlinear autonomous system are then employed to derive the asymptotically stabilizing controllers by solving an algebraic equation. Existence conditions of the asymptotic stabilizing feedback controller for the driftless systems by employing the stability criteria on linear systems and a class of third-order homogeneous systems are obtained to demonstrate the application of the approach. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Driftless systems; Feedback controller; Time-invariant stabilizer; Lyapunov function

1. Introduction

In the recent years, feedback stabilization of nonlinear systems, specifically, the nonlinear critical systems, has attracted lots of attention (see e.g., [1,5,10,11]). Critical systems are the systems of which the linearized models possess eigenvalues lying on the imaginary axis with the remaining eigenvalues in the open left half of the complex plane. For the most degenerate case, the system linearization may possess only zero eigenvalues. A class of such systems is the so-called "driftless systems". Practical examples of driftless systems

[*] Corresponding author. E-mail: dcliaw@cc.nctu.edu.tw

include the control model of a synchronous satellite's orbital motion [2,3] and the car-like robot [17].

The recent study of the asymptotic stabilization of nonlinear driftless systems include the existence conditions of time-invariant smooth stabilizers [4,12–14], design of time-varying stabilizers [7,15], design of time-invariant piecewise smooth stabilizers [6] and numerically finding controls which achieve a desired state transfer [16]. In the design of time-invariant asymptotic stabilizers, in general, the derived control laws explicitly depend on the choice of Lyapunov function. However, in general, it is not easy to construct an appropriate Lyapunov function. Furthermore, the stabilizer might be very complicated due to the highly nonlinearity of given system dynamics.

In this paper, instead of seeking Lyapunov functions for the stabilization design of driftless systems as given by $\dot{x} = g(x)u$, we propose another approach. By assuming the controller to be a function of system states, that is, letting $u = u(x)$, we can then transform the stabilization design into a stability problem. The known stability results on general nonlinear systems can then be employed to derive the corresponding state feedback asymptotic stabilizers for the original driftless systems. That is, suppose for some vector fields $f(x)$ we know the origin of system $\dot{x} = f(x)$ is asymptotically stable. By letting $g(x)u(x) = f(x)$, we can then transform the stabilization design into an algebraic equation for solving $u(x)$. If such a solution $u(x)$ exists, it will also stabilize the original driftless system. Two examples of $f(x)$ are given to verify the application of the approach. One is to have $f(x) = Ax$, where A is an arbitrary Hurwitz matrix. The other is to have $f(x) = C(x, x, x)$, a trilinear system with $-x^{\mathrm{T}}C(x, x, x)$ being a locally positive definite function.

There are several advantages of the approach. Firstly, the stabilization problem of the driftless systems can be transformed into the problem of solving an algebraic equation. Secondly, the stabilizability conditions and the design of corresponding controllers can be derived from the well-documented stability results. Thirdly, this approach is entirely independent of the construction of Lyapunov function.

The organization of this paper is as follows. The main approach of the study is given in Section 2. It is followed by two case studies, which demonstrate the application of the main approach. Finally, Section 4 gives the conclusions.

2. Main approach

Consider a class of nonlinear driftless systems as given by

$$\dot{x} = g(x)u = \sum_{k=1}^{m} u_k g_k(x), \tag{1}$$

where $x \in \mathbb{R}^n$, $u = (u_1, \ldots, u_m)^{\mathrm{T}} \in \mathbb{R}^m$ and $g(x) = (g_1(x), \ldots, g_m(x)) \in \mathbb{R}^{n \times m}$. In addition, $g_i(x)$ are assumed to be smooth vector fields of $\mathscr{R}^n$ for $1 \leqslant i \leqslant m$. Various results have been presented regarding the asymptotic stabilization of the operating point of system (1) (see e.g., [6,7,9,12–17]). Two practical examples of the driftless systems are also given by Ahmed and Sen [2,3] and Walsh et al. [17]. The former cites the control model of the orbital motion of synchronous satellites; while the latter cites the motion equation of car-like robots. In the sequel, for simplicity and without loss of generality, the origin is assumed to be the operating point of interest.

