Explicit Solution of Partitioning Problems over a 1-Dimensional Parameter Space

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Abstract: We consider a class of partitioning problems where the partitioned set is a finite set of real numbers and the objective function of a partition is a function of the vector whose coordinates are the sums of the elements in each part of the given partition (the number of such parts is assumed given). We obtain an explicit solution of such partitioning problem with polynomial complexity bounds. © 2000 John Wiley & Sons, Inc. Naval Research Logistics 47: 531–540, 2000

1. INTRODUCTION—PROBLEM FORMULATION AND REVIEW OF MAIN RESULTS

Throughout, we let n be a positive integer and $N \equiv \{1, \ldots, n\}$. These parameters will be fixed throughout this paper.

A *partition* is an ordered collection of sets $\pi = (\pi_1, \ldots, \pi_p)$, where $\pi_1, \ldots, \pi_p$ are pairwise disjoint, *nonempty* subsets of N whose union is N. Given such a partition π, we refer to p as its *size* and to the sets $\pi_1, \ldots, \pi_p$ as its *parts*. Also, if the number of elements in the parts of a partition $\pi = (\pi_1, \ldots, \pi_p)$ are $n_1, \ldots, n_p$, respectively, we refer to $(n_1, \ldots, n_p)$ as the *shape* of π; of course, in this case $\sum_{j=1}^{p} n_j = |N| = n$. Partitions of size p are called *p-partitions* and partitions of shape $(n_1, \ldots, n_p)$ are called $(n_1, \ldots, n_p)$-*partitions*.

Let A be an n-vector. For a p-partition $\pi = (\pi_1, \ldots, \pi_p)$ we define the π-*summation-vector of* A, denoted A^π, by

$$A^\pi \equiv \left[\sum_{t \in \pi_1} A_t, \ldots, \sum_{t \in \pi_p} A_t \right]^T \in R^p. \tag{1}$$

Our main goal is to study partitioning problems defined as follows:

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Partitioning problem over a 1-dimensional parameter space: Maximize a real-valued *function* $F(\cdot)$ over a set of p-partitions Π, where the function $F(\cdot)$ is defined on Π through an n-vector A and a function $C(\cdot)$ on R^p (or a subset thereof) by $F(\pi) = C(A^\pi)$ for each $\pi \in \Pi$. A partition that maximizes $F(\cdot)$ over Π is then called *optimal over* Π. Special cases of interest are *convex* and *linear partitioning problems* corresponding to cases, where $C(\cdot)$ is convex or linear, respectively. In particular, if $C(\cdot)$ is linear, it is represented by a vector $C \in R^p$ such that $C(X) = C^T X$ for each $X \in R^p$. $\square$

Partitioning problems where the partitioned elements are vectors, rather than real numbers, are considered in the literature; see [5, 6]. These, more general problems, are referred to as the *partitioning problem over multidimensional parameter spaces*.

Sets of partitions of particular interest are those whose shape is constrained to be in a prescribed set. Specifically, if Γ is a set of positive integer p-vectors with coordinate-sum n (that is, Γ is a set of potential shapes of p-partitions), we refer to the set of all p-partitions whose shape is in Γ as *the set of Γ-shape partitions*; for convenience, we suppress the explicit dependence on Γ and refer generically to *constrained-shape partition-sets*. If L and U are positive integer p-vectors satisfying $L \leq U$ and $\sum_{j=1}^{p} L_j \leq n \leq \sum_{j=1}^{p} U_j$, the (nonempty) set of positive integer p-vectors $(n_1, \ldots, n_p)$ with coordinate-sum n that satisfy $L_j \leq n_j \leq U_j$ for each $j = 1, \ldots, p$ is denoted $\Gamma^{(L,U)}$; the corresponding set of partitions is denoted $\Pi^{(L,U)}$ and, with the dependence on L and U suppressed, referred to as a *bounded-shape partition set*. The set consisting of a single vector $(n_1, \ldots, n_p)$ is denoted $\Gamma^{(n_1, \ldots, n_p)}$; the corresponding set of partitions is denoted $\Pi^{(n_1, \ldots, n_p)}$ and, with the dependence on $(n_1, \ldots, n_p)$ suppressed, referred to as a *single-shape partition set*.

