# Equivalent Nondegenerate L-Shapes of Double-Loop Networks

**Chiuyuan Chen**
*Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan, Republic of China*

**F.K. Hwang**
*Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan, Republic of China*

**Double-loop networks have been widely studied as architecture for local area networks. The L-shape is an important tool for studying the distance properties of double-loop networks. Two L-shapes are equivalent if the numbers of nodes $k$ steps away from the origin are the same for every $k$. Hwang and Xu first studied equivalent L-shapes through a geometric operation called 3-rectangle transformation. Fiol et al. proposed three equivalent transformations. Rödseth gave an algebraic operation, which was found by Huang et al. to correspond to 3-rectangle transformations. In this paper, we show that all equivalent nondegenerate L-shapes are determined by four basic geometric operations. We also discuss the algebraic operations corresponding to these geometric operations.** 

**Keywords:** double-loop network; L-shape; diameter; Euclidean algorithm

## 1. INTRODUCTION

A double-loop network $DL(N; a, b)$ has $N$ nodes $0, 1, \ldots, N-1$ and $2N$ links of two types:

$$a\text{-links: } i \to i + a \,(\text{mod } N), i = 0, 1, \ldots, N-1,$$
$$b\text{-links: } i \to i + b \,(\text{mod } N), i = 0, 1, \ldots, N-1.$$

Double-loop networks have been widely studied (see [7] for literature) as architecture for local area networks.

The minimum-distance diagram of a double-loop network $DL(N; a, b)$ gives a shortest path from node $u$ to node $v$ for any $u, v$. Since a double-loop network is node-symmetric, it suffices to give a shortest path from node 0 to any other node. Let 0 occupy cell (0, 0). Then, $v$ occupies cell $(i, j)$ if and only if $ia + jb \equiv v$ (mod $N$) and $i + j$ is the minimum among all $(i', j')$ satisfying the congruence, where $\equiv$ means congruent modulo $N$, namely, a shortest path from 0 to $v$ is through taking $i$ $a$-links and $j$ $b$-links (in any order). Note that in a cell $(i, j), i$ is the column index and $j$ is the row index. A minimum-distance diagram includes every node exactly once (in case of two shortest paths, the convention is to choose the cell with the smaller row index, i.e., the smaller $j$). Wong and Coppersmith [9] proved that the diagram is always an L-shape (a rectangle is considered a degeneration). See Figure 1 for two examples.

Let $d(k)$ denote the number of cells $(i, j)$ in an L-shape such that $i + j = k$. Hwang and Xu [6] defined two double-loop networks, or two L-shapes, to be *equivalent* if they have the same $d(k)$ for every $k$. Note that two equivalent double-loop networks have the same diameter and average distance. Two double-loop networks $DL(N; a, b)$ and $DL(N; a', b')$ are called *strongly isomorphic* [5] if there exists a $z$ prime to $N$ such that $a' \equiv az, b' \equiv bz$ (mod $N$). It can be easily seen that two strongly isomorphic double-loop networks are equivalent, but the reverse is not true.

Hwang and Xu [6] proved that $DL(N; 1, s)$ and $DL(N; 1, N+1-s)$ are equivalent by showing that they correspond to different ways of piling up three

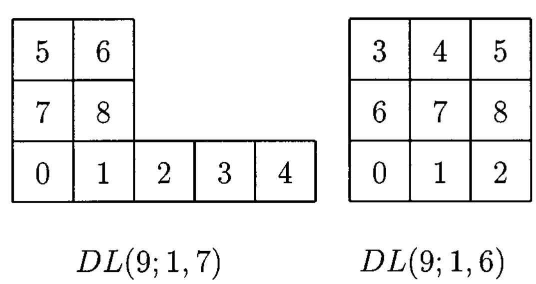


FIG. 1. Two examples of L-shapes.

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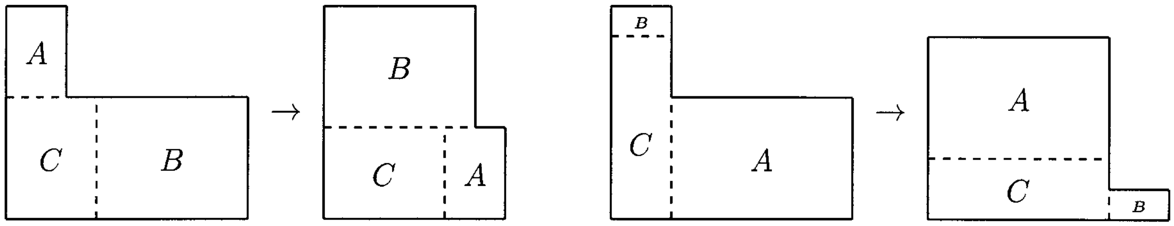


FIG. 2. The 3-rectangle transformations.