For most of the existing stabilization results above, the stabilizing controllers for system (1) were obtained by constructing appropriate Lyapunov functions. However, in general, it is not easy to construct such a Lyapunov function. To relax the burden of finding Lyapunov functions, in this paper, we consider another approach. This is achieved by employing the known stability criteria for general nonlinear autonomous systems to derive the stabilizing control laws for system (1). Details are given as follows.

It is known that the origin of system (1) is asymptotically stabilizable if there exists a function $u = u(x)$ such that the origin of system (2) below is asymptotically stable:

$$\dot{x} = g(x)u(x). \tag{2}$$

Denote $\mathscr{S}$ a subset of the set of the vector field $\mathscr{V}$ as defined by

$$\mathscr{V} := \{f(x) | f(0) = 0 \text{ and the origin of } \dot{x} = f(x) \text{ is asymptotically stable}\}. \tag{3}$$

We then have the following obvious result.

Lemma 2.1. *The origin of system* (1) *is asymptotically stabilizable if there exists a function* $u = u(x)$ *such that* $g(x)u(x) \in \mathscr{S}$. *Moreover, suppose for some given* $f(x) \in \mathscr{S}$ *such that Eq.* (4) *below holds:*

$$g(x)u(x) = f(x). \tag{4}$$

Then the solution $u(x)$ *is an asymptotic stabilizer for system* (1).

Lemma 2.1 above provides a guideline for the stabilization design. In the following, we will discuss how to apply Lemma 2.1 to system (1). First, we consider the case of which $g(0)$ is a nonsingular matrix. It is obvious that Eq. (4) is solvable around a neighborhood of the origin no matter what $f(x) \in \mathscr{S}$ is chosen. However, if $\mathrm{rank}(g(0)) < n$ (i.e., $m < n$ or $g(0)$ loses rank), the origin of system (1) may still be asymptotically stabilizable while Eq. (4) may not be solvable for some specific vector field $f(x) \in \mathscr{S}$. Thus, in general, it is not easy to check the asymptotical stabilizability of the origin of system (1) through the solvability of Eq. (4) for some specific $f(x) \in \mathscr{S}$. This motivates us

to work on the solvability of Eq. (4) for any $f(x) \in \mathscr{S}$. One example of $\mathscr{S}$ can be chosen as

$$\mathscr{S} = \{ f(x) | f(0) = A_f x = Q_f(x, x) = 0$$
$$\text{and } -x^{\mathrm{T}} C_f(x, x, x) \text{ is an lpdf for } x \in \mathbb{R}^n \}, \tag{5}$$

where A_f, $Q_f(x, x)$ and $C_f(x, x, x)$ denote the Jacobian matrix, the quadratic terms and the cubic terms of $f(x)$, respectively. The asymptotic stabilizability condition of the origin of system (1) as stated in Lemma 2.1 can then be determined by whether the intersection of given $\mathscr{S}$ and span$\{g_1, \ldots, g_m\}$ is nonempty.

There are several advantages of the application of Lemma 2.1. Firstly, the stabilization problem of the driftless systems (1) can be transformed into the problem of solving algebraic equation as in Eq. (4). Secondly, the stabilizability conditions on $g(x)$ and the design of corresponding controllers can be derived from the well-documented stability results. Thirdly, this approach is entirely independent of the construction of the Lyapunov function. Suppose the chosen $\mathscr{S}$ is defined for polynomial type systems. Then the asymptotic stabilizers can be chosen as a polynomial function too. The checking conditions can hence be simplified by invoking Taylor's series expansion on $g(x)$ and the asymptotic stabilizer can be obtained by checking the coefficients of the corresponding order of polynomials. Two cases are studied in the next section to demonstrate the application of the main approach.

3. Case study

By taking Taylor's series expansion on $g_i(x)$ up to the third order for each i, $1 \leqslant i \leqslant m$, we have

$$g_i(x) = g_i(0) + L_i x + Q_i(x, x) + C_i(x, x, x) + o(||x||^3). \tag{6}$$

Here, $o(||x||^k)$ denotes terms of order higher than k. In the following, with $g_i(x)$ as given in Eq. (6), two special classes of $\mathscr{S}$ are selected for the demonstration of possible application of Lemma 2.1. One is defined in Eq. (5) and the other is associated with Hurwitz stability for linear system. Details are given as follows.