The above partitioning problems constitute combinatorial optimization problems with applications in diverse fields that include clustering, statistics, scheduling, reliability and system assembly; see [5, 6] and references therein. In particular, it is shown in [5] that, with $p = n$, $C(\cdot)$ convex and Π as the set of partitions with a single prescribed shape, our 1-dimensional partitioning problem captures the Traveling Salesperson Problem which is known to be NP-complete; thus, these partitioning problems are computationally prohibitive. Still, with p fixed, a polynomial algorithm in the number of partitioned vectors n for solving the convex constrained-shape partitioning problem was developed in [5]; the polynomial bound on computational complexity concerns the number of arithmetic operations, oracle evaluations of $C(\cdot)$ and oracle tests on whether or not an integer vector $(n_1, \ldots, n_p)$ is in the given set of shapes. In this paper we develop a polynomial algorithm for linear bounded-shape partitioning problems with p considered as a variable part of the input.

Our algorithmic results can be summarized in Table 1, which specifies the number of arithmetic operations (and evaluations of C in the case it is an arbitrary convex function).

Table 1. Complexity bounds.

$C(\cdot)$	Single-Shape	Bounded-Shape
Linear	$O[np + p(\log p)]$	$O[np^2 + n(\log n)]$
Convex	$O[npp!]$	$O[n^2 p^2 p!]$
	or	or
	$O[np! + n(\log n)]$	$O[n^2 pp! + n(\log n)]$

2.   SOLUTION METHODS

We consider several specializations of the partitioning problem. For each we give a structural theorem which characterizes optimal solutions, followed by an algorithmic solution. We start with single-shape problems.

THEOREM 1: Let $A \in R^n, C \in R^p, n_1, \ldots, n_p$, be positive integers with $\sum_{j=1}^{p} n_j = n$, and $F \colon \Pi^{(n_1,\ldots,n_p)} \to R$ with $F(\pi) = C^T A^\pi$ for each $\pi \in \Pi^{(n_1,\ldots,n_p)}$. Further, assume that

$$A_1 \geq A_2 \geq \cdots \geq A_n, \tag{2}$$

$$C_1 \geq C_2 \geq \cdots \geq C_p. \tag{3}$$

Then the p-partition π^* with $\pi_j^* = \{\sum_{u=1}^{j-1} n_u + 1, \ldots, \sum_{u=1}^{j} n_u\}$ for $j = 1, \ldots, p$ maximizes $F(\cdot)$ over $\Pi^{(n_1,\ldots,n_p)}$; further, if the inequalities of (2) and (3) hold strictly, then π^* is the only maximizer.

PROOF: For each partition $\pi \in \Pi^{(n_1,\ldots,n_p)}$ and $j = 1, \ldots, p, \sum_{u=1}^{j} |\pi_u^*| = \sum_{u=1}^{j} n_u = \sum_{u=1}^{j} |\pi_u|$; consequently, (2) implies that

$$\sum_{u=1}^{j} (A^{\pi^*})_u \geq \sum_{u=1}^{j} (A^\pi)_u. \tag{4}$$

Also, with arbitrary selection of C_{p+1}, we have that, for each π

$$F(\pi) \;=\; C^T A^\pi = \sum_{j=1}^{p} C_j (A^\pi)_j = \sum_{j=1}^{p} C_j \left[\sum_{u=1}^{j} (A^\pi)_u - \sum_{u=1}^{j-1} (A^\pi)_u \right]$$

$$= \sum_{j=0}^{p} (C_j - C_{j+1}) \left[\sum_{u=1}^{j} (A^\pi)_u \right] + C_{p+1} \left[\sum_{i=1}^{n} A_i \right], \tag{5}$$

where we use the fact that $\sum_{u=1}^{p} (A^\pi)_u = \sum_{i=1}^{n} A_i$. Let C_{p+1} be selected so that $C_p > C_{p+1}$, in which case the inequalities $(C_j - C_{j+1}) \geq 0$ extend from $j = 1, \ldots, p-1$ to $j = p$. The application of (5) to an arbitrary π in $\Pi^{(n_1,\ldots,n_p)}$ and to $\pi = \pi^*$ combines with (4) and the nonnegativity of the $(C_j - C_{j+1})$'s to show that

$$F(\pi^*) - F(\pi) = \sum_{j=1}^{p} (C_j - C_{j+1}) \left[\sum_{u=1}^{j} (A^{\pi^*})_u - \sum_{u=1}^{j} (A^\pi)_u \right] \geq 0. \tag{6}$$

Finally, suppose the inequalities of (2) and (3) are strict. Then, for each partition $\pi \neq \pi^*$, a strict inequality holds in (4) for at least one j and the $(C_j - C_{j+1})$'s are positive, it then follows from (5) (applied to π and π^*) that (6) holds as a strict inequality.   □

COROLLARY 2 (Solution of Linear Single-Shape Partitioning Problems): Single-shape linear partitioning problems can be solved in time $O[np + p(\log p)]$, in particular, for fixed p it is solvable in linear time $O(n)$.