rectangles. We call this the *3-rectangle transformation*. Fiol et al. [3] proposed three equivalent transformations, which are called $T, B$, and $FV$ in this paper (see Section 4). Rödseth [8] considered the multiloop networks $ML(N; S)$, where $S = \{s_1, s_2, \ldots, s_l\}$ and the type-$j$ links are $i \to i + s_j \pmod N$, $j = 1, 2, \ldots, l$. Let $\bar{S} = S \cup \{0\}$. He proved that $ML(N; S)$ and $ML(N; S')$ are equivalent if $S' = \{s_i - s | s_i \in \bar{S}, s_i \neq s\}$ for some $s \in \bar{S}$. In particular, $DL(N; a, b)$ is equivalent to $DL(N; N - a, b - a)$ and $DL(N; a - b, N - b)$. Since $-1$ is prime to $N, DL(N; (-1)(N - a), (-1)(b - a)) = DL(N; a, a - b)$ is equivalent to $DL(N; a, b)$. The Hwang–Xu result then corresponds to the special case $a = 1$. Similarly, $DL(N; b-a, b)$ is also equivalent to $DL(N; a, b)$. Huang et al. [4] proved that Rödseth's theorem yields only the 3-rectangle transformations [the original one and a dual-type corresponding to $DL(N; b - a, b)$; see Fig. 2].

In this paper, we determine the spectrum of all equivalent transformations for nondegenerate L-shapes and prove that they can all be derived from four transformations. In particular, the 3-rectangle transformations can be obtained by a composition of two such transformations. We also discuss how the transformations affect the parameters $(a, b)$. We give an algorithm to compute the new $(a, b)$, but we are unable to obtain a Rödseth-like theorem for these transformations.

## 2. SOME PRELIMINARY REMARKS

Since there is only one nondegenerate L-shape with three cells, from now on we will only talk about nondegenerate L-shapes with at least four cells.

Let the segments on the boundary of an L-shape be labeled as shown in Figure 3. Since $h$ and $l$ can be determined from the other four parameters, an L-shape can also be denoted by $L(m, n, p, q)$ using its geometric parameters. Note that $N = lh - pn$.

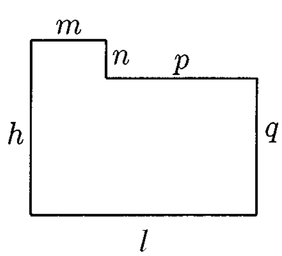


FIG. 3. An L-shape with parameters.

Fiol et al. [3] showed that an L-shape always tessellates the plane. By considering their relative positions of lattice points occupied by node 0, they derived the following congruences:

$$\begin{aligned} la - nb &\equiv 0 \pmod N \\ -pa + hb &\equiv 0 \pmod N). \end{aligned} \qquad (1)$$

They also stated that the solution $(a, b)$ of (1) is unique up to strong isomorphism.

For a given L-shape $L(m, n, p, q)$, we call the set of cells of distance $d$ from the origin the *$d$-diagonal*. A $d$-diagonal is *complete* if it contains $d + 1$ cells. Note that if $L$ has a complete $d$-diagonal then it has a complete $d'$-diagonal for $0 \leq d' < d$. The set of complete diagonals form a staircase $S(L)$ whose order $\|S(L)\|$ is defined to be $\max\{d :$ the $d$-diagonal is complete$\}$. It is easily verified that $\|S(L)\| = \min\{n + q, m + q, m + p\} - 1$. A staircase of order $d$ is called a *$d$-staircase*.

A $(k, d), k \leq d$, *jigsaw piece* is a $(d - 1)$-staircase missing the lines of lengths $1, 2, \ldots, k-1$. A $(k, d)$ jigsaw piece with $k = 1$ will be treated as a staircase, while a $(k, d)$ jigsaw piece with $k > 1$ will be called *wide*. The *length-set* of a $(k, d)$ jigsaw piece $J$ is $l(J) = \{k, k + 1, \ldots, d\}$. For example, consider the L-shape in Figure 4(a): $\|S(L)\| = 3, S(L)$ is a 3-staircase, $J_t(L)$ is a (1, 2) jigsaw piece (a staircase), $J_b(L)$ is a (2, 3) jigsaw piece (a wide piece), $l(J_t(L)) = \{1, 2\}$, and $l(J_b(L)) = \{2, 3\}$.

It is easily verified that $L \backslash S(L)$ consists of either one or two jigsaw pieces. If $L \backslash S(L)$ consists of two jigsaw pieces, then the two jigsaw pieces are either separated [Fig. 4(a)], including touching at one point [Fig. 4(b)] or one piggybacking on another [Fig. 4(c) and (d)]; $J_t(L)$ and $J_b(L)$ will denote the top and bottom jigsaw pieces, respectively. If $L \backslash S(L)$ consists of only one jigsaw piece $J$, then we set $J_t(L) = \emptyset$ and $J_b(L) = J$ [Fig. 4(e)]. For sets $A = \{a_1, a_1, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}, A * B$ denotes the multiset $\{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n\}$.

**Lemma 1.** *$L$ and $L'$ are equivalent if and only if $S(L) = S(L')$ and $l(J_t(L)) * l(J_b(L)) = l(J_t(L')) * l(J_b(L'))$.*

**Proof.** Suppose that $S(L) \neq S(L')$, say, $\|S(L)\| = d > \|S(L')\|$. Then, the number of cells of distance $d$ from the origin is $d + 1$ in $L$ but less than $d + 1$ in $L'$. Hence, $L$ and $L'$ are not equivalent.