3.1. Quadratic-plus-cubic state feedback controller

First, we seek for the stabilizability condition on $g(x)$ and the corresponding asymptotic stabilizers for the driftless system by the use of Lemma 2.1 with $\mathscr{S}$ as defined in Eq. (5). For simplicity, let the control be given by

$$u_i(x) = x^{\mathrm{T}} q_i x + c_i(x, x, x) \quad \text{for each } i. \tag{7}$$

It follows that

$$u_i(x)g_i(x) = (x^T q_i x)g_i(0) + (x^T q_i x)(L_i x) + c_i(x,x,x)g_i(0) + \text{o}(||x||^3). \quad (8)$$

It is known that the lowest order of a smooth asymptotically stable dynamics cannot be an even number (see e.g., Corollary 2.1 of Koditschek and Narendra [8]. From the definition of $\mathscr{S}$ as in Eq. (5) and Lemma 2.1, we then have the next result.

Theorem 3.1. *Suppose the control input is a quadratic-plus-cubic function of state x in the form of Eq.* (7). *Then the origin of system* (1) *is asymptotically stable if the following two conditions hold*:
(i) $\sum_{i=1}^{m}(x^T q_i x)g_i(0) \equiv 0$, *and*
(ii) $-\sum_{i=1}^{m}\{(x^T q_i x)(x^T L_i x) + c_i(x,x,x)x^T g_i(0)\}$ *is a locally positive definite function*.

When the column vectors $g_1(0), \ldots, g_m(0)$ are linearly independent, the Condition (i) of Theorem 3.1 can never be satisfied unless $q_i = 0$ for all $i = 1, \ldots, m$. This leads to the next result.

Corollary 3.1. *Suppose $g(0)$ is of full rank. Then there exists no purely quadratic asymptotic stabilizer for the origin of system* (1).

It is known from Brockett [4] that the origin of system (1) is not stabilizable by any smooth time-invariant control law if $g(0)$ is of full rank and $m < n$ (i.e., the number of input is less than that of system states). Corollary 3.1 further claims that system (1) does not possess *purely quadratic* asymptotic stabilizer even when $\text{rank}(g(0)) = n$.

Though system (1) does not possess quadratic asymptotic stabilizer when $g(0)$ is of full rank, it does possess a cubic asymptotic stabilizer as given in the next result which follows directly from Theorem 3.1.

Corollary 3.2. *Suppose $g(0)$ is of full rank. Then the origin of system* (1) *is asymptotically stabilizable by a purely cubic controller in the form of $u_i(x) = c_i(x,x,x)$ for $1 \leqslant i \leqslant m$ if and only if $m = n$. Moreover, a set of candidates of $c_i(x,x,x)$ are*

$$c_i(x,x,x) = -(x^T M_i x)x^T g_i(0), \quad (9)$$

where M_i denotes an arbitrary symmetric positive definite matrices.

Next, we consider the case of which the constant matrix $g(0)$ loses rank. Denote by $N(g(0))$ the null space of $g(0)$. To satisfy Condition (i) of Theorem 3.1, q_i can be chosen as $q_i = \alpha_i H$, where the vector $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in N(g(0))$ and H is a square matrix. In addition, choose $c_i(x,x,x)$ to be $x^T H x x^T g_i(0)$. Then Condition (ii) of Theorem 3.1 becomes

$$-(x^{\mathrm{T}}Hx)x^{\mathrm{T}}\sum_{i=1}^{m}\{\alpha_i L_i + g_i(0)g_i^{\mathrm{T}}(0)\}x \text{ is an lpdf.} \tag{10}$$

Denote by L_i^s the symmetric part of the matrix L_i. It is known that $x^{\mathrm{T}}L_i x = x^{\mathrm{T}}L_i^s x$ for all x, we then have the next obvious result.