PROOF: Let $A, C, n_1, \ldots, n_p$, and F be as in Theorem 1. One can achieve condition (2) and (3) by sorting the coordinates of A and C, respectively, and renumbering indices; as A has n coordinates and C has p coordinates, these tasks can be accomplished in time $O[n(\log n)]$ and $O[p(\log p)]$, respectively. Once the indices are renumbered, Theorem 1 provides an explicit solution of the partitioning problem. The total effort is $O[n(\log n)]$. Of course, the renumbering of the indices of C requires the rearrangement of $n_1, \ldots, n_p$.

We next explain how computation can actually be accelerated. Indeed, suppose a sorting of the coordinates of C is first executed, requiring time $O[p(\log p)]$, and an index-enumeration $j_1, \ldots, j_p$ satisfying $C_{j_1} \geq C_{j_2} \geq \cdots \geq C_{j_p}$ is obtained. It is then not necessary to fully sort $A_1, \ldots, A_n$ in order to determine the optimal partition; all that is needed is to determine the set of n_{j_1}-largest coordinates of A, the next n_{j_2}-largest coordinates, and so on. This block-sorting can be executed with $O(pn)$ comparisons (e.g., Knuth [7]), yielding an improved complexity bound of $O(pn)$. If the data is given with (2) in force, Theorem 1 provides an explicit solution of the partitioning problem requiring only the sorting of the coordinates of C; so, in this case the problem is solvable in time $O[p(\log p)]$.   □

The above method with $n_j = 1$ for $j = 1, \ldots, p$ is well known (e.g., [4]). Also, an explicit representation of the convex hull of the A^π's with prescribed shape was derived by Theorem 1 in [3].

Suppose (2) is in force. A p-partition π is then called *consecutive* if its parts consist of consecutive integers, that is, if there is an enumeration of the parts of π, say $\pi_{j_1}, \ldots, \pi_{j_p}$, such that if $(n_1, \ldots, n_p)$ is the shape of π we have $\pi_{j_s} = \{\sum_{u=1}^{s-1} n_{j_u} + 1, \ldots, \sum_{u=1}^{s} n_{j_u}\}$ for $s = 1, \ldots, p$. When the A_t's are distinct, consecutiveness is equivalent to the assertion that the convex hulls of elements corresponding to distinct parts are disjoint. Theorem 1 yields a new proof for the following known result asserting the optimality of consecutive partitions {cf. [1, Theorem 2] and [3, Theorem 1] (the former reference provides a multidimensional generalization)}.

THEOREM 3: Let $A \in R^n$ satisfy (2), $C: R^p \to R$ be a convex function, $n_1, \ldots, n_p$ be positive integers with $\sum_{j=1}^p n_j = n$, and $F: \Pi^{(n_1,\ldots,n_p)} \to R$ with $F(\pi) = C(A^\pi)$ for each $\pi \in \Pi^{(n_1,\ldots,n_p)}$. Then there exists a consecutive partition which maximizes $F(\cdot)$ over $\Pi^{(n_1,\ldots,n_p)}$.

PROOF: Let $\Pi \equiv \Pi^{(n_1,\ldots,n_p)}$. If $C(\cdot)$ is linear, the conclusion of the corollary is immediate from the algorithm resulting from Theorem 1. From standard results each vertex of $P_A^\Pi = \text{conv}\{A^\pi: \pi \in \Pi\}$, say v, has a representation $v = A^\pi$ for some $\pi \in \Pi$, and such a vertex is the unique maximizer of some linear functional over P_A^Π, say $F_v(\cdot)$. We conclude that each vertex v of P_A^Π has a representation $v = A^\pi$ with π as a consecutive partition. The conclusion of the corollary now follows from a standard result that assures that a maximum of a convex function over a polytope is attained at one of the polytope's extreme points.   □

The proof of Theorem 3 demonstrates that, with Π as the set of all partitions with a prescribed shape, the vertices of the polytope $\text{conv}\{A^\pi: \pi \in \Pi\}$ correspond to consecutive partitions. A multidimensional generalization of this result is given in [1] while a (one-dimensional) inverse result is established in [3].

COROLLARY 4 (Solution of Convex Single-Shape Partitioning Problems): Convex single-shape partitioning problems can be solved in $O[npp!]$ arithmetic operations and $p!$ evaluations of the underlying convex function C; an alternative bound is $O[np! + n(\log n)]$ arithmetic operations and $p!$ evaluations of C.

