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The Parametric Solutions of Eigenstructure Assignment for Controllable and Uncontrollable Singular Systems[1]

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The previous results about the parametric solutions of eigenstructure assignment for singular systems applied by pure proportional or proportional-plus-derivative state feedback can only apply to controllable systems. It is because the parametric solutions of the right generalized eigenvectors corresponding to the infinite and uncontrollable finite eigenvalues have still not been found. In this paper, the parametric solutions of the right generalized eigenvectors for finite controllable (or uncontrollable) eigenvalues and infinite controllable (or uncontrollable) eigenvalues, when the system is applied by pure proportional or proportional-plus-derivative state feedback, are given. Hence, the parametric solution of eigenstructure assignment can be used to design the state feedback of both the controllable and uncontrollable systems. The condition for detecting the regularity of the resulting system is also given.

1. INTRODUCTION

The parametric solutions of eigenstructure assignment for *normal systems* [1] have been studied by many researchers [2–9]. If the state feedback is designed by eigenstructure assignment, not only the eigenvalues but also the right generalized eigenvectors can be assigned. These solutions have been further generalized to singular systems [10–14] of the type

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, and Rank $E = q, q \le n$. The *regularity* of $(A - \lambda E)$ is assumed. Normal systems are special cases of (1) where $q = n$.

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Singular systems have the ability to capture the dynamic behavior of many physical phenomena, so they are applied in many fields, such as network theory, robotics, and economics. In the research about linear controller design of singular systems, two types of feedback are frequently used. One is the *pure proportional state feedback*

$$u(t) = K_1 x(t), \qquad K_1 \in R^{m\times n}, \tag{2}$$

and the other is *proportional-plus-derivative state feedback*

$$u(t) = K_1 x(t) - K_2 \dot{x}(t), \qquad K_1, K_2 \in R^{m\times n}. \tag{3}$$

Among those who have researched the parametric solutions of the eigenstructure problem for singular systems applied by pure proportional state feedback, Fahmy and O'Reilly [10] developed solutions that were applicable only to the controllable system and the eigenvalues of the resulting system cannot coincide with those of the original system. Duan [11] gave a parametric solution calculated by the Smith form of the matrix pencil; however, his solutions were not parametric if the system was uncontrollable. Chen and Chang [12] generalized the results of Fahmy and O'Reilly [10] to the strongly controllable system. This condition was more general than the controllable system (see [15–17]). But their results cannot be applied to the strongly uncontrollable systems.

Chen and Chang [13] and Jing [14] have developed the parametric solutions of eigenstructure assignment for singular systems applied by the special case of proportional-plus-derivative state feedback (3) where $K_2 = \alpha K_1$ (i.e., the constant-ration-proportional-derivative state feedback). However, their solutions also can only be applied to controllable systems.

Furthermore, in all of the previous research, the parametric solutions of the eigenstructure assignment problem for infinite eigenvalues have never been considered.

Therefore, if a system is uncontrollable, its state feedback cannot be designed using the previous results about the parametric solutions of eigenstructure assignment. In this paper, we obtain parametric solutions of eigenstructure assignment that can be used to design the pure proportional state feedback and proportional-plus-derivative state feedback for both controllable and uncontrollable singular systems. The parametric solutions of right generalized eigenvectors for finite controllable (or uncontrollable) eigenvalue and infinite controllable (or uncontrollable) eigenvalues, when the system is applied by pure proportional and proportional-plus-derivative state feedback, are all obtained. For ensuring the uniqueness of the state responses $x(t)$, the singular system should be regular. In this paper, the condition for detecting the regularity of the resulting systems is also given.

The organization of this paper is as follows. In Section 2, some concepts and notations about the controllability and uncontrollability of finite and infinite eigenvalues are discussed. The eigenstructure assignment problem and a condition for detecting the regularity of the resulting system are stated in Section 3. In Section 4, the parametric solutions of right generalized eigenvectors for controllable and uncontrollable finite eigenvalues are given. The parametric solutions of right generalized eigenvectors for the controllable and uncontrollable infinite eigenvalues are given in Section 5. In Section 6, we give an example where the proportional-plus-derivative feedback is designed for an uncontrollable system. The state feedback design in this example cannot be achieved by any previous results about the parametric solutions of eigenstructure assignment. Section 7 concludes the paper.