Corollary 3.3. *Suppose there exists a nonzero vector* $\alpha = (\alpha_1, \ldots, \alpha_m)^{\mathrm{T}} \in N(g(0))$ *such that*

$$\sum_{i=1}^{m}\{\alpha_i L_i^s + g_i(0)g_i^{\mathrm{T}}(0)\} \text{ is a positive definite matrix.} \tag{11}$$

Then the origin of system (1) *is asymptotically stabilizable by a quadratic-plus-cubic control in the form of Eq.* (7). *Moreover, asymptotic stabilizers can be selected with* $q_i = \alpha_i H$ *and* $c_i(x,x,x) = x^{\mathrm{T}}Hxx^{\mathrm{T}}g_i(0)$ *for all* $i = 1, \ldots, m$, *where* H *is a negative definite matrix.*

Suppose there exists some $k \in \{1, \ldots, m\}$ such that $g_k(0) = 0$ and L_k^s is a definite matrix. The next result follows readily from Corollary 3.3 and Theorem 3.1.

Corollary 3.4. *Suppose there exists some* $k \in \{1, \ldots, m\}$ *such that* $g_k(0) = 0$ *and* L_k^s *is a definite matrix. Then the origin of system* (1) *is asymptotically stabilizable by a purely nonlinear control in the form of Eq.* (7). *Moreover, the control input can be selected with* $q_k = \alpha_k H$; $q_i = 0$ *if* $i \neq k$ *and* $c_i(x,x,x)$ *can be zero or* $c_i(x,x,x) = x^{\mathrm{T}}Hxx^{\mathrm{T}}g_i(0)$ *for all* $i = 1, \ldots, m$. *Here,* H *may be any negative definite matrix,* $\alpha_k > 0$ (*resp.* $\alpha < 0$) *if* L_k^s *is a positive definite matrix* (*resp. if* L_k^s *is a negative definite matrix*).

On the other hand, if all the matrices L_i are not definite, then the cubic terms $c_i(x,x,x)$ in Condition (ii) of Theorem 3.1 should be selected to compensate $-\sum_{i=1}^{m}(x^{\mathrm{T}}q_i x)(x^{\mathrm{T}}L_i x)$ for forming an lpdf. For this purpose, we decompose L_k^s as

$$L_k^s = V^{\mathrm{T}}DV = \sum_{j=1}^{n}d_j v_j v_j^{\mathrm{T}}, \tag{12}$$

where $V^{\mathrm{T}} = [v_1, \ldots, v_n] \in \mathbb{R}^{n \times n}$ denotes an orthogonal matrix and $D = \mathrm{diag}(d_1, \ldots, d_n)$ is a diagonal matrix with

$$d_j > 0 \quad \text{for } 1 \leqslant j \leqslant r_1, \tag{13}$$

$$d_j = 0 \quad \text{for } r_1 + 1 \leqslant j \leqslant r_2, \tag{14}$$

$$d_j < 0 \quad \text{for } r_2 + 1 \leqslant j \leqslant n. \tag{15}$$

We then have the next theorem.

Theorem 3.2. *Suppose there exists some $k \in \{1, \ldots, m\}$ such that $g_k(0) = 0$ and L_k^s is decomposed into the form as in Eq. (12) with*

$$\text{span}\{v_1, \ldots, v_{r_2}\} \subseteq \text{span}\{g_1(0), \ldots, g_m(0)\}. \tag{16}$$

Then the origin of system (1) is asymptotically stabilizable by nonlinear control in the form of Eq. (7). Moreover, the controller can be chosen with $q_i = 0$ if $i \neq k$; q_k being a symmetric positive definite matrix and $c_i(x, x, x) = (x^{\mathrm{T}} M x) x^{\mathrm{T}} g_i(0)$. Here, M is a symmetric matrix satisfying

$$\lambda_{\max}(M) < -\frac{\lambda_{\max}(L_k^s) \lambda_{\max}(q_k)}{\alpha} \tag{17}$$

with α being the smallest nonzero eigenvalue of $\sum_{i=1}^{m} g_i(0) g_i^{\mathrm{T}}(0)$.