PROOF: Let $A, C, n_1, \ldots, n_p$, and F be as in Theorem 3. The goal is to maximize $F(\cdot)$ over $\Pi^{(n_1,\ldots,n_p)}$. If (2) is satisfied, Theorem 3 implies the existence of a consecutive partition which is optimal. Each consecutive $(n_1, \ldots, n_p)$-partition is determined by a permutation of the integers $1, \ldots, p$; hence there exists exactly $p!$ such partitions. For each such partition π, A^π is computable by n additions and $F(\pi)$ is available by a function evaluation of C. Finding the maximizer of the resulting $p!$ values of $F(\pi)$ will require $O(p!)$ comparisons (e.g., [7]). So, an optimal partition can be determined by $np!$ additions, $p!$ evaluations of C and $O(p!)$ comparisons. If (2) is not satisfied *a priori*, one can preprocess the data and sort the elements $A_1, \ldots, A_n$ requiring additional time $O[n(\log n)]$, or block-sort $A_1, \ldots, A_n$ with respect to each of the permutation of $\{1, \ldots, p\}$ requiring additional time $p!O(np)$. $\square$

We say that a vector B is *nonnegative*, written $B \geq 0$, if all of B's coordinates are nonnegative, and we say that B is nonpositive, written $B \leq 0$, if $-B \geq 0$. We will extend the analysis of linear (1-dimensional) partitioning problems from single-shape to bounded-shape. The first step is to consider the cases where A is either nonnegative or nonpositive.

We need some further definitions. Let L and U be positive integer p-vectors satisfying $L \leq U$ and $\sum_{j=1}^p L_j \leq n \leq \sum_{j=1}^p U_j$ and let $\Pi \equiv \Pi^{(L,U)}$. Consider first the case where $\sum_{j=1}^p L_j < n < \sum_{j=1}^p U_j$. As $\sum_{u=1}^p L_u < n$ and $\sum_{u=1}^p L_u + \sum_{u=1}^p (U_u - L_u) = \sum_{u=1}^p U_u > n$, there exists an index $j \in \{1, \ldots, p\}$ with $\sum_{u=1}^p L_u + \sum_{u=1}^j (U_u - L_u) \geq n$; let j^+ be the first such index, let $\mu^+ \equiv n - \sum_{u=1}^{j^+-1} U_u - \sum_{u=j^++1}^p L_u$ and let $(n_1^+, \ldots, n_p^+) \equiv (U_1, \ldots, U_{j^+-1}, \mu^+, L_{j^++1}, \ldots, L_p)$. The selection of j^+ assures $L_{j^+} < \mu^+ \leq U_{j^+}$ and therefore $(n_1^+, \ldots, n_p^+) \in \Gamma^{(L,U)}$. Similarly, let j^- be the first index $i = 1, \ldots, p$ with $\sum_{u=1}^p U_u - \sum_{u=1}^j (U_u - L_u) \leq n$ and let $\mu^- \equiv n - \sum_{u=1}^{j^--1} U_u - \sum_{u=j^-+1}^p L_u$ and $(n_1^-, \ldots, n_p^-) \equiv (L_1, \ldots, L_{j^--1}, \mu^-, U_{j^-+1}, \ldots, U_p)$. Then $L_{j^-} \leq \mu^- < U_{j^-}$ and $(n_1^-, \ldots, n_p^-) \in \Gamma^{(L,U)}$. Finally, if $\sum_{j=1}^p L_j = n$, then the shape of all partitions in Π is $(L_1, \ldots, L_p)$ and in this case we set $j^+ = 1, j^- = p, \mu^+ = L_1$, and $\mu^- = L_p$; and if $\sum_{j=1}^p U_j = n$, then the shape of all partitions in Π is $(U_1, \ldots, U_p)$ and we set $j^+ = 1, j^- = p, \mu^+ = U_1$, and $\mu^- = U_p$.

THEOREM 5: Let $A \in R^n$ satisfy (2), $C \in R^p$ satisfy (3), L and U be positive integer p-vectors satisfying $L \leq U$ and $\sum_{j=1}^p L_j \leq n \leq \sum_{j=1}^p U_j$, and $F: \Pi^{(L,U)} \to R$ with $F(\pi) = C^T A^\pi$ for each $\pi \in \Pi^{(L,U)}$. Then:

   (i) If $A \geq 0$, then the p-partition π^+ with $\pi_j^+ = \{\sum_{u=1}^{j-1} n_u^+ + 1, \ldots, \sum_{u=1}^j n_u^+\}$
        for $j = 1, \ldots, p$ maximizes $F(\cdot)$ over $\Pi^{(L,U)}$.
  (ii) If $A \leq 0$, then the p-partition π^- with $\pi_j^- = \{\sum_{u=1}^{j-1} n_u^- + 1, \ldots, \sum_{u=1}^j n_u^-\}$
        for $s = 1, \ldots, p$ maximizes $F(\cdot)$ over $\Pi^{(L,U)}$.