2. SOME CONCEPTS AND NOTATIONS ABOUT CONTROLLABILITY AND UNCONTROLLABILITY

If the system (1) is uncontrollable, the uncontrollable eigenvalues of $(A - \lambda E)$ and some left generalized eigenvectors with these uncontrollable eigenvalues cannot be altered by any state feedback (2) or (3). However, by a similar reason as stated by Moore [2], the right generalized eigenvectors with the uncontrollable eigenvalues still can be changed by the state feedback. The uncontrollable left generalized eigenvectors are important for the development of our solutions. So they are first discussed in this section.

By [1, p. 29, Theorem 2-2.1], whether the system is controllable can be characterized as follows:

LEMMA 2.1. *The system* (1) *is controllable if and only if* $\mathrm{Rank}[A - \lambda E \quad B] = n$, $\forall \lambda \in C$, λ *is finite*, *and* $\mathrm{Rank}[E \quad B] = n$.

Two nonsingular matrices $Q, P \in R^{n \times n}$ can be selected to consider the uncontrollability of the system (1). if $x(t) = P\bar{x}(t)$ and we left multiply (1) by Q, then (1) can be transformed into the form

$$\begin{bmatrix} \overline{E}_c & \overline{E}_{12} \\ 0 & \overline{E}_{\bar{c}} \end{bmatrix} \dot{\bar{x}}(t) = \begin{bmatrix} \overline{A}_c & \overline{A}_{12} \\ 0 & \overline{A}_{\bar{c}} \end{bmatrix} \bar{x}(t) + \begin{bmatrix} \overline{B}_c \\ 0 \end{bmatrix} u(t),$$

where $\overline{E}_c, \overline{A}_c \in R^{n_c \times n_c}$, $\overline{E}_{\bar{c}}, \overline{A}_{\bar{c}} \in R^{n_{\bar{c}} \times n_{\bar{c}}}$, $n_c + n_{\bar{c}} = n$, and $(\overline{E}_c, \overline{A}_c, \overline{B}_c)$ is controllable. The uncontrollable eigenvalues of $(A - \lambda E)$ are the eigenvalues of $(\overline{A}_{\bar{c}} - \lambda \overline{E}_{\bar{c}})$. (For details, see [1, pp. 50–55].) Assume that $(\overline{A}_{\bar{c}} - \lambda \overline{E}_{\bar{c}})$ has infinite eigenvalue and π distinct finite eigenvalues $\lambda_i \in$

C, $i = 1, \ldots, \pi$, and it also has the left generalized eigenvectors $\hat{h}_{ij}^k \in C^{1 \times n_{\bar{c}}}$, $i = 1, \ldots, \pi, \infty$; $j = 1, \ldots, \theta_i^{\bar{c}}$; $k = 1, \ldots, \rho_{ij}^{\bar{c}}$, satisfying the relations

$$\hat{h}_{ij}^k \left(\overline{A}_{\bar{c}} - \lambda_i \overline{E}_{\bar{c}} \right) = \hat{h}_{ij}^{k-1} \overline{E}_{\bar{c}}, \qquad \hat{h}_{ij}^0 = 0, i = 1, \ldots, \pi;\ j = 1, \ldots, \theta_i^{\bar{c}};$$
$$k = 1, \ldots, \rho_{ij}^{\bar{c}},$$

$$\hat{h}_{\infty j}^k \overline{E}_{\bar{c}} = \hat{h}_{ij}^{k-1} \overline{A}_{\bar{c}}, \qquad \hat{h}_{\infty j}^0 = 0,\ j = 1, \ldots, \theta_\infty^{\bar{c}};\ k = 1, \ldots, \rho_{\infty j}^{\bar{c}}.$$

It is satisfied that $\sum_{i=1}^{\pi} \sum_{j=1}^{\theta_i^{\bar{c}}} \rho_{ij}^{\bar{c}} + \sum_{j=1}^{\theta_\infty^{\bar{c}}} \rho_{\infty j}^{\bar{c}} = n_{\bar{c}}$. Then by the theory of linear algebra, the following lemma can be obtained.