Proof. Let

$$\phi(x) = (x^{\mathrm{T}} q_k x)(x^{\mathrm{T}} L_k^s x) + x^{\mathrm{T}} M x \sum_{i=1}^{m} x^{\mathrm{T}} g_i(0) g_i^{\mathrm{T}}(0) x, \tag{18}$$

where L_k^s is given in Eq. (12). According to Theorem 3.1, we only need to check whether the function $-\phi(x)$ is an lpdf. Let $\mathscr{W}_1 = \text{span}\{g_1(0), \ldots, g_m(0)\}$ and $\mathscr{W}_2 = \text{span}\{v_1, \ldots, v_{r_2}\}$. Denote by $\mathscr{W}_1^{\perp}$ and $\mathscr{W}_2^{\perp}$ the orthogonal complement of $\mathscr{W}_1$ and $\mathscr{W}_2$, respectively. Then for any nonzero vector ξ, we have

$$\xi = \xi_{w_1} + \xi_{w_1^{\perp}}, \qquad \xi = \xi_{w_2} + \xi_{w_2^{\perp}}, \tag{19}$$

where $\xi_{w_1} \in \mathscr{W}_1$, $\xi_{w_2} \in \mathscr{W}_2$, $\xi_{w_1^{\perp}} \in \mathscr{W}_1^{\perp}$ and $\xi_{w_2^{\perp}} \in \mathscr{W}_2^{\perp}$. It is clear from (16) that $\mathscr{W}_2 \subseteq \mathscr{W}_1$, $\mathscr{W}_1^{\perp} \subseteq \mathscr{W}_2^{\perp}$ and $\|\xi_{w_2}\| \leqslant \|\xi_{w_1}\|$.

If $\xi_{w_1} = 0$, then we have $\xi = \xi_{w_1^{\perp}}$. This implies that $\xi^{\mathrm{T}} g_i(0) g_i^{\mathrm{T}}(0) \xi = 0$ for all i. Thus, we have

$$\phi(\xi) = (\xi^{\mathrm{T}} q_k \xi)(\xi^{\mathrm{T}} L_k^s \xi) = (\xi^{\mathrm{T}} q_k \xi)(\xi_{w_1^{\perp}}^{\mathrm{T}} L_k^s \xi_{w_1^{\perp}}) < 0,$$

where the inequality follows from $\xi_{w_1^{\perp}} \in \mathscr{W}_2^{\perp}$ and Eqs. (13)–(15). On the other hand, if $\xi_{w_1} \neq 0$, we then have

$$\phi(\xi) = (\xi^{\mathrm{T}} q_k \xi)(\xi^{\mathrm{T}} L_k^s \xi) + \xi^{\mathrm{T}} M \xi \sum_{i=1}^{m} \xi^{\mathrm{T}} g_i(0) g_i^{\mathrm{T}}(0) \xi$$

$$= (\xi^{\mathrm{T}} q_k \xi)(\xi_{w_2}^{\mathrm{T}} L_k^s \xi_{w_2}) + \xi^{\mathrm{T}} M \xi \sum_{i=1}^{m} \xi_{w_1}^{\mathrm{T}} g_i(0) g_i^{\mathrm{T}}(0) \xi_{w_1}$$

$$< \lambda_{\max}(q_k) \|\xi\|^2 \cdot \lambda_{\max}(L_k^s) \|\xi_{w_2}\|^2 - \frac{\lambda_{\max}(q_k) \lambda_{\max}(L_k^s)}{\alpha} \|\xi\|^2 \cdot \alpha \|\xi_{w_1}\|^2$$

$$\leqslant 0$$

since $\|\xi_{w_2}\| \leqslant \|\xi_{w_1}\|$. The conclusion of Theorem 3.2 is hence implied. $\quad \square$

Note that, the results of Theorem 3.2 can be easily applied to the case of which there exists some $k \in \{1, \ldots, m\}$ such that L_k^s is semidefinite matrix. Details are omitted.

