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Theory and Methodology

Multiaction maintenance subject to action-dependent risk and stochastic failure

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Abstract

Although Markov maintenance problems have received extensive attention, relatively few studies have addressed multiple nonreplacement maintenance actions for a multistate system with action-dependent risk and stochastic failure. In this study, we closely examine such multiple maintenance actions by viewing a coherent multicomponent system as a multistate system and assuming that each maintenance action regarding state improvement has a probabilistic risk which is ineffective. The above problem is also modeled as a Markov maintenance problem, demonstrating that the optimal maintenance policy has easily computed and implemented control limit rule. More specially, results in this study confirm the practical nature of the proposed model with respect to its ability than most conventional Markov maintenance models for some particular systems. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Rapid technological advances have led to increasingly sophisticated fabrication of systems and expensive life cycle cost, resulting in a high demand for the specialized maintenance actions to upgrade operational conditions and ultimately reduce the life cycle cost.

Numerous investigations have modeled the equipment maintenance/replacement problems based on the Markov chain theory. Preinreich (1940) and Terborgh (1949) undertook some of the earliest studies on economic life of available equipment. Those investigations focused on replacing the equipment when the costs involving operations and maintenance in net present value terms were high enough to justify a replacement. This approach has found diverse applications (e.g. Meyer, 1971; White, 1978; Christer and

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Goodbody, 1980; Hatoyama, 1984; Kusaka, 1985; Hopp, 1988; Hopp and Wu, 1990; Bylka et al., 1992; Wu and Chang, 1996; Hontelez et al., 1996). Other noteworthy surveys include those of McCall (1965), Pierskalla and Voelker (1976), Wu (1986), White (1991), Cho and Parlar (1991), Pintelon and Gelders (1992). Despite the applications of above theoretical results, some weaknesses are still found in practice.

Limitations of previous studies based on Markov theory are summarized as follows. (i) For structures, such as buildings and bridges, a situation possibly arises which is worse than the previous one before maintenance owing to the imperfect nature of the maintenance process. Such a situation is referred to herein as an imperfect maintenance action. The above concept differs from the imperfect maintenance models (i.e. imperfect maintenance is normally attribute to the extremely malfunctioning of a system or the incorrect choice of maintenance actions) which have been developed by numerous investigations based on renewal theory and average cost rate function (e.g., Brown and Proschan, 1983; Yun and Bai, 1987; Nakagawa, 1988; Srivastava and Wu (1993); Wang and Pham, 1996). However, all Markov theory-based models do not address this situation. (ii) For expensive and complex systems, the life cycle years are extremely long. In such case, the multiple nonreplacement actions must be considered to reduce the ownership cost over planning life years. However, except for Hopp and Wu (1986, 1988, 1990), no work has been done. (iii) For reducing the unavailability of a system, the unplanned maintenance actions due to the failures between observed epochs should be considered. Although some Markov-based investigations (e.g. Chen and Feldman, 1997) also considered such failures, combining the action-dependent risk, multiple nonreplacement maintenance, and stochastic failures within observed epochs is a more general means of explaining a system's maintenance policy.

To reduce above weaknesses, this study assumes not only the system can be categorized as a list of operational states (not including failed situations) but also that a probabilistic risk occurs when performing a specific maintenance action to improve its existing state (i.e. imperfect maintenance). This study also models this complex problem to obtain an optimal preventive maintenance policy, thereby minimizing the total expected discounted cost over planning horizon (planning life cycle years). The rest of this paper is organized as follows. Section 2 thoroughly describes the problem addressed herein. Section 3 presents the proposed model. Next, Section 4 reveals the structural characteristics of the optimal policy. Section 5 then outlines the solution procedure and illustrations. Actual examples are provided in Section 6. Conclusions and areas for future research are finally made in Section 7.

2. Problem description

Initially, we consider an enterprise that owns a list of multistate systems. Let $S = \{1, 2, \dots, n\}$ be the operational state space of a system (not including failed situations), where 1 denotes the "best" state and n represents the "worst" state. Also, states 1 and n are treated herein as nonrepairable conditions since they are the best and the worst, respectively. On the other hand, except for states 1 and n , any state $i \in S$ will reverse to a better condition than current after performing a specific nonreplacement maintenance action. Given any state $i \in S$ at beginning of each period, the owners must choose an action from an available set $A = \{0, 1, \dots, \omega\}$, $\omega \leq n - 1$, where action 0 represents "do nothing" and maintenance action $0 < a < i$ stochastically moves the system to state a or i or n . When the action $a \geq i$ the system depends on the action type stochastically moves to state n or still remains on state i . Based on the defined action space, we then let partition $\Omega = \{A_0, A_1, A_2\}$ of action space A , be given. That is, $A = A_0 \cup A_1 \cup A_2$, where $A_0 = \{0\}$, $A_1 = \{1, \dots, \omega_0\}$, and $A_2 = \{\omega_0 + 1, \dots, \omega\}$. Given any state $i \in S$ at the beginning of each period, action $a \in A_1$ either with the constant probability \bar{q} moves the system to state a or with the constant probability $(1 - \bar{q})$ moves the system to state n if $a < i$. Also, action $a \in A_1$ and $a \geq i$, the system with the probability $(1 - \bar{q})$ leaves state i to state n and with probability \bar{q} still remains on the state i . Similarly, action $a \in A_2$ with the constant probability q moves the system to state a or with the probability $(1 - q)$ still remains on i

if $a < i$. Furthermore, when $a \in A_2$ and $a \geq i$, the system deterministically still remains on state i . In addition, when the system is not in operational conditions (failure) within a period, the decision maker must choose an unplanned maintenance action from the one element set $\bar{A} = \{\text{minimal repair}\}$. Herein the maintenance action “minimal repair” is meant to adopt an instant therapy to correct stochastic failure of a system. We then let $o_t(i)$ be the total expected unplanned maintenance cost for pertaining the normal operations in time interval $[t, t + 1]$ under given state i at beginning of period t . Owing to that a more worse operational state implies a higher system’s failure rate, the unplanned maintenance cost is nondecreasing in i , $i \in S$.