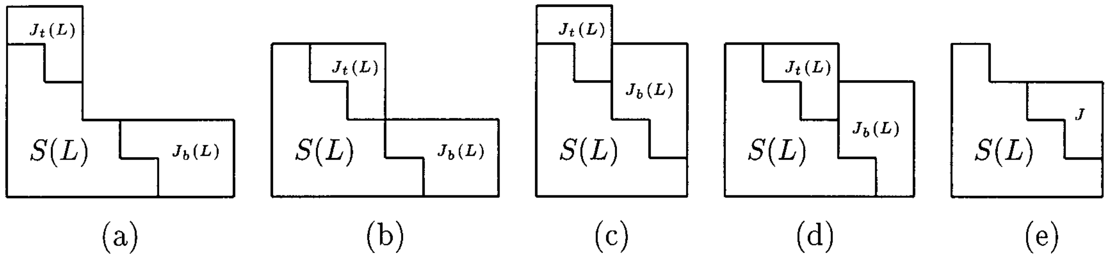


FIG. 4. Decomposing L-shapes into staircases and jigsaw pieces.

When a jigsaw piece is fitted into the ladder, it occupies the cells as close to the origin as possible. Therefore, if we line up the two jigsaw pieces such that the zigzag lines are on the same side (see Fig. 5), then the number of cells in the $i$th column represents the number of cells in the L-shape distance-$(d + i)$ from the origin. Let $\{|C_i(J_t(L) \cup J_b(L))|\}$ denote the set of column sizes. It is easily verified that if $l(J_t(L)) * l(J_b(L)) \neq l(J_t(L')) * l(J_b(L'))$ then $\{|C_i(J_t(L) \cup J_b(L))|\} \neq \{|C_i(J_t(L') \cup J_b(L'))|\}$. Consequently, $L$ and $L'$ are not equivalent.

On the other hand, similar arguments also show that if equalities hold for $S$ and $l$ then $L$ and $L'$ are equivalent. ∎

The following lemma is easily verified; therefore, we omit the proof.

**Lemma 2.** $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\| - 1$ *or* $\|S(L)\|$ *or* $\|S(L)\| + 1$.

Since $l(J)$ is a set of consecutive integers, there are three cases:

(i) $l(J_t(L)) \cap l(J_b(L)) = \emptyset$ and $l(J_t(L)) * l(J_b(L))$ is not consecutive.
(ii) $l(J_t(L)) \cap l(J_b(L)) = \emptyset$ and $l(J_t(L)) * l(J_b(L))$ is consecutive.
(iii) $l(J_t(L)) \cap l(J_b(L)) \neq \emptyset$.

**Lemma 3.** *No $DL(N; a, b)$ exists which yields an L-shape for case (i).*

**Proof.** Suppose that $l(J_t(L)) = \{k, k + 1, \ldots, d\}$ and $l(J_b(L)) = \{k', k' + 1, \ldots, d'\}$, where $k' \geq d + 2$. We set $\|S(L)\| = s$ for easy writing. Consider two subcases (see Fig. 6):

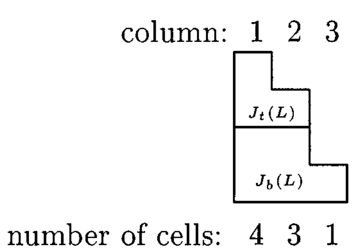


FIG. 5.

(a) $J_t(L)$ appears in column 1 of the L-shape. Then, since $k' > d + 1$,

$$h = s + 1 + k < s + 1 + k + k' - (d + 1) = s + 1 + k' - (d - k + 1) = l - m = p.$$

(b) $J_t(L)$ does not appear in column 1. Then, since $k' > d + 1$,

$$h = s + 1 < s + 1 + k' - (d + 1) = l - m = p.$$

It was proved in [1] that $h \geq p$ is a necessary condition for the existence of $DL(N; a, b)$. Hence, the lemma follows. ∎

For case (ii), consider the following subcases:

(a) $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\| - 1$ [see Fig. 7(a)].
(b) $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\|$ [see Fig. 7(b) and (b′)]. Let $l(J_t(L)) * l(J_b(L)) = \{k, k+1, \ldots, k + \|S(L)\| - 1\}$. If $k \geq 2$, then we will have $h < p$ and this violates the fact that $h \geq p$ is a necessary condition for an L-shape [1]. Hence, $k = 1$ and it is easily verified that the L-shape in Figure 7(b′) is equivalent to that in Figure 7(b), which is a degenerate L-shape (a square).
(c) $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\| + 1$ [see Fig. 7(c)]. Then, there exists a fitting which yields a degenerate L-shape (a rectangle).

Therefore, it suffices to consider subcase (a) of case (ii) and case (iii) for the equivalence of nondegenerate L-shapes.

## 3. EQUIVALENT L-SHAPES

Let $F$ denote the *flipping* operation, that is, $F$ transforms $L(m, n, p, q)$ to $L(q, p, n, m)$. It is easily seen that flipping preserves equivalence, and if $L$ is the L-shape of $DL(N; a, b)$, then $F(L)$ is the L-shape of $DL(N; b, a)$. We determine how many equivalent L-shapes can be formed.