For the special case of which $g_i(0) = 0$ for all $1 \leqslant i \leqslant m$, the next result follows readily from Theorem 3.1.

Corollary 3.5. *Suppose $g_i(0) = 0$ for all $1 \leqslant i \leqslant m$. Then the origin of system* (1) *is asymptotically stabilizable by the nonlinear control in the form of Eq.* (7) *if*

$$-\sum_{i=1}^{m}(x^{\mathrm{T}}q_ix)(x^{\mathrm{T}}L_ix) \text{ is an lpdf.} \tag{20}$$

Moreover, a set of candidates for q_i are $q_i = -L_i$ for all $i = 1, \ldots, m$.

Remark 3.1. The checking condition of Corollary 3.5 as in Eq. (20) can be simplified as

$$\sum_{i=1}^{m}(x^{\mathrm{T}}L_ix)^2 \text{ being an lpdf} \tag{21}$$

due to the following reason. Suppose $\sum_{i=1}^{m}(x^{\mathrm{T}}L_ix)^2$ is not an lpdf. Then for any neighborhood of the origin, there exists a nonzero point x_1 such that $\sum_{i=1}^{m}(x_1^{\mathrm{T}}L_ix_1)^2 = 0$. This implies that $x_1^{\mathrm{T}}L_ix_1 = 0$ for each i. Thus, Condition (20) can never hold regardless of the value of q_i. On the other hand, if Condition (21) holds, then one set of candidates of q_i can be $q_i = -L_i$ for $1 \leqslant i \leqslant m$. However, this choice is not unique for system stabilization.

3.2. Constant-plus-linear state feedback controller

Next, we seek for the stabilizability condition on $g(x)$ and the corresponding asymptotic stabilizer for the driftless system by the use of Lemma 2.1 with $\mathscr{S}$ as defined by

$$\mathscr{S} = \{f(x)|f(0) = 0 \text{ and } A_f \text{ is a Hurwitz matrix for } x \in \mathbb{R}^n\}. \tag{22}$$

For simplicity, choose

$$u_i(x) = a_i + l_i^{\mathrm{T}}x \quad \text{for all } 1 \leqslant i \leqslant m. \tag{23}$$

It follows that

$$u_i(x)g_i(x) = a_ig_i(0) + \{g_i(0)l_i^{\mathrm{T}} + a_iL_i\}x + \mathrm{o}(||x||^1). \tag{24}$$

In general, if a purely constant controller is taken, that is, $u_i = a_i$ for all $1 \leqslant i \leqslant m$, the problem of stabilizability of the origin of system (1) is then reduced to determine the stability of the origin of the system

$$\dot{x} = \sum_{i=1}^{m} a_i g_i(x). \tag{25}$$

We have the next obvious result.

Lemma 3.1. *Suppose there exists some k such that* $g_k(0) = 0$ *and the origin of the system* $\dot{x} = g_k(x)$ *or* $\dot{x} = -g_k(x)$ *is stable. Then the origin of system (1) is asymptotically stabilizable by constant control.*

From Lemma 2.1, we then have the next theorem.

Theorem 3.3. *Suppose the control laws are in the form of (23). Then the origin of system (2) is asymptotically stable if the following two conditions hold*:
(i) $\sum_{i=1}^{m} a_i g_i(0) = 0$; *and*
(ii) $\sum_{i=1}^{m} \{g_i(0) l_i^{\mathrm{T}} + a_i L_i^s\}$ *is a Hurwitz matrix.*

It is observed that the stabilizability conditions of Theorem 3.3 above are very similar to those in Theorem 3.1. Thus, the results presented in Section 3.1 might be applicable to the systems defined by Eq. (24).

Suppose $g(0)$ is of full rank. It is obvious from Condition (i) of Theorem 3.3 that the origin of system (1) does not possess constant asymptotic stabilizer. Analogous to the discussion in Section 3.1, we have the following trivial result for the case of which $g(0)$ is of full rank.