Within the above framework and adhering to the principle of easy implementation and control, we are of particular interest to find the optimal preventive maintenance strategy $(X_0^*, X_1^*, \dots, X_{T-1}^*)$, where $X_t^* = (a_1^*(t), a_2^*(t), \dots, a_n^*(t))$ denotes the decision rule (policy) and $a_i^*(t)$ denotes the optimal maintenance action on the state i at decision epoch t , to minimize the expected total discounted cost over planning horizon.

Moreover, to characterize the structure of an optimal maintenance policy, the other assumptions are listed as follows:

A.0. Whenever the action “minimal repair” is performed, then the failure will be corrected and the system is reversed to the previous operational state before failing.

A.1. Let $P_t = [p_{ij}^t]$ be the Markovian deterioration matrix at period t . If the system is in state i at beginning of period t and no system failure occurs in the time interval $[t, t + 1]$, $0 \leq t \leq T - 1$, then the system deteriorates to state j , $j \geq i$, with probability p_{ij}^t when no maintenance action in A is performed. Moreover, P_t is upper triangular and $\sum_{j=k}^n p_{ij}^t$ is nondecreasing in i for all $k \geq i$. Restated, the transition rate of operational states has increasing failure rate (IFR) properties (Derman, 1963).

A.2. All operational states, although unobservable directly, can be known by instantaneous inspections. Moreover, performing an inspection action at the beginning of each period is the periodic policy and the action cost is independent of the current state.

A.3. All operational states can be well-defined to account for the aged-dependent system failure.

A.4. Any maintenance action in A , except for “do nothing”, must be ready for one period and subject to the capacity, at beginning of each period. Each action at most can be performed once.

A.5. If the maintenance actions in A are all available then the complete time of maintenance actions in A is extremely shorter than the defined period.

3. Model formulation

To construct the model, relevant notations are defined as follows:

$\psi_t^T(i)$	total expected minimal discounted cost in periods t to T when the operational state at the beginning of period t is i
β	discounted factor, $\beta \in (0, 1]$
$c(a)$	expected maintenance cost of action a , $a \in A$
N_t	denotes the number of failures in time interval $[0, t]$
Z_t	denotes the operational state at the beginning of period t
$(X_0^*, \dots, X_{T-1}^*)$	an optimal preventive maintenance strategy over the finite horizon T , where $X_t^* = (a_1^*(t), \dots, a_n^*(t))$, $a_i^*(t)$ justifies the optimal preventive maintenance action when the operational state at the beginning of period t is i

Next, we need to construct the dynamic programming equations (DPEs) of optimal preventive maintenance strategy $(X_0^*, \dots, X_{T-1}^*)$ over the finite horizon T . After performing an action a , $a \in A$, the system deteriorates to state j , $j \geq i$, with probability $p_{ij}^t(a)$, where

$$p'_{ij}(a) = E\{P\{Z_{t+1} = j|Z_t = i; a_i(t) = a; N_{t+1} - N_t|Z_t = i; \text{performing minimal repairs in } [t, t + 1]\}\}. \tag{1}$$

Owing to that a perfect maintenance action moves the system to another better state in a significantly shorter time than a defined period (assumption A.5), the transition probability $p'_{ij}(a)$ can be rewritten as

$$p'_{ij}(a) = E\{P\{Z_{t+1} = j|Z_t = a; a_a(t) = 0; N_{t+1} - N_t|Z_t = a; \text{performing minimal repairs in } [t, t + 1]\}\} = p'_{aj}(0) \text{ for all } a \in A - \{0\}. \tag{2}$$

Moreover, based on assumption A.2, the inspection cost is constant and it will be taken at the beginning of each period. Thus, it can be omitted in the model (since it does not impact the optimal preventive maintenance policy). Hence, the total expected minimal discounted cost in periods t to T can be written as follows:

$$\psi_t^T(1) = o_t(1) + \beta \sum_{j \in S} p'_{1j}(0) \psi_{t+1}^T(j) \text{ for all } 0 \leq t \leq T - 1, \tag{3}$$

$$\psi_t^T(i) = \text{Min} \begin{cases} k_t^T(i) \\ f_t^T(i) \\ g_t^T(i) \end{cases} \text{ for all } 0 \leq t \leq T - 1 \text{ and } i \in S - \{1, n\}, \tag{4}$$

$$\psi_t^T(n) = o_t(n) + \beta \sum_{j \in S} p'_{nj}(0) \psi_{t+1}^T(j) \text{ for all } 0 \leq t \leq T - 1, \tag{5}$$

and

$$\psi_T^T(i) = 0 \text{ for all } i \in S, \tag{6}$$

where

$$k_t^T(i) = o_t(i) + \beta \sum_{j \in S} p'_{ij}(0) \psi_{t+1}^T(j),$$

$$f_t^T(i) = \text{Min} \begin{cases} \text{Min}_{a \in A_2, a < i} c(a) + q \left\{ o_t(a) + \beta \sum_{j \in S} p'_{aj}(0) \psi_{t+1}^T(j) \right\} + (1 - q) \left\{ o_t(i) + \beta \sum_{j \in S} p'_{ij}(0) \psi_{t+1}^T(j) \right\}, \\ \text{Min}_{a \in A_2, a \geq i} c(a) + o_t(i) + \beta \sum_{j \in S} p'_{ij}(0) \psi_{t+1}^T(j), \end{cases}$$

and

$$g_t^T(i) = \text{Min} \begin{cases} \text{Min}_{a \in A_1, a < i} c(a) + \bar{q} \left\{ o_t(a) + \beta \sum_{j \in S} p'_{aj}(0) \psi_{t+1}^T(j) \right\} + (1 - \bar{q}) \left\{ o_t(n) + \beta \sum_{j \in S} p'_{nj}(0) \psi_{t+1}^T(j) \right\}, \\ \text{Min}_{a \in A_1, a \geq i} c(a) + \bar{q} \left\{ o_t(i) + \beta \sum_{j \in S} p'_{ij}(0) \psi_{t+1}^T(j) \right\} + (1 - \bar{q}) \left\{ o_t(n) + \beta \sum_{j \in S} p'_{nj}(0) \psi_{t+1}^T(j) \right\}. \end{cases}$$