**Lemma 4.** *Suppose that subcase (a) of case (ii) occurs and flipping is not considered. Then, there is only one equivalent L-shape.*

**Proof.** It is easily verified that $l(J_t(L)) * l(J_b(L)) = \{1, 2, \ldots, \|S(L)\| - 1\}$ and the only way to fit the jigsaw pieces with $S(L)$ is to combine the two jigsaw pieces into one staircase [see Fig. 7(a)]. Thus, only two equivalent L-shapes, mutually obtainable by flipping, can be

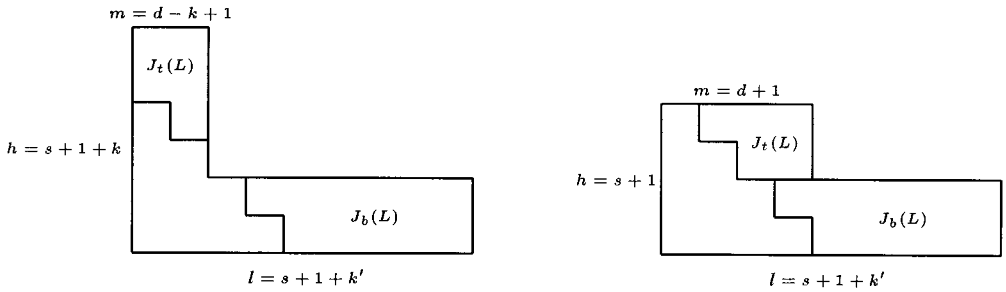


FIG. 6. Two subcases for case (i).

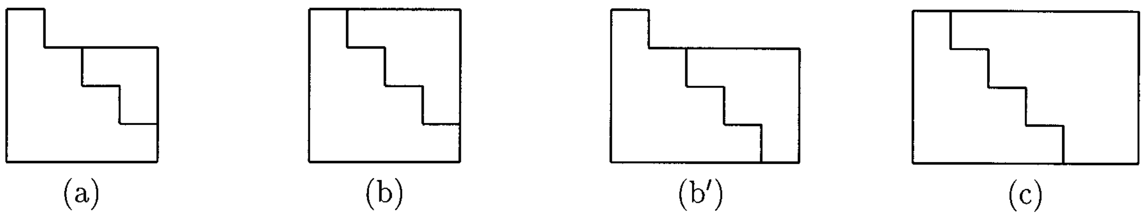


FIG. 7. Three subcases for case (ii).

formed. When flipping is not considered, there is only one equivalent L-shape, that is, $L$ itself. ▪

Now consider case (iii). Since flipping an L-shape clearly preserves equivalence, we will only discuss the nonflipping types. First, we determine how many equivalent L-shapes can be formed with fixed $J_t(L)$ and $J_b(L)$.

**Lemma 5.** *Suppose that case* (*iii*) *occurs* [*i.e.*, $l(J_t(L)) \cap l(J_b(L)) \neq \emptyset$], $J_t(L)$ *and* $J_b(L)$ *are fixed, and flipping is not considered*:

*If* $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\| - 1$, *then at most one equivalent L-shape can be formed.*

*If* $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\|$, *then at most two* (3) *equivalent L-shapes can be formed if* $(J_t(L), J_b(L))$ *consists of one staircase and one wide piece* (*two staircases*).

*If* $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\| + 1$, *then at most four* (3, 3) *L-shapes can be formed if* $(J_t(L), J_b(L))$ *consists of one staircase and one wide piece* (*two staircases*, *two wide pieces*).

**Proof.** Suppose that $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\| - 1$. Then, it is easily verified that $(J_t(L), J_b(L))$ consists of two staircases and at most one equivalent L-shape can be formed (see Fig. 8).

Suppose that $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\|$. Then, it is impossible that $(J_t(L), J_b(L))$ consists of two wide pieces, since, otherwise, we will have $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\| + 1$. If $(J_t(L), J_b(L))$ consists of one staircase and one wide piece, then at most two equivalent L-shapes can be formed (see Fig. 9). If $(J_t(L), J_b(L))$ consists of two staircases, then at most three equivalent L-shapes can be formed (see Fig. 10).

Suppose that $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\| + 1$. If $(J_t(L), J_b(L))$ consists of one staircase and one wide piece; then, at most four equivalent L-shapes can be formed [see Fig. 4(a)–(d)]. If $(J_t(L), J_b(L))$ consists of two staircases, then at most three equivalent L-shapes can be formed (see Fig. 11). If $(J_t(L), J_b(L))$ consists of two wide pieces, then at most three equivalent L-shapes can be formed (see Fig. 12). ▪

Finally, we replace the condition that $J_t(L)$ and $J_b(L)$ are fixed by the condition that $l(J_t(L)) * l(J_b(L))$ is fixed. As before, we will only discuss the nonflipping types. More specifically, we assume that if the parameters of $J_t(L)$ and $J_b(L)$ are $(k, d)$ and $(k', d')$, respectively, then $d \leq d'$.