Corollary 3.6. *Suppose* $g(0)$ *is of full rank with* $m = n$. *Then the origin of system (1) is asymptotically stabilizable by a purely linear controller with* $u_i(x) = l_i^{\mathrm{T}} x$ *for all* $i = 1, \ldots, m$. *One candidate of linear stabilizers is to have* $l_i = g_i(0)$ *for all* $i = 1, \ldots, m$.

If $g_i(0) = 0$ for all $i = 1, \ldots, m$, we can seek for constant controller which makes the matrix $\sum_{i=1}^{m} a_i L_i$ being Hurwitz [13].

Let the control function u_i be a purely linear function of x. Then system (1) becomes

$$\dot{x} = g(x) Ax, \quad \text{where } A = \begin{pmatrix} l_1^{\mathrm{T}} \\ \vdots \\ l_m^{\mathrm{T}} \end{pmatrix} \in \mathbb{R}^{m \times n}. \tag{26}$$

It is noted that if $m < n$, there are at least $n - m$ dimensional vector space which makes $Ax = 0$. This implies that system (26) possesses at least $n - m$ dimensional equilibrium points passing through the origin. We then have the next result.

Lemma 3.2. *Suppose $u_i(x) = l_i^T x$ for $1 \leqslant i \leqslant m$ and $m < n$. Then the origin of system* (26) *is not an isolated equilibrium point and it is hence not asymptotically stable.*

According to Lemma 3.2, in order that system (1) has a linear asymptotic stabilizer, we must have the condition of $m = n$. Moreover, in addition to the condition $m = n$, the matrix A should be selected to be a nonsingular matrix so that the origin is an isolated equilibrium point. Thus, we have the next result.

Lemma 3.3. *Suppose $u_i(x) = l_i^T x$ for $1 \leqslant i \leqslant m$ and $m = n$. Then the origin of system* (1) *is asymptotically stable if $g(0)$ is of full rank and $g(0) \cdot A$ is a Hurwitz matrix, where A is as given in Eq.* (26). *Moreover, a candidate of A is $A = -g^T(0)$.*

For the general study of system stabilization when $g(0)$ is not of full rank, due to the similarity, it is not difficult to check that the stabilizability conditions presented in the preceding subsection for the design of nonlinear stabilizers are also true for the design of constant-plus-linear stabilizers. Since the derivations are very similar, in the following, we only present the results.

Corollary 3.7. *Suppose there exists a nonzero vector $\alpha = (\alpha_1, \ldots, \alpha_m) \in N(g(0))$ such that*

$$\sum_{i=1}^{m} (\alpha_i L_i^s) + g(0)g^T(0) \text{ is a positive definite matrix.} \tag{27}$$

Then the origin of system (1) *is asymptotically stabilizable by a control of the form* (23). *Moreover, a set of candidates of a_i and l_i are*

$$a_i = -\alpha_i \quad and \quad l_i = -g_i(0) \quad for \ all \ i = 1, \ldots, m. \tag{28}$$

If there exists a $k \in \{1, \ldots, m\}$ such that $g_k(0) = 0$ and L_k^s is a definite matrix, then we have the next result which is similar to those of Corollary 3.4.

Corollary 3.8. *Suppose there exists some $k \in \{1, \ldots, m\}$ such that $g_k(0) = 0$ and L_k^s is a definite matrix. Then the origin of system* (1) *is asymptotically stabilizable by a control of the form* (23). *Moreover, a set of candidates of a_i and l_i are*

$$a_k = \alpha_k; \quad a_i = 0 \quad if \ i \neq k, \tag{29}$$

$$l_i = 0 \ or \ -g_i(0) \quad for \ all \ i = 1, \ldots, m. \tag{30}$$

Here, $\alpha_k < 0$ (resp. $\alpha_k > 0$) if L_k^s is a positive definite matrix (resp. if L_k^s is a negative definite matrix).