Further, if the inequalities of (2) hold strictly and the coordinates of C are distinct, then π^+ and π^- are, respectively, the only optimal partitions.

PROOF: We establish only the (weak and strict) inequalities corresponding to case (i) as those corresponding to (ii) follow from similar arguments. The definition of π^+ assures that

$$\sum_{u=1}^j |\pi_u^+| = \sum_{u=1}^j n_u^+ \geq \sum_{u=1}^j |\pi_u| \qquad \text{for each partition } \pi \in \Pi^{(L,U)} \text{ and } j = 1, \ldots, p; \qquad (7)$$

hence, the nonnegativity of the A^i's and (2) imply that

$$\sum_{u=1}^{j} (A^{\pi^+})_u \geq \sum_{u=1}^{j} (A^{\pi})_u \quad \text{for } j = 1, \ldots, p. \tag{8}$$

Also, with arbitrary selection of C_{p+1}, we have that for each partition π

$$F(\pi) \;=\; \langle C, A^{\pi} \rangle = \sum_{j=1}^{p} C_j (A^{\pi})_j = \sum_{j=1}^{p} C_j \left[\sum_{u=1}^{j} (A^{\pi})_u - \sum_{u=1}^{j-1} (A^{\pi})_u \right]$$

$$= \sum_{j=0}^{p} (C_j - C_{j+1}) \left[\sum_{u=1}^{j} (A^{\pi})_u \right] + C_{p+1} \left[\sum_{i=1}^{n} A_i \right], \tag{9}$$

where we use the fact that $\sum_{u=1}^{p}(A^{\pi})_u = \sum_{i=1}^{n} A_i$. Let C_{p+1} be selected so that $C_{p+1} < C_p$, in which case the inequalities $(C_j - C_{j+1}) \geq 0$ extend from $j = 1, \ldots, p-1$ to $j = p$. Now, with π as an arbitrary partition in $\Pi^{(L,U)}$, the application of (9) to π and π^+ combines with (8) and the nonnegativity of the $(C_j - C_{j+1})$'s to show that

$$F(\pi^+) - F(\pi) = \sum_{j=1}^{p} (C_j - C_{j+1}) \left[\sum_{u=1}^{j} (A^{\pi^+})_u - \sum_{u=1}^{j} (A^{\pi})_u \right] \geq 0. \tag{10}$$

Finally, suppose that the inequalities of (2) are strict and the coordinates of C are distinct. Then for each partition $\pi \neq \pi^+$, a strict inequality holds in (8) for at least one j; as the $(C_j - C_{j+1})$'s are positive, it follows from (9) (applied to π and π^+) that (10) holds as a strict inequality. $\quad\square$

COROLLARY 6 (Solution of Nonnegative/Nonpositive Linear Bounded-Shape Partitioning Problems): Linear bounded-shape partitioning problems with A nonnegative or nonpositive can be solved in time $O[pn + p(\log p)]$, in particular, for fixed p it is solvable in linear time $O(n)$.

PROOF: We consider only the nonnegative case. Let A, C, L, U, and F be as in Theorem 5 with $A \geq 0$. As for the single-shape case, conditions (2) and (3) can be achieved by sorting the coordinates of A and C and renumbering indices, tasks that can be accomplished in time $O[n(\log n)]$ and $O[p(\log p)]$, respectively. Once the indices are renumbered, Theorem 5 provides an optimal solution of the partitioning problem that can be computed in time $O(p)$ [needed to determine the partial sums of the L_j's and of the $(U_j - L_j)$'s and the corresponding index j^+]. The total effort requires time $O[n(\log n) + p(\log p)]$. Acceleration is possible by determining the shape ( ) (after sorting the L_j's and U_j's) and then block-sorting of the A_i's with $O(pn)$ arithmetic operations; this will lead to an improved complexity bound of $O[pn + p(\log p)]$. If the data is given with (2) in force, the effort reduces to $O[p(\log p)]$. $\square$

We are ready to consider general linear bounded shape partitioning problems. The idea is to determine potential shapes of optimal partitions; these are identified by using Theorem 5 for the index sets corresponding to the nonnegative and to the negative A_i's, respectively.