**Lemma 6.** *Suppose that case* (*iii*) *occurs* [*i.e.*, $l(J_t(L)) \cap l(J_b(L)) \neq \emptyset$] *and flipping is not considered*:

*If* $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\| - 1$, *then at most one equivalent L-shape can be formed.*

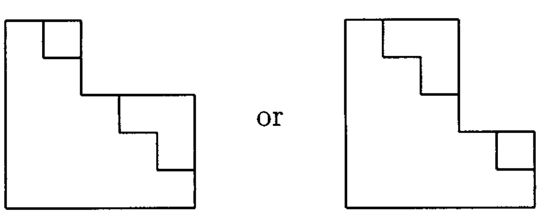


FIG. 8.

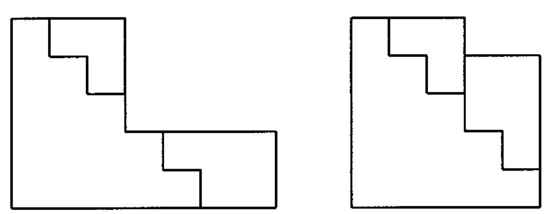

FIG. 9.

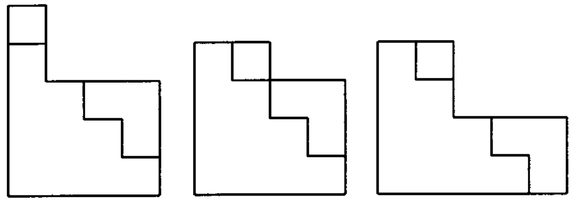

FIG. 10.

*If* $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\|$, *then at most four* (3) *equivalent L-shapes can be formed if* $(J_t(L), J_b(L))$ *consists of one staircase and one wide piece* (*two staircases*).

*If* $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\| + 1$, *then at most six* (3, 6) *L-shapes can be formed if* $(J_t(L), J_b(L))$ *consists of one staircase and one wide piece* (*two staircases*, *two wide pieces*).

**Proof.** Suppose that $l(J_t(L)) = \{k, k+1, \ldots, d\}$ and $l(J_b(L)) = \{k', k'+1, \ldots, d'\}$. Then, $l(J_t(L)) * l(J_b(L))$ can be decomposed into three parts:

(i) The set $M_1 = \{\min\{k,k'\}, \min\{k,k'\}+1, \ldots, \max\{k,k'\}-1\}$.
(ii) The multiset $M_2 = \{\max\{k,k'\}, \max\{k,k'\}, \max\{k,k'\}+1, \max\{k,k'\}+1, \ldots, d, d\}$ in which every number appears twice.
(iii) The set $M_3 = \{d+1, d+2, \ldots, d'\}$.

Note that $M_2$ is always nonempty, $M_1$ is empty if $k = k'$, and $M_3$ is empty if $d = d'$.

In splitting $l(J_t(L)) * l(J_b(L))$ into two length sets $l(J_t(L'))$ and $l(J_b(L'))$, $M_2$ must be split into two identical sets, say, $M_2'$ and $M_2''$. By convention, $M_3$ goes to $J_b(L')$. There are two conditions of $L'$ depending on whether $M_1$ goes to $J_t(L')$ or $J_b(L')$. Hence, there are at most two ways of splitting $l(J_t(L)) * l(J_b(L))$. Moreover, if $(J_t(L), J_b(L))$ consists of two staircases, then $k = k' = 1$ and $M_1 = \emptyset$; thus, there is only one way of splitting $l(J_t(L)) * l(J_b(L))$.

Suppose that $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\| - 1$. Then, $(J_t(L), J_b(L))$ consists of two staircases. Hence, there is only one way of splitting $l(J_t(L)) * l(J_b(L))$; by Lemma 5, at most one equivalent L-shape can be formed.

Suppose that $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\|$. As was mentioned in the proof of Lemma 5, it is impossible that $(J_t(L), J_b(L))$ consists of two wide pieces. If $(J_t(L), J_b(L))$ consists of one staircase and one wide piece, then there are two ways of splitting $l(J_t(L)) * l(J_b(L))$; by Lemma 5, at most four equivalent L-shapes can be formed [see Fig. 13(a)–(d)]. If $(J_t(L), J_b(L))$ consists of two staircases, then there is only one way of splitting $l(J_t(L)) * l(J_b(L))$; by Lemma 5, at most three equivalent L-shapes can be formed.

Suppose that $|l(J_t(L))| + |l(J_b(L))| = \|S(L)\| + 1$. If $(J_t(L), J_b(L))$ consists of one staircase and one wide piece, then there are two ways of splitting $l(J_t(L)) * l(J_b(L))$; by Lemma 5, at most eight equivalent L-shapes can be formed. But a type [Fig. 4(c)] with the split

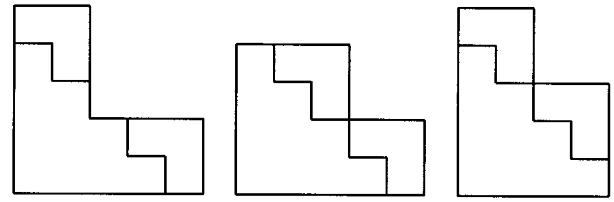

FIG. 11.