To study the structural characteristics of above model, the following inferences are made in the properties of transition probability $p'_{ij}(0)$ and unplanned maintenance cost $o_t(i)$. Since after performing a

minimal repair has no effect on $p_{ij}^t(0)$ (assumption A.0), the system is restored to the previous operational state before failing instantaneously and therefore the transition probability $p_{ij}^t(0)$ can be rewritten as

$$\begin{aligned}
 p_{ij}^t(0) &= E\{P\{Z_{t+1} = j|Z_t = i; a_i(t) = 0; N_{t+1} - N_t|Z_t = i; \text{performing minimal repairs in } [t, t + 1]\}\} \\
 &= P\{Z_{t+1} = j|Z_t = i; a_i(t) = 0; \text{no system failure occurs in } [t, t + 1]\} = p_{ij}^t.
 \end{aligned}
 \tag{7}$$

Owing to that Markovian deterioration matrix $P^t = [p_{ij}^t]$ has IFR properties (assumption A.1), Eq. (7) implies that Markovian deterioration matrix $P^t(0) = [p_{ij}^t(0)]$ also has IFR properties.

Furthermore, the fact that the operational states can be well-defined to account for the age-dependent system failure (assumption A.3) suggests that $p_{ij}^t(0)$ is independent of t and the following Eq. (8) is obtained.

$$p_{ij}^t = p_{ij}, \tag{8}$$

where

$$p_{ij} = P\{Z_1 = j|Z_0 = i; a_i(0) = 0; \text{no system failure occurs in } [0, 1]\} \quad \forall i, j \in S, j \geq i.$$

Also, the unplanned maintenance cost is

$$\begin{aligned}
 o_t(i) &= o_t(i) = E\{\text{cost of each minimal repair in } [t, t + 1]\} \cdot E\{N_{t+1} - N_t|Z_t = i\} \\
 &= E\{\text{cost of each minimal repair in } [0, 1]\} \cdot E\{N_1 - N_0|Z_0 = i\} = o_0(i).
 \end{aligned}
 \tag{9}$$

(For convenience, will write $o_0(i) = o(i)$.)

Maintenance problems considered herein are those systems with extremely large planning life cycle years. The term “life cycle” used herein implies that the system will be planned to be replaced/reconstructed either by a new identical/similar system or by an advanced system. Owing to that the planning horizon (T) has extremely large size, it appears so complex that it is difficult to obtain an optimal preventive maintenance policy $(X_0^*, \dots, X_{T-1}^*)$. However, when $T \gg t$ and T is extremely large, we can transform the problem as the infinite case. That is, the stationary strategy (X^*, \dots, X^*) is used to substitute $(X_0^*, \dots, X_{T-1}^*)$. In addition, if the planning life cycle is large, any action chosen at the end of the life cycle only slightly affects the total expected discounted cost function with respect to infinite horizon. Therefore, the following proposed model, developing to obtain stationary strategy (X^*, \dots, X^*) over T periods, loses the problems occurring at the end of the life cycle.

Since $\lim_{T \rightarrow \infty} \psi_t^T(i) = \lim_{T \rightarrow \infty} \psi_{t+1}^T(i) = \dots = \lim_{T \rightarrow \infty} \psi_k^T(i)$ for all finite integer k , assume that $\lim_{T \rightarrow \infty} \psi_t^T(i) = h(i)$. Thus, DPE(4) can be rewritten as

$$h(i) = \text{Min} \left\{ \begin{aligned} & o(i) + \beta \sum_{j \geq i} p_{ij} h(j), \\ & \text{Min} \left\{ \begin{aligned} & \text{Min}_{a \in A_2, a < i} c(a) + q \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} h(j) \right\} + (1 - q) \left\{ o(i) + \beta \sum_{j \geq i} p_{ij} h(j) \right\}, \\ & \text{Min}_{a \in A_2, a \geq i} c(a) + o(i) + \beta \sum_{j \geq i} p_{ij} h(j), \end{aligned} \right. \\ & \text{Min} \left\{ \begin{aligned} & \text{Min}_{a \in A_1, a < i} c(a) + \bar{q} \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} h(j) \right\} + (1 - \bar{q}) \{ o(n) + \beta h(n) \}, \\ & \text{Min}_{a \in A_1, a \geq i} c(a) + \bar{q} \left\{ o(i) + \beta \sum_{j \geq i} p_{ij} h(j) \right\} + (1 - \bar{q}) \{ o(n) + \beta h(n) \}, \end{aligned} \right. \end{aligned} \right. \tag{10}$$

for all $i \in S - \{1, n\}$

and DPEs (3) and (5) can be rewritten as

$$h(1) = o(1) + \beta \sum_{j \geq 1} p_{1j} h(j), \tag{11}$$

$$h(n) = o(n) + \beta h(n). \tag{12}$$

Our problem is converted into the infinite case in which the optimal maintenance actions at each decision epoch are independent of all time periods. Therefore, we focus our attention on elucidating the structural characteristics of optimal policy.