THEOREM 7: Let $A \in R^n$ satisfy (2), $C \in R^p$ satisfy (3), L and U be positive integer p-vectors satisfying $L \leq U$ and $\sum_{j=1}^{p} L_j \leq n \leq \sum_{j=1}^{p} U_j$, and $F: \Pi^{(L,U)} \to R$ with $F(\pi) =$

$C^T A^\pi$ for each $\pi \in \Pi^{(L,U)}$. Also, let $N^+ \equiv \{i : A_i \geq 0\} \neq \emptyset$ and let $N^- \equiv N \backslash N^+ \neq \emptyset$. Then there exists a partition π that maximizes $F(\cdot)$ over $\Pi^{(L,U)}$ such that for some positive integers j^+ and j^- satisfying $j^+ \leq j^-$, $\sum_{u=1}^{j^+-1} U_u < |N^+|$ and $\sum_{u=j^-+1}^{p} U_u < |N^-|$, we have that

$$
|\pi_j| = \begin{cases} U_u & \text{for } u = 1, \ldots, j^+ - 1, \\ L_u & \text{for } u = j^+ + 1, \ldots, j^- - 1, \\ U_u & \text{for } u = j^- + 1, \ldots, p. \end{cases} \tag{11}
$$

Further, if the inequalities of (2) and (3) hold strictly, then the shape of every optimal partition has the above structure.

PROOF: We start by assuming that the inequalities of (2) and (3) hold strictly. Let π be an optimal partition. By considering the single-shape partitioning problem with prescribed shape $(n_1, \ldots, n_p) = (|\pi_1|, \ldots, |\pi_p|)$, the strict version of Theorem 1 implies that each of the sets $\pi_1, \pi_2, \ldots, \pi_p$ consists of consecutive integers, in order; in particular, with $i^* \equiv |N^+|$ and j^* as the index with $i^* \in \pi_{j^*}$, we have that $\cup_{u=1}^{j^*-1} \pi_u = N^+ \backslash \pi_{j^*}$ and $\cup_{u=j^*+1}^{p} \pi_u = N^- \backslash \pi_{j^*}$. We consider four separate cases.

Case I—$\pi_{j^*} \cap N^- = \emptyset$: In this case π_{j^*} is contained in N^+ and $\pi_1, \ldots, \pi_{j^*}$ partition N^+; thus, the optimality of π implies that this partition is optimal for the problem of partitioning N^+ into j^* parts with $L_1, \ldots, L_{j^*}$ and $U_1, \ldots, U_{j^*}$ as lower and upper bounds on the part sizes, respectively, and with the original A_i's. It now follows from the strict version of Theorem 5 that for some $1 \leq j^+ \leq j^*$:

  (i) $|\pi_s| = U_s$ for $s = 1, \ldots, j^+ - 1$,
  (ii) $|\pi_s| = L_s$ for $s = j^+ + 1, \ldots, j^*$.

Examining the corresponding partitioning problem of N^-, we get from similar arguments (using the results of Theorem 5 about nonpositive A) that for some $j^* + 1 \leq j^- \leq p$:

  (iii) $|\pi_s| = U_s$ for $s = j^- + 1, \ldots, p$,
  (iv) $|\pi_s| = L_s$ for $s = j^* + 1, \ldots, j^- - 1$.

In particular, π satisfies (11).

Case II—$\pi_{j^*} \cap N^- \neq \emptyset$, $|\pi_{j^*}| = L_{j^*}$ and $1 < j^* < p$: As $\pi_1, \pi_2, \ldots, \pi_{j^*-1}$ partition $N^+ \backslash \pi_{j^*}$, the optimality of π implies that this partition is optimal for the problem of partitioning $N^+ \backslash \pi_{j^*}$ into $j^* - 1$ parts with $L_1, \ldots, L_{j^*-1}$ and $U_1, \ldots, U_{j^*-1}$ as lower and upper bounds on the part sizes, respectively, and with the original A_i's. The arguments of Case I establish the existence of an index $j^+ \geq 1$ satisfying conditions (i)–(ii), except that we have $j^+ < j^*$. The arguments of Case I also establish the existence of an index j^- satisfying (iii)–(iv), where as in Case I we have $j^- > j^*$. So, again, π satisfies (11).

Case III—$\pi_{j^*} \cap N^- \neq \emptyset$, $|\pi_{j^*}| = L_{j^*}$, and $j^* = 1$ and/or $j^* = p$: The arguments of Case II apply, except that $j^+ = 1$ in the case where $j^* = 1 < p$ and $j^- = p$ in the case where $j^* = p > 1$. If $j^* = 1 = p$, π satisfies (11) trivially with $j^- = j^+ = 1$.