$(M_1, M_2')$ and $(M_2'', M_3)$ overlaps a type with the split $(M_1, M_2', M_3)$ and $(M_2'')$, and a type [Fig. 4(d)] with the split $(M_1, M_2')$ and $(M_2'', M_3)$ also overlaps a type with the split $(M_1, M_2', M_3)$ and $(M_2'')$ (see Fig. 14). Therefore, at most six equivalent L-shapes can be formed. If $(J_t(L), J_b(L))$ consists of two staircases, then there is only one way of splitting $l(J_t(L)) * l(J_b(L))$; by Lemma 5, at most three equivalent L-shapes can be formed. If $(J_t(L), J_b(L))$ consists of two wide pieces, then there are two ways of splitting $l(J_t(L)) * l(J_b(L))$; by Lemma 5, at most six equivalent L-shapes can be formed. ∎

**Theorem 7.** *There are at most six equivalent L-shapes* (*not considering flipping*).

**Proof.** This theorem follows from Lemma 4 and Lemma 6. ∎

We give an example of six equivalent L-shapes in Figure 15. In this example, $(J_t(L), J_b(L))$ consists of one staircase and one wide piece.

## 4. GEOMETRICAL INTERPRETATION AND ALGEBRAIC RELATIONS OF EQUIVALENT L-SHAPES

From Theorem 7, there are at most six equivalent L-shapes when flipping is not considered. In Figure 15, we illustrate three geometric operations $T, B, V$ and their relation to the six equivalent L-shapes. $T(L)$ can be seen to be obtained from $L$ by turning the top rectangle 90 deg around [see Fig. 16(a)] and $B(L)$ is obtained from $L$ by turning the bottom rectangle [see Fig. 16(b)]. $V(L)$ is obtained from $L$ by interchanging the horizontal side and the vertical side, namely, from $L(m,n,p,q)$ to $L(n+q-p, n, p, m+p-n)$ [see Fig. 16(c)]. The other two L-shapes can each be obtained from three different ways as marked.

By definition, the following is easily seen:

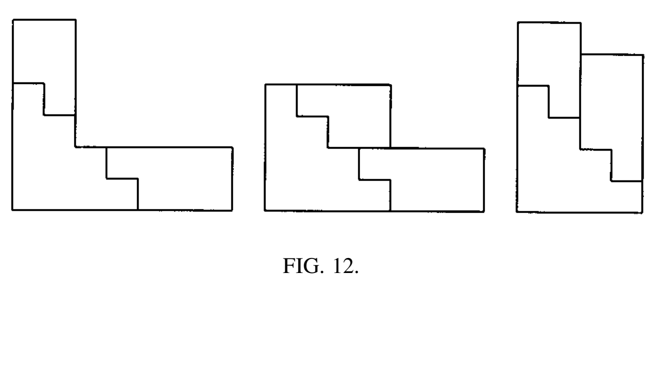

FIG. 12.

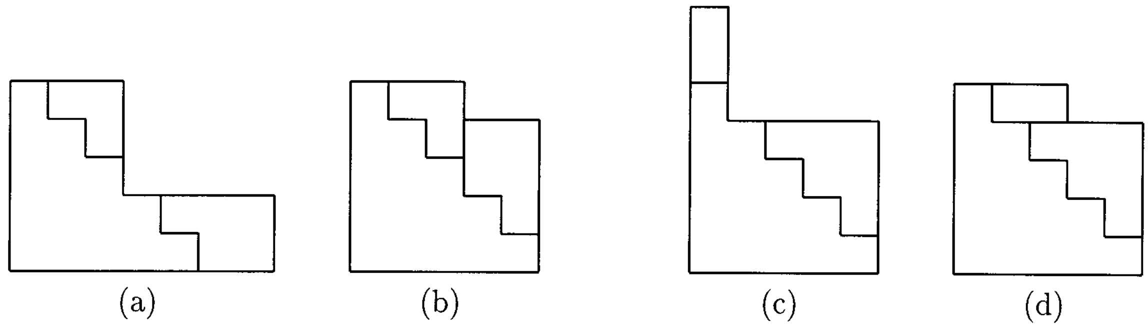


FIG. 13.

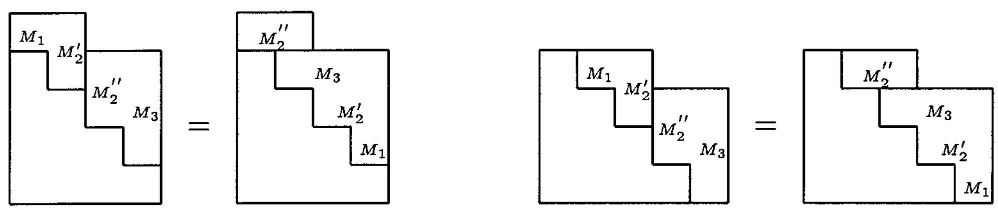


FIG. 14.