4. Structural characteristics

Theorem 1. $h(i) = \bar{h}(i)$ for all $i \in S$, where

$$\bar{h}(i) = \text{Min} \begin{cases} o(i) + \beta \sum_{j \geq i} p_{ij} \bar{h}(j), \\ \text{Min}_{a \in A_2, a < i} c(a) + q \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} \bar{h}(j) \right\} + (1 - q) \left\{ o(i) + \beta \sum_{j \geq i} p_{ij} \bar{h}(j) \right\}, \\ \text{Min}_{a \in A_1, a < i} c(a) + \bar{q} \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} \bar{h}(j) \right\} + (1 - \bar{q}) \left\{ o(n) + \beta \bar{h}(n) \right\}, \end{cases} \tag{13}$$

for all $i \in S - \{1, n\}$,

$$\bar{h}(1) = o(1) + \beta \sum_{j \geq 1} p_{1j} \bar{h}(j), \tag{14}$$

$$\bar{h}(n) = o(n) + \beta \bar{h}(n). \tag{15}$$

Proof.

(i) Since $c(a) > 0$ for all $a \in A_2$, we get

$$\text{Min}_{a \in A_2, a \geq i} c(a) + o(i) + \beta \sum_{j \geq i} p_{ij} h(j) > o(i) + \beta \sum_{j \geq i} p_{ij} h(j)$$

(ii) Since only the “do nothing” action can be chosen for state n , $c(a) > 0$ for all $a \in A_1$, and $o(n) > o(i)$ for all $i \in S$, it implies that $o(n) + \beta h(n) > o(i) + \sum_{j \geq i} p_{ij} h(j)$ and

$$\text{Min}_{a \in A_1, a \geq i} c(a) + \bar{q} \left\{ o(i) + \beta \sum_{j \geq i} p_{ij} h(j) \right\} + (1 - \bar{q}) \{ o(n) + \beta h(n) \} > o(i) + \beta \sum_{j \geq i} p_{ij} h(j).$$

Putting (i) and (ii) together, enables proof of Theorem 1. \square

Theorem 1 demonstrates that if DPEs (3)–(5) represent the maintenance system 1 and DPEs (13)–(15) represent the maintenance system 2, then system 1 is equivalent to system 2. To study the optimal policy characteristics, we further demonstrate that the system 2 is equivalent to another system 3 which is defined on the following DPEs (16)–(18).

Lemma 1. $\hat{h}(i)$ is nondecreasing in i for all $i \in S$, where

$$\hat{h}(i) = \text{Min} \begin{cases} o(i) + \beta \sum_{j \geq i} p_{ij} \hat{h}(j), \\ \text{Min}_{a \in A_2} c(a) + q \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} \hat{h}(j) \right\} + (1 - q) \left\{ o(i) + \beta \sum_{j \geq i} p_{ij} \hat{h}(j) \right\}, \\ \text{Min}_{a \in A_1} c(a) + \bar{q} \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} \hat{h}(j) \right\} + (1 - \bar{q}) \left\{ o(n) + \beta \hat{h}(n) \right\}, \end{cases} \quad (16)$$

for all $i \in S - \{1, n\}$,

$$\hat{h}(1) = o(1) + \beta \sum_{j \geq 1} p_{1j} \hat{h}(j), \quad (17)$$

$$\hat{h}(n) = o(n) + \beta \hat{h}(n). \quad (18)$$

Proof. By letting $\hat{h}(i) = \lim_{T \rightarrow \infty} \pi_i^T(i)$, it is sufficient to prove that $\pi_i^T(i)$ is nondecreasing in i for any finite period T and result of this lemma follows by induction as $T \rightarrow \infty$. Without a loss of generality, take $\pi_{T-1}^T(i) = 0$ for any finite period T ,

$$\pi_{T-1}^T = \text{Min} \begin{cases} o(i), \\ \text{Min}_{a \in A_2} c(a) + qo(a) + (1 - q)o(i), \\ \text{Min}_{a \in A_1} c(a) + \bar{q}o(a) + (1 - \bar{q})o(n). \end{cases}$$

Since $o(i)$ is nondecreasing in i , we can infer that $\text{Min}_{a \in A_2} c(a) + qo(a) + (1 - q)o(i)$ and $\text{Min}_{a \in A_1} c(a) + \bar{q}o(a) + (1 - \bar{q})o(n)$ are also nondecreasing in i . If $\pi_{t+1}^T(i)$ is nondecreasing in i , then

$$\pi_i^T(i) = \text{Min} \begin{cases} o(i) + \beta \sum_{j \geq i} p_{ij} \pi_{t+1}^T(j), \\ \text{Min}_{a \in A_2} c(a) + q \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} \pi_{t+1}^T(j) \right\} + (1 - q) \left\{ o(i) + \beta \sum_{j \geq i} p_{ij} \pi_{t+1}^T(j) \right\}, \\ \text{Min}_{a \in A_1} c(a) + \bar{q} \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} \pi_{t+1}^T(j) \right\} + (1 - \bar{q}) \left\{ o(n) + \beta \pi_{t+1}^T(n) \right\}. \end{cases}$$

Since the transition rate of operational states has IFR properties, i.e., $\sum_{j \geq i} p_{ij} \pi_{t+1}^T(j)$ is nondecreasing in i (Derman, 1963), we conclude that $\pi_i^T(i)$ is nondecreasing in i and the results of this lemma follows by induction as $T \rightarrow \infty$. \square

Lemma 2. $k(i)$ is nondecreasing in i for all $i \in S$, where

$$k(i) = o(i) + \beta \sum_{j \geq i} p_{ij} \hat{h}(j).$$

Proof. Since the transition rate has IFR properties and $\hat{h}(i)$ is nondecreasing in i for all $i \in S$ (Lemma 1), we obtain

$$\beta \sum_{j \geq l} p_{lj} \hat{h}(j) - \beta \sum_{j \geq m} p_{mj} \hat{h}(j) = \beta \left\{ \sum_{j \geq l} p_{lj} \hat{h}(j) - \sum_{j \geq m} p_{mj} \hat{h}(j) \right\} \geq 0 \quad \text{for any } l, m \in S, l \geq m.$$