Case IV—$\pi_{j^*} \cap N^- \neq \emptyset$ and $|\pi_{j^*}| > L_{j^*}$: As $\pi_1, \pi_2, \ldots, \pi_{j^*-1}, \pi_{j^*} \cap N^+$ partition N^+, the optimality of π implies that this partition is optimal for the problem of partitioning N^+ into j^* parts with $L_1, \ldots, L_{j^*-1}, |\pi_{j^*} \cap N^+| - 1$ and $U_1, \ldots, U_{j^*-1}, |\pi_{j^*} \cap N^+|$ as lower and upper bounds on the part sizes, respectively, and with the original A_i's. In the optimal partition $\pi_1, \pi_2, \ldots, \pi_{j^*-1}, \pi_{j^*} \cap N^+$, the size of the last set is strictly greater than its lower bound; consequently, the strict version of Theorem 5 implies that $|\pi_s| = U_s$ for $s = 1, \ldots, j^* - 1$.

Considering a corresponding partitioning problem of N^-, we get from similar arguments that $|\pi_s| = U_s$ for $s = j^* + 1, \ldots, p$. So, π satisfies (11) with $j^- = j^+$.

Finally, if either of the inequalities of (2) or (3) does not hold strictly, then the coordinates of A and C may be perturbed so that all of the inequalities of (2) and (3) hold strictly. Specifically, for each positive ϵ, let $A^\epsilon \in R^n$ and $C^\epsilon \in R^p$ be defined by $(A^\epsilon)_i = A_i + \epsilon^i$ for $i = 1, \ldots, n$ and $(C^\epsilon)_j = C_j + \epsilon^j$ for $j = 1, \ldots, p$, and let π^ϵ be (any selection of) a corresponding optimal partition; in particular, our earlier arguments show that each π^ϵ satisfies (11) with corresponding integers $(j^+)^\epsilon$ and $(j^-)^\epsilon$. As the number of partitions is finite, there exists a sequence of positive numbers $\epsilon_1, \epsilon_2, \cdots$ with $\lim_{k \to \infty} \epsilon_k = 0$, where π^{ϵ_k} is independent of k. Let π be a common optimal partitions corresponding to the elements of such a sequence and let j^+ and j^- be corresponding integers. It then follows from continuity arguments that π is optimal for the original partitioning problem with unperturbed data; further, π satisfies (11). $\square$

The shape structure and the arguments of the proof of Theorem 7 resemble the notion of quasiconsecutiveness of [2] and the arguments used therein.

The following example demonstrates no uniqueness result for optimal partitions is available for the general linear one-dimensional bounded-shape partitioning problems, even when the inequalities of (2) and (3) are strict.

EXAMPLE 1: Let $n = 4, p = 3, L = (1,1,1), U = (2,1,2), A = (4,1,-2,-3)$, and $C = (4,2,1)$ and consider the corresponding linear bounded-shape partitioning problem. By the strict version of Theorem 1 all optimal partitions are consecutive, namely, it suffices to consider $\pi^1 \equiv (\{1,2\}, \{3\}, \{4\})$ and $\pi^2 \equiv (\{1\}, \{2\}, \{3,4\})$. As $C(\pi^1) = C(\pi^2) = 13$, the optimal partitions are precisely π^1 and π^2, both of which have the structure of (11).

COROLLARY 8 (Solution of Linear Bounded-Shape Partitioning Problems): Linear bounded-shape partitioning problems can be solved in time $O[n(\log n) + p^2 n]$.

PROOF: Let A, C, L, U, and F be as in Theorem 8. We will assume that $\sum_{j=1}^{p} L_j < n < \sum_{j=1}^{p} U_j$ [the cases where $\sum_{j=1}^{p} L_j = n$ or $\sum_{j=1}^{p} U_j = n$ are ignored as they imply that $\Gamma^{(L,U)}$ contains a single vector, namely, $(L_1, \ldots, L_p)$ or $(U_1, \ldots, U_p)$, respectively, and the solution of single-shape problems has already been provided]. Our approach is to identify a set of shapes that includes one of an optimal partition; such a set facilitates the solution of the partitioning problem as a partition with the highest objective among those that share a prescribed shape is available from Theorem 1. Again, conditions (2) and (3) can be achieved by sorting the coordinates of A and C and renumbering indices, tasks that can be accomplished in time $O[n(\log n)]$ and $O[p(\log p)]$, respectively.