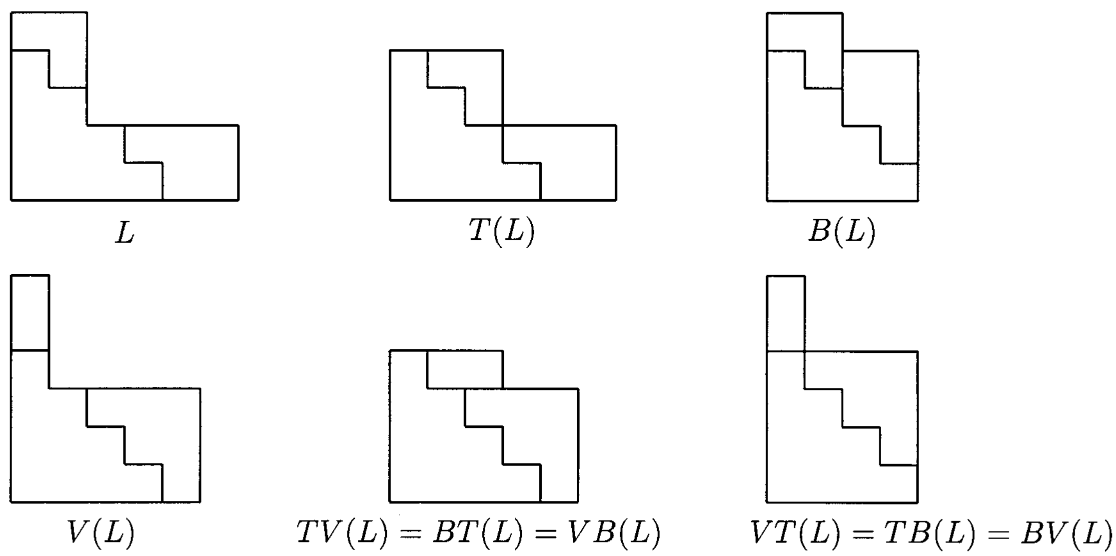


FIG. 15. The six equivalent L-shapes.

**Lemma 8.** $F^{-1} = F, T^{-1} = T, B^{-1} = B, V^{-1} = V$.

Thus, all equivalent L-shapes can be obtained through the four transformations $F, T, B$, and $V$. In particular, the 3-rectangle transformations $H$ and $\bar{H}$ can also be expressed in terms of $F, T, B$, and $V$.

**Theorem 9.** $FH = TV(= BT = VB)$. $F\bar{H} = VT(= TB = BV)$.

**Proof.**

$$L(m,n,p,q) \xrightarrow{V} L(n+q-p, n, p, m+p-n) \xrightarrow{T} L(n, n+q-p, q, m+p-n).$$

$$L(m,n,p,q) \xrightarrow{T} L(n, m, m+p-n, q) \xrightarrow{V} L(n+q-p, m, m+p-n, p).$$

We show the changes in parameters for $H$ and $\bar{H}$ in Figure 17(a) and (b), respectively. Theorem 9 follows immediately. ■

Rödseth proved that if $L' = H(L)$ then $a' = N - a, b' \equiv b - a \pmod N$; if $L' = \bar{H}(L)$, then $a' \equiv a - b \pmod N$, $b' = N - b$. It is also easily seen that if $L' = F(L)$ then $a' = b$ and $b' = a$. In this section, we study the same relation between $L$ and $FV(L)$. Note that once this relation is determined then the relation between $L$ and $V(L)$ is known from the equation $V = F^{-1}FV$ and the relations between $L$ and $T(L)$ and between $L$ and $B(L)$ are known from the equation $FH = TV = BT = VB$.

Esqué et al. [2] proposed a method of computing $a$ and $b$ such that $DL(N; a, b)$ realizes $L(m,n,p,q)$. They

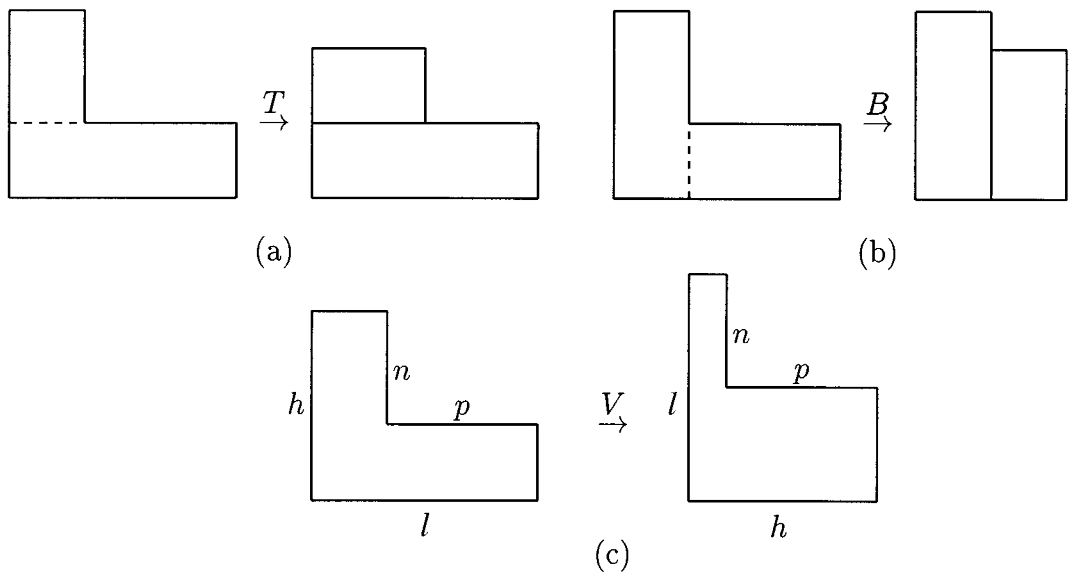


FIG. 16. Geometric interpretations.