Corollary 3.9. *Suppose there exists $k \in \{1, \ldots, m\}$ such that $g_k(0) = 0$ and L_k^s has the form* (12)–(15) *satisfying*

$$\text{span}\{v_1, \cdots, v_{r_2}\} \subseteq \text{span}\{g_1(0), \ldots, g_m(0)\}. \tag{31}$$

Then the origin of system (1) *is asymptotically stabilizable by a control of the form* (23). *Moreover, a set of candidates for* a_i *and* l_i *are*

$$a_k = 1; \quad a_i = 0 \quad \text{if } i \neq k; \text{ and } \quad l_i = \beta g_i(0). \tag{32}$$

Here,

$$\beta < -\frac{\lambda_{\max}(L_k^s)}{\alpha}, \alpha = \text{the smallest nonzero eigenvalue of } \sum_{i=1}^m g_i(0)g_i^{\mathrm{T}}(0). \tag{33}$$

4. Conclusions

In this paper, we have studied the stabilization problem of driftless systems from the stability point of view. It is achieved by taking the control input as a function of system states and then transforms the stabilization design into the problem of solving corresponding algebraic equation. We have obtained stabilizability conditions and the corresponding asymptotic stabilizers for two special cases to demonstrate the application of the proposed approach. It is believed that through such an approach, asymptotic stabilizability conditions for the driftless system can be easily obtained from the well-documented stability results without constructing Lyapunov functions.

References

[1] D. Aeyels, Stabilization of a class of nonlinear systems by a smooth feedback control, Systems and Control Lett. 5 (1985) 289–294.

[2] N.U. Ahmed, A.K. Sen, Stability and control of a trap mode in synchronous orbit, IEEE Trans. Aerosp. Electron. Systems AES-16 (1980) 474–480.

[3] N.U. Ahmed, A.K. Sen, Control of synchronous orbit using Lyapunov stability theory, IEEE Trans. Aerosp. Electron. Systems AES-17 (1981) 48–53.

[4] R. Brockett, Asymptotic stability and feedback stabilization, in: Differential Geometric Control Theory, Birkhauser, Boston, 1983, pp. 181–191.

[5] S. Behtash, S. Sastry, Stabilization of nonlinear systems with uncontrollable linearization, IEEE Trans. Automat. Control AC-33 (1988) 585–590.

[6] C. Canudas de Wit, O.J. Sordalen, Examples of piecewise smooth stabilization of driftless NL systems with less input than states, in: Proceedings of the IFAC Nonlinear Control Systems Design Symposium, 1992, pp. 26–30.

[7] J.M. Coron, Global asymptotic stabilization for controllable systems without drift, Math. Control Signals Systems 5 (1992) 295–312.

[8] D.E. Koditschek, K.S. Narendra, The stability of second-order quadratic differential equations, IEEE Trans. Automat. Control (1982) 783–798.

[9] G. Lafferriere, A general strategy for computing steering controls of systems without drift, in: Proceedings of the 30th IEEE Conference on Decision and Control, Brighton, England, 1991, pp. 1115–1120.

[10] D.-C. Liaw, Composite Liapunov functions for nonlinear critical systems, in: Proceedings of the Second European Control Conference, Groningen, The Netherlands, 1993, pp. 1453–1458.

[11] D.-C. Liaw, Application of center manifold reduction to nonlinear system stabilization, Appl. Math. Comput. 91 (1998) 243–258.

[12] D.-C. Liaw, Y.-W. Liang, Asymptotic stabilization of driftless system, Int. J. Control, accepted.

[13] D.-C. Liaw, Y.-W. Liang, Feedback stabilization of nonlinear driftless systems with applications to homogeneous-type systems, Int. J. Systems Sci. 28 (2) (1996) 173–182.

[14] D.-C. Liaw, Y.-W. Liang, Stabilization of driftless systems via a time-invariant approach, J. Control Systems and Technol. 4 (3) (1996) 205–210.

[15] J.B. Pomet, Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift, Systems and Control Lett. 18 (1992) 147–158.

[16] E. Sontag, Control of systems without drift via generic loops, IEEE Trans. Automat. Control 40 (7) (1995) 1210–1219.

[17] G. Walsh, D. Tilbury, S. Sastry, R. Murray, J.P. Laumond, Stabilization of trajectories for systems with nonholonomic constraints, IEEE Trans. Automat. Control (1994) 216–222.