Since

$$k(l) - k(m) = [o(l) - o(m)] + \beta \left\{ \sum_{j \geq l} p_{lj} \hat{h}(j) - \sum_{j \geq m} p_{mj} \hat{h}(j) \right\} \geq 0 \quad (\text{since } o(l) - o(m) \text{ for any } l \geq m),$$

the results of this lemma are obtained. \square

Lemma 3. $f(i)$ is nondecreasing in i for all $i \in S$, where

$$f(i) = \text{Min}_{a \in A_2} c(a) + q \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} \hat{h}(j) \right\} + (1 - q) \left\{ o(i) + \beta \sum_{j \geq i} p_{ij} \hat{h}(j) \right\}.$$

Proof. Define

$$f(i, a) = c(a) + q \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} \hat{h}(j) \right\} + (1 - q) \left\{ o(i) + \beta \sum_{j \geq i} p_{ij} \hat{h}(j) \right\}$$

and $\forall l, m \in S, l \geq m$, then

$$f(l, a) - f(m, a) = (1 - q) \left\{ o(l) - o(m) + \beta \left\{ \sum_{j \geq l} p_{lj} \hat{h}(j) - \sum_{j \geq m} p_{mj} \hat{h}(j) \right\} \right\} \geq 0.$$

This allows us to prove this lemma. \square

Lemma 4. $g(i)$ is nondecreasing in i for all $i \in S$, where

$$g(i) = \text{Min}_{a \in A_1} c(a) + \bar{q} \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} \hat{h}(j) \right\} + (1 - \bar{q}) \left\{ o(n) + \beta \hat{h}(n) \right\}.$$

Proof. Define

$$g(i, a) = c(a) + \bar{q} \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} \hat{h}(j) \right\} + (1 - \bar{q}) \left\{ o(n) + \beta \hat{h}(n) \right\}$$

and $\forall l, m \in S, l \geq m$, then $g(l, a) - g(m, a) = 0$. This completes the proof. \square

Theorem 2. $h(i) = \bar{h}(i) = \hat{h}(i)$ for all $i \in S$.

Proof.

(i) Since $k(i)$ is nondecreasing in i for all $i \in S$, $c(a) + qk(a) + (1 - q)k(i) \geq k(i)$ for all $i \in S$ and $a \geq i$.

(ii) Since $g(i)$ is nondecreasing in i for all $i \in S$, $c(a) + \bar{q}k(a) + (1 - \bar{q})k(n) \geq k(i)$ for all $i \in S$ and $a \geq i$.

Putting (i), (ii) and Theorem 1 together, the results of this theorem are obtained. \square

Theorem 2 demonstrates that the maintenance system 1 is equivalent to the maintenance system 3 and, hence, the main structural characteristics of system 1 are the same as in system 3. Also, the main structural characteristics of system 1 can be obtained by investigating system 3. Obviously, Theorems 1 and 2 also indicate that the worse the operational state implies a higher maintenance cost of system 1 over the system’s life cycle years.

Lemma 5. *There exists an action a^* , $a^* \in A_2$, which is optimal for all $f(i)$, $i \in S$.*

Proof. To prove this result, it is sufficient to prove that states m and $m - 1$ have the same solution. If action $a^* \in A_2$ is optimal for $f(m - 1)$ and use $h(i)$ to substitute $\hat{h}(i)$, then

$$\begin{aligned} f(m) - f(m - 1) &= f(m, a) - f(m - 1, a^*) \\ &= c(a) + q \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} h(j) \right\} + (1 - q) \left\{ o(m) + \beta \sum_{j \geq m} p_{mj} h(j) \right\} \\ &\quad - c(a^*) - q \left\{ o(a^*) + \beta \sum_{j \geq a^*} p_{a^*j} h(j) \right\} - (1 - q) \left\{ o(m - 1) + \beta \sum_{j \geq m-1} p_{m-1,j} h(j) \right\} \\ &\geq c(a) + q \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} h(j) \right\} + (1 - q) \left\{ o(m) + \beta \sum_{j \geq m} p_{mj} h(j) \right\} \\ &\quad - c(a) - q \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} h(j) \right\} - (1 - q) \left\{ o(m - 1) + \beta \sum_{j \geq m-1} p_{m-1,j} h(j) \right\} \\ &= (1 - q) \{ k(m) - k(m - 1) \} \geq 0. \end{aligned}$$

Based on above results, we can infer that

$$f(m, a) \geq f(m - 1) + (1 - q) \{ k(m) - k(m - 1) \},$$

and when $a = a^*$, $f(m, a)$ reaches its lower bound. Thus, $f(m) = \text{Min}_{a \in A_2} f(m, a) = f(m, a^*)$, and a^* is optimal for $f(m)$. Since if a^* is optimal for $f(m)$ but not optimal for $f(m - 1)$ then it contradicts the previous result. This completes the proof of Lemma 5. \square

Lemma 6. *There exists an action a^* , $a^* \in A_1$, which is optimal for all $g(i)$, $i \in S$.*

Proof. If action a^* , $a^* \in A_1$ which is optimal for $g(m - 1)$ and use $h(i)$ to substitute $\hat{h}(i)$, then

$$\begin{aligned} g(m) - g(m - 1) &= g(m, a) - g(m - 1, a^*) \\ &= c(a) + \bar{q} \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} h(j) \right\} + (1 - \bar{q}) \{ o(n) + \beta h(n) \} \\ &\quad - c(a^*) - \bar{q} \left\{ o(a^*) + \beta \sum_{j \geq a^*} p_{a^*j} h(j) \right\} - (1 - \bar{q}) \{ o(n) + \beta h(n) \} \\ &= c(a) - c(a^*) + \bar{q} \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} h(j) \right\} - \bar{q} \left\{ o(a^*) + \beta \sum_{j \geq a^*} p_{a^*j} h(j) \right\} \\ &= c(a) - c(a^*) + \bar{q} \{ k(a) - k(a^*) \}. \end{aligned}$$