A shape of an optimal partition is available from (11) and the specification of the four parameters, namely, $j^+, j^-, |\pi_{j^+}|$ and $|\pi_{j^-}|$; in fact, as the sum of the sizes of the parts of a partition must be n, we have that j^+ and $|\pi_{j^+}|$ uniquely determine the potential shape through (11) with j^- as the first index t with $\sum_{u=1}^{j^+ - 1} U_u + |\pi_{j^+}| + \sum_{u=j^+ + 1}^{p} L_u + \sum_{u=j^+ + 1}^{t} (U_u - L_u) \geq n$ (we note that though the shape is uniquely determined, j^- need not be uniquely determined, e.g., it could be the last index t satisfying the above inequality).

Let $N^+ \equiv \{i : A_i \geq 0\}$ and $N^- \equiv N \backslash N^+$. Let $x \in \{1, \ldots, p\}$ satisfy $\sum_{u=1}^{x-1} U_u < |N^+|$ and $\sum_{u=1}^{x-1} U_u + \sum_{u=x}^{p} L_u \leq |N|$ and $v \in \{L_w, L_w + 1, \ldots, U_w\}$ satisfy $\sum_{u=1}^{x-1} U_u + v + \sum_{u=x+1}^{p} L_u \leq n$ and $\sum_{u=1}^{x-1} U_u + v + \sum_{u=x+1}^{p} U_u \geq n$. The argument of the above paragraph (with x replacing j^+ and v replacing $|\pi_{j^+}|$) show that there exists a unique integer vector

$(n_1, \ldots, n_p)$ in $\Gamma^{(L,U)}$ having structure as in (11) and with $n_x = v$. For each such pair (x, v), we let $\pi(x, v)$ be the partition with $\pi(x, v)_j = \{\sum_{u=1}^{j-1} n_u + 1, \ldots, \sum_{u=1}^{j} n_u\}$ for $j = 1, \ldots, p$ and we evaluate $C[\pi(x, v)]$. Theorems 7 and 1 assure that there is an optimal partition among those generated by the above scheme; hence the generated partition with the maximal $C(\cdot)$-value is optimal. [We observe that, in view of Theorem 7, some further pairs (x, v) can be discarded, for example, those for which there exists an index $z > x$ with $\sum_{u=1}^{z-1} n_u < |N^+|$ and $n_z > L_z$.]

The evaluation of the shape associated with a pair (x, v) requires effort $O(p)$ (see the paragraph preceding Theorem 5). However, if for a given x, shape vectors corresponding to multiple v's are to be determined, the evaluations following the first one require only a single shift and can each be done by effort requiring time $O(1)$ with total effort of $O(p + U_x - L_x)$. Also, the evaluation of the objective of a partition corresponding to a particular shape is $O(n)$, but the effort needed for updates corresponding to a unit increase of v is $O(p)$. So, using n as a bound on $U_x - L_x$, the total effort for determining an optimal partition is $O(p^2 n)$.

Block sorting can be incorporated into the above algorithm, but partitions corresponding to numerous shapes have to be evaluated, each requiring a separate block-sorting. Consequently, the use of block sorting does not improve complexity bounds; further, its usefulness for practical computation seems to be of limited interest.   □

COROLLARY 9 (Solution of Convex Bounded-Shape Partitioning Problems):   Convex bounded-shape partitioning problems can be solved in time $O[n^2 p^2 p!]$ arithmetic operations and $O[npp!]$ evaluation of the underlying convex function C; an alternative bound is $O[n^2 pp! + n(\log n)]$ arithmetic operations and $O[npp!]$ evaluation of C.

PROOF: As in the proofs of Theorem 3 and Corollary 4, a set of partitions that is known to contain an optimal solution for linear problems contains solutions to convex problems. In the proof of Corollary 8 we constructed such a set for linear bounded-shape partitioning problems. For each potential permutation of the C_j's, we get a set of at most np partitions determined by the corresponding parameters j^+ and $|\pi_{j^+}|$, totaling $npp!$ partitions. For each partition, A^π can be evaluated with n additions and $F(\pi)$ can be determined with an evaluation of C. The analysis of the proof of Corollary 4 shows that when the A_i's are fully sorted (with $O[n(\log n)]$ comparisons), we get a complexity bound of $O[n^2 pp! + n(\log n)]$ arithmetic operations and $O[npp!]$ evaluations of C; when the full sorting of the A_i's is avoided and block sorting is used for evaluating each of the corresponding $O[npp!]$ partitions, we get a complexity bound of $O[n^2 p^2 p!]$ arithmetic operations and $O[npp!]$ evaluation of C.   □

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