LEMMA 2.2. *The system* (1) *has uncontrollable eigenvalues as stated above if and only if there are a series of linear independent row vectors* $h_{ij}^k \in C^{1 \times n}$, $i = 1, \ldots, \pi, \infty$; $j = 1, \ldots, \theta_i^{\bar{c}}$; $k = 1, \ldots, \rho_{ij}^{\bar{c}}$, *that satisfy the relations*

$$h_{ij}^k [A - \lambda_i E \quad B] = h_{ij}^{k-1} [E \quad 0], \qquad h_{ij}^0 = 0,\ i = 1, \ldots, \pi, \tag{4}$$
$$h_{\infty j}^k [E \quad B] = h_{\infty j}^{k-1} [A \quad 0], \qquad h_{\infty j}^0 = 0. \tag{5}$$

By Lemma 2.2, the following definition can be given.

DEFINITION 2.1. A series of row vectors $h_{ij}^k \in C^{1 \times n}$, $i = 1, \ldots, \pi$; $j = 1, \ldots, \theta_i^{\bar{c}}$; $k = 1, \ldots, \rho_{ij}^{\bar{c}}$, satisfying (4) are called uncontrollable left generalized eigenvectors with the finite eigenvalues. Also, a series of vectors $h_{\infty j}^k \in C^{1 \times n}$, $j = 1, \ldots, \theta_\infty^{\bar{c}}$; $k = 1, \ldots, \rho_{\infty j}^{\bar{c}}$, satisfying (5) are called uncontrollable left generalized eigenvectors with the infinite eigenvalues. h_{ij}^k (all possible values of i are $1, \ldots, \pi, \infty$) is called an uncontrollable left generalized eigenvector of grade k. For h_{ij}^k, if there exists a row vector h_{ij}^{k+1} such that both h_{ij}^{k+1} and h_{ij}^k satisfy (4) or (5), then we say that h_{ij}^k has a next uncontrollable left generalized eigenvector, otherwise, we say that h_{ij}^k has no next uncontrollable left generalized eigenvector.

The following notation is defined within $i = 1, \ldots, \pi, \infty$, i.e., for finite and infinite uncontrollable eigenvalues. Assume that $\rho_{i1}^{\bar{c}} \leq \rho_{i2}^{\bar{c}} \leq \cdots \leq \rho_{i\theta_i^{\bar{c}}}^{\bar{c}}$. Denote ϕ_i as the number of all distinct element of the set4b $\{\rho_{i1}^{\bar{c}}, \rho_{i2}^{\bar{c}}, \ldots, \rho_{i\theta_i^{\bar{c}}}^{\bar{c}}\}$. The notations $\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^{\phi_i}$, satisfying $\sigma_i^1 < \sigma_i^2 < \cdots < \sigma_i^{\phi_i}$, represent all distinct elements of the set $\{\rho_{i1}^{\bar{c}}, \rho_{i2}^{\bar{c}}, \ldots, \rho_{i\theta_i^{\bar{c}}}^{\bar{c}}\}$. Assume that there are η_{il} elements with value σ_i^l, $l = 1, \ldots, \phi_i$, in the set $\{\rho_{i1}^{\bar{c}}, \rho_{i2}^{\bar{c}}, \ldots, \rho_{i\theta_i^{\bar{c}}}^{\bar{c}}\}$. Then $\eta_{i1} + \eta_{i2} + \cdots + \eta_{i\phi_i} = \theta_i$. Denote H_{il}^k as the matrix whose row vectors are the uncontrollable left generalized eigenvectors of grade k with the uncontrollable eigenvalue