Since $g(m, a) \geq g(m - 1)$, for all $a \in A_1$, and when $a = a^*$, $g(m, a) = g(m - 1)$. Thus, $g(m) = \text{Min}_{a \in A_1} g(m, a) = g(m, a^*)$ and a^* is optimal for $g(m)$. If a^* is optimal for $g(m)$ but not optimal for $g(m - 1)$ then it contradicts the previous result. This completes the proof of Lemma 6. \square

Theorem 3. *If action a_i^* denotes the optimal action for $h(i) \forall i \in S$, then*

$$a_i^* = \begin{cases} 0 & \text{if } i \in S_0, \\ a^* & \text{if } i \in S_1, \\ \hat{a}^* & \text{if } i \in S_2, \end{cases}$$

where $S = S_0 \cup S_1 \cup S_2$, $S_i \cap S_j = \emptyset \forall i, j \in \{0, 1, 2\}$; and

$$a_i^* \in \begin{cases} A_0 & \text{if } i \in S_0, \\ A_1 & \text{if } i \in S_1, \\ A_2 & \text{if } i \in S_2. \end{cases}$$

Proof. By Theorem 2, we can obtain

$$h(i) = \begin{cases} k(i) & \text{if } i \in S_0, \\ f(i) & \text{if } i \in S_1, \\ g(i) & \text{if } i \in S_2, \end{cases}$$

and hence we can obtain the result of this theorem by Lemmas 5 and 6. \square

Theorem 4. *If action $\delta^* \in A_0$ is optimal for $h(i)$, $\forall i \in S - \{n\}$, then δ^* is also optimal for each $h(j)$, $j \leq i$.*

Proof. Define $\Delta(m) = f(m) - k(m)$ for all $m \in S$. If $\delta^* \in A_0$ is optimal for $h(m)$ and $a^* \in A_2$ is optimal for $f(m)$, then $\Delta(m) \geq 0$ and

$$\begin{aligned} \Delta(m) &= c(a^*) + q \left\{ o(a^*) + \beta \sum_{j \geq a^*} p_{a^*j} h(j) \right\} + (1 - q) \left\{ o(m) + \beta \sum_{j \geq m} p_{mj} h(j) \right\} - o(m) + \beta \sum_{j \geq m} p_{mj} h(j) \\ &= c(a^*) + q \{ o(a^*) - o(m) \} - q \left\{ \beta \sum_{j \geq a^*} p_{a^*j} h(j) - \beta \sum_{j \geq m} p_{mj} h(j) \right\}. \end{aligned}$$

Moreover, if there exists an action $\bar{a}^* \in A_2$ which is optimal for $h(m - 1)$, then it is also optimal for $f(m - 1)$. Therefore, $\Delta(m - 1) \leq 0$ and

$$\begin{aligned} \Delta(m - 1) &= c(\bar{a}^*) + q \left\{ o(\bar{a}^*) + \beta \sum_{j \geq \bar{a}^*} p_{\bar{a}^*j} h(j) \right\} + (1 - q) \left\{ o(m - 1) + \beta \sum_{j \geq m-1} p_{m-1,j} h(j) \right\} \\ &\quad - o(m - 1) + \beta \sum_{j \geq m-1} p_{m-1,j} h(j) \\ &= c(\bar{a}^*) + q \{ o(\bar{a}^*) - o(m - 1) \} - q \left\{ \beta \sum_{j \geq \bar{a}^*} p_{\bar{a}^*j} h(j) - \beta \sum_{j \geq m-1} p_{m-1,j} h(j) \right\}. \end{aligned}$$

Also, we know that $a^* = \bar{a}^*$ by Lemma 5, then

$$\begin{aligned}
 \Delta(m) - \Delta(m - 1) &= c(a^*) + q\{o(a^*) - o(m)\} + q\left\{\beta \sum_{j \geq a^*} p_{a^*j}h(j) - \beta \sum_{j \geq m} p_{mj}h(j)\right\} \\
 &\quad - c(\bar{a}^*) - q\{o(\bar{a}^*) - o(m - 1)\} - q\left\{\beta \sum_{j \geq \bar{a}^*} p_{\bar{a}^*j}h(j) - \beta \sum_{j \geq m-1} p_{m-1,j}h(j)\right\} \\
 &= q\{o(m - 1) - o(m)\} + q\left\{\beta \sum_{j \geq m-1} p_{m-1,j}h(j) - \beta \sum_{j \geq m} p_{mj}h(j)\right\} \\
 &= q\{k(m - 1) - k(m)\} \leq 0.
 \end{aligned}$$

However, since $\Delta(m) \geq 0$ and $\Delta(m - 1) \leq 0$ which implies that $\Delta(m) - \Delta(m - 1) \geq 0$.

It obviously contradicts the above result. Similarly, let $\bar{\Delta}(m) = g(m) - k(m)$ for all $m \in S$, the results as same as above can also be obtained. This completes the proof. \square

Theorem 4 demonstrates that if an optimal action is “do nothing” then it can optimally perform the same action as this state for those states which are a better state than this state.

Theorem 5. *If $\delta^* \in A_1$ is optimal for $h(i)$, then δ^* is also optimal for $h(j)$, $j \geq i$.*

Proof. Define $\Theta(m) = g(m) - f(m)$ for all $m \in S$. If $\delta^* \in A_1$ is optimal for $h(m)$ and $a^* \in A_2$ is optimal for $f(m)$, then $\Theta(m) \geq 0$ and

$$\begin{aligned}
 \Theta(m) &= c(\delta^*) + \bar{q}\left\{o(\delta^*) + \beta \sum_{j \geq \delta^*} p_{\delta^*j}h(j)\right\} + (1 - \bar{q})\{o(n) - \beta h(n)\} - c(a^*) \\
 &\quad + q\left\{o(a^*) + \beta \sum_{j \geq a^*} p_{a^*j}h(j)\right\} - (1 - q)\left\{o(m) + \beta \sum_{j \geq m} p_{mj}h(j)\right\}.
 \end{aligned}$$

Furthermore, if there exists an action $\bar{a}^* \in A_2$ which is optimal for $h(m + 1)$, then it is also optimal for $f(m + 1)$. Thus, $a^* = \bar{a}^*$ (by Lemma 5), $\Theta(m + 1) \geq 0$, and

$$\begin{aligned}
 \Theta(m + 1) &= c(\delta^*) + \bar{q}\left\{o(\delta^*) + \beta \sum_{j \geq \delta^*} p_{\delta^*j}h(j)\right\} + (1 - \bar{q})\{o(n) + \beta h(n)\} - c(a^*) \\
 &\quad - q\left\{o(a^*) + \beta \sum_{j \geq a^*} p_{a^*j}h(j)\right\} - (1 - q)\left\{o(m + 1) + \beta \sum_{j \geq m+1} p_{m+1,j}h(j)\right\}.
 \end{aligned}$$

Also,

$$\begin{aligned}
 \Theta(m + 1) - \Theta(m) &= (1 - q)\left\{o(m) - o(m + 1) - \beta \sum_{j \geq m+1} p_{m+1,j}h(j) + \beta \sum_{j \geq m} p_{mj}h(j)\right\} \\
 &= q\{k(m) - k(m + 1)\} \leq 0 \quad (\text{since } k(i) \text{ is nondecreasing in } i, \text{ Lemma 2}).
 \end{aligned}$$

However, the fact that $\Theta(m+1) \geq 0$ and $\Theta(m) \leq 0$ which implies that $\Theta(m+1) - \Theta(m) \geq 0$. It obviously contradicts the above result. In addition, according to Theorem 4, it is impossible that the optimal action regarding $h(i)$, $i \geq m$, is “do nothing” when the optimal action $h(m)$ belongs to A_1 . Moreover, putting the above results and Theorem 3 together, allows us to prove this theorem. \square

Theorem 5 demonstrates that if an optimal action is an element of A_1 then it is optimal to perform the same action as this state for those states which are worse than this state.

Theorem 6. *If two actions $a_i^* \in A_2$, $a_j^* \in A_2$ are optimal for $h(i)$ and $h(j)$, $j \geq i$, then*

- (i) $a_i^* = a_j^* = a^*$,
- (ii) a^* is also optimal for $h(m)$, $i < m < j$.

Proof.

- (i) Results obtained from (i) are a straightforward induction of Theorem 3.
- (ii) If $a^* \in A_1$ is not optimal for $h(m)$, $i < m < j$ then there exists an action \bar{a}^* , $\bar{a}^* = 0$ or $\bar{a}^* \in A_1$, implying that \bar{a}^* is optimal for $h(m)$, $i < m < j$. Obviously, this result contradicts Theorems 1 and 2 and hence it enables proof of Theorem 6. \square

Theorem 6 demonstrates that if we know any two optimal actions which belong to A_2 then the two actions are the same and it is optimal to perform the same action as these two states for those states are among these two states.

Theorem 7. *At most there exist two control limits i_1 , i_2 , and two actions $\delta_1 \in A_1$, $\delta_2 \in A_2$ such that*

$$a_i^* = \begin{cases} 0 & \text{if } i \leq i_1, \\ \delta_2 & \text{if } i_1 < i \leq i_2, \\ \delta_1 & \text{if } i_2 < i \leq n-1, \\ 0 & \text{if } i = n. \end{cases}$$

Theorem 7 demonstrates that the optimal policy has a well-defined structure. On the other hand, the optimal policy for a multi-action maintenance problem subject to action-risk has an easily implementable control limit rule.

5. Solution procedure and illustrations

5.1. Solution procedure

In this section, we propose a solution procedure to apply to the proposed model.

Step 1. Define a period length which satisfies the conditions of assumption A.5.

Step 2. Use the following equation to estimate the transition probability p_{ij} for all $i, j \in S$, $j \geq i$.

$$\hat{p}_{ij} = \hat{p}\{Z_1 = j | Z_0 = i; a_i(0) = 0; \text{performing minimal repairs in } [0, 1]\}.$$

Step 3. Use Eq. (9) to estimate the unplanned maintenance cost $o(i)$ for all $i \in S$.

Step 4. Use the following LP (Linear Programming) form to solve $h(i)$ for all $i \in S$:

$$\text{Max } z = \sum_{i=1}^n h(i)$$

subject to

$$h(i) \leq o(i) + \beta \sum_{j \geq i} p_{ij} h(j) \quad \forall i \in S - \{1, n\};$$

$$h(i) \leq c(a) + q \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} h(j) \right\} + (1 - q) \left\{ o(i) + \beta \sum_{j \geq i} p_{ij} h(j) \right\} \quad \forall a \in A_2, i > a;$$

$$h(i) \leq c(a) + \bar{q} \left\{ o(a) + \beta \sum_{j \geq a} p_{aj} h(j) \right\} + (1 - \bar{q}) \{ o(n) + \beta h(n) \} \quad \forall a \in A_1, i > a;$$

$$h(i) \leq \frac{o(n)}{1 - \beta} \quad \forall i \in S - \{n\};$$

$$h(1) = o(1) + \beta \sum_{j \geq 1} p_{1j} h(j);$$

$$h(n) = \frac{o(n)}{1 - \beta}.$$

(How to use the LP formulation to solve Markov maintenance problem can be found in White and White (1989), Serin and Avsar (1997).)

Step 5. Use the control limit rule of Theorem 6 to find the optimal maintenance policy.

5.2. Numerical illustration

Consider the following satellite maintenance problem. The satellite can be categorized by either two modes: “Safe Hold Mode” and “Science Mode”. The satellite in “Safe Hold Mode” implies its inability to perform the given missions owing to the malfunctioning of components. The malfunctioning of components may originate from variations of environmental stress such as pressure, temperature and the impacts of natural scenarios such as water impact, and ground impact. Whenever the satellite is in “Safe Hold Mode”, a recovery procedure (an unplanned maintenance action) must be performed to reverse the “Safe Hold Mode” to the “Science Mode (normal operations)”. Moreover, the “Science Mode” can be categorized as eight underlying operational states, i.e., $S = \{1, 2, \dots, 8\}$. Each operational states represents the various capacities of achieving the given missions. In addition, allow the preventive maintenance action $A = A_0 \cup A_1 \cup A_2$ where $A_0 = \{0\}$, $A_1 = \{1, 2\}$ and $A_2 = \{3, 4\}$. Each maintenance action represents an alternative level of maintenance and has a probabilistic risk to be ineffective owing to the imperfect maintenance process. Also, assume that the inspection occurs at the beginning of every period and each operational state after the inspection is known. According to various operational conditions and the various risks of maintenance actions, a search must be made for the optimal stationary control limit rule, thereby minimizing total expected discounted cost. The Markov transition matrix P is defined as

Table 1
Relevant data of this example

β	q	\bar{q}		$c(1)$	$c(2)$	$c(3)$	$c(4)$
0.99	0.8	0.7		100US\$	70US\$	45US\$	40US\$
$o(1)$	$o(2)$	$o(3)$	$o(4)$	$o(5)$	$o(6)$	$o(7)$	$o(8)$
20US\$	20US\$	25US\$	40US\$	50US\$	85US\$	90US\$	100US\$

Table 2
Total expected discounted cost $h(i)$ and optimal action a_i^*

$h(1)$	$h(2)$	$h(3)$	$h(4)$	$h(5)$	$h(6)$	$h(7)$	$h(8)$
9664.68US\$	9686.72US\$	9742.18US\$	9804.74US\$	9830.42US\$	9850.70US\$	9850.70US\$	10000US\$
a_1^*	a_2^*	a_3^*	a_4^*	a_5^*	a_6^*	a_7^*	a_8^*
Action 0	Action 0	Action 0	Action 3	Action 3	Action 2	Action 2	Action 0

$$P = \begin{bmatrix} 0.1 & 0.3 & 0.3 & 0.1 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0.3 & 0.35 & 0.1 & 0.1 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & 0.4 & 0.11 & 0.09 & 0.2 & 0.1 & 0.1 \\ 0 & 0 & 0 & 0.1 & 0.1 & 0.2 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0.2 & 0.7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Table 1 summarizes other relevant data. Following Steps 4 and 5 of the proposed procedure, Table 2 presents the total expected discounted cost over infinite horizon and the optimal action for each operational state.

Above results strongly suggest that optimal preventive maintenance policy (X^*, \dots, X^*) over infinite horizon is $((0, 0, 0, 3, 3, 2, 2, 0), \dots, (0, 0, 0, 3, 3, 2, 2, 0))$. That is, it is optimal to perform the action 0 (do nothing) when the current state is not greater than state 3, or it reaches nonrepairable state n . Moreover, it is optimal to perform action 2 when the state reaches states 6 and 7, and to perform action 3 when the state reaches 4 and 5.

6. Actual examples

Example 1. NBA basketball games draw fans worldwide. A NBA basketball team can be viewed as a system. The operational state space consists of different levels of a NBA basketball team. The preventive maintenance actions refer to those available assignment strategies in a playing game. Assume herein that the operational states follows the Markovian deterioration rule. In addition, the maintenance action “minimal repair” can be viewed as an instant action of encouraging morale if the system failure is defined as the worse morale. Since a different assignment strategy always has a probabilistic risk to move the team to a worse level than original level in a playing game, this example is appropriate for this model. In this case, maintenance cost items are substituted by the rewards. Moreover, the minimum problem is substituted by the maximum problem.

Example 2. Maintaining the health of a human body is important. Consider the maintenance of a football player's feet. The operational state space consists of different scenarios involving a player's feet; in addition, the feet failure occurs if the player felt a serious feet ache. Moreover, the preventive maintenance actions of a player's feet consist of options such as medication, surgery, and cortisone shots and unplanned maintenance action of a player's feet (minimal repair) is the instantaneous relief of ache (e.g. cold packing and antipyretic injection by intramuscular route). Owing to that any feet maintenance always has a probabilistic risk to move the player's feet to a worse situation than the previous condition, the proposed model can be applied to resolve this problem.

7. Conclusions and further research

To compensate for the gap between the theories and applications on Markov maintenance problem, this study presents two novel views: (a) the unplanned maintenance actions (minimal repairs) and multiple preventive maintenance actions must be considered concurrently; and (b) imperfect maintenance must be considered. While considering the two concepts, Markov maintenance models are more practical than the conventional models. Our results also demonstrate that the optimal policy for multi-action maintenance problem with action-dependent risk (imperfect maintenance) and stochastic failure has an easily computed and implemented control limit rule. Results in this study used in future studies to more closely examine a situation in which an action's risk depends not only on the action itself but also on the current state. Moreover, the proposed model herein assumes that an inspection is performed at the beginning of every period, equidistant periods limits allotted times for inspections as well. However, such limitations should be reduced in some real world situations. For instance, in a civil structure, measuring the condition of the concrete elements or the corrosion of the metal elements can be an enormous task. Therefore, the inspections and the allotted times for inspections may be significantly impacting optimal cost structure. Future research should focus on incorporating the inspection as a decision variable and treating the times between observations as non-equidistant periods.

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