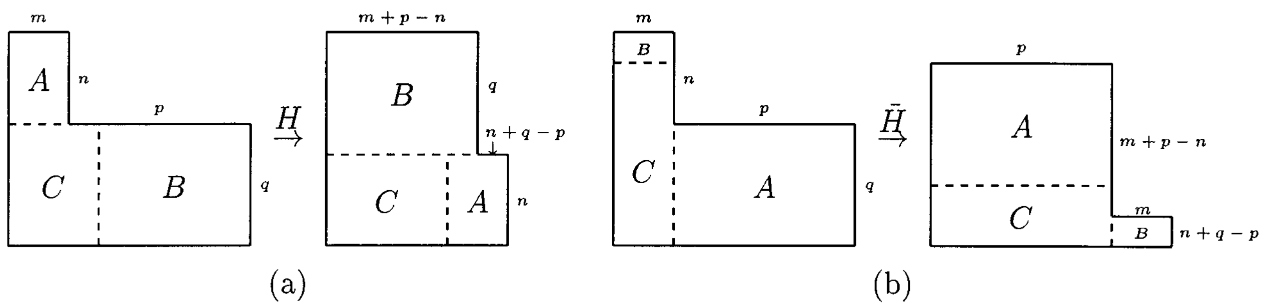


FIG. 17.

considered the integral matrix

$$\mathcal{M} = \begin{pmatrix} l & -p \\ -n & h \end{pmatrix}$$

and computed the Smith normal form of $\mathcal{M}, \mathcal{S}(\mathcal{M}) = \text{diag}(1, N)$. Then, $\mathcal{S}(\mathcal{M}) = \mathcal{L}\mathcal{M}\mathcal{R}$, where $\mathcal{L}$ and $\mathcal{R}$ are two nonsingular unimodular integral matrices. They proved that if

$$\mathcal{L} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

then $a = \gamma \pmod N$ and $b = \delta \pmod N$ in $DL(N; a, b)$.

If $\mathcal{L}\mathcal{M}\mathcal{R} = \mathcal{S}(\mathcal{M})$, then $\mathcal{R}^T\mathcal{M}^T\mathcal{L}^T = [\mathcal{S}(\mathcal{M})]^T = \text{diag}(1, N) = \mathcal{S}(\mathcal{M}^T)$, where $T$ indicates the transpose of a matrix. Let $DL(N; a', b')$ be the double-loop network whose L-shape is $FV(L(m, n, p, q))$. Then, $\mathcal{M}^T$ is the matrix of $FV(L)$ in Esqué et al.'s computation [2]. Therefore, if

$$\mathcal{R}^T = \begin{pmatrix} \alpha' & \gamma' \\ \beta' & \delta' \end{pmatrix},$$

then $a' = \beta' \pmod N$ and $b' = \delta' \pmod N$ in $DL(N; a', b')$. These observations lead to the following algorithm of computing $a'$ and $b'$ without the need of computing the Smith normal form $\mathcal{S}(\mathcal{M})$ first.

**Theorem 10.** *Suppose that* $1 \le a, b \le N - 1$. *Let* $x$ *and* $y$ *be integers such that* $bx - ay = 1$. *Then,* $DL(N; a', b')$ *with* $a' = px - hy$ *(mod* $N$*) and* $b' = lx - ny \pmod N$ *realizes* $FV(L)$.

**Proof.** Suppose that $1 \le a, b \le N - 1$. If $\gcd(a, b) = d$, take $a^* = a/d$ and $b^* = b/d$. Since $\gcd(N, a, b) = 1$ implies that $\gcd(d, N) = 1, DL(N; a, b)$ is strongly isomorphic to $DL(N; a^*, b^*)$. Therefore, we may assume that $d = 1$, since, otherwise, we could work with $(a^*, b^*)$. Let $x$ and $y$ be integers such that $bx - ay = 1$ and let

$$\mathcal{L} = \begin{pmatrix} \frac{hb-pa}{N} & \frac{-la+nb}{N} \\ px - hy & lx - ny \end{pmatrix},$$

$$\mathcal{M} = \begin{pmatrix} l & -n \\ -p & h \end{pmatrix}, \quad \text{and } \mathcal{R} = \begin{pmatrix} x & a \\ y & b \end{pmatrix}.$$

By (1), $(hb - pa)/N$ and $(-la + nb)/N$ are integers. $\mathcal{M}$ is the corresponding matrix of $FV(L)$. It is easily verified that both $\mathcal{L}$ and $\mathcal{R}$ are nonsingular unimodular integral matrices and $\mathcal{L}\mathcal{M}\mathcal{R} = \text{diag}(1, N)$, the Smith normal form $\mathcal{S}(\mathcal{M})$ of $\mathcal{M}$. By the argument in [2], $DL(N; a', b')$ with

$a' = px - hy \pmod N$ and $b' = lx - ny \pmod N$ realizes $FV(L)$. ▪

Note that $(x, y)$ in $bx - ay = 1$ can be solved by the Euclidean algorithm which takes $O(\log N)$ time. However, we are unable to obtain a Rödseth-like theorem for $FV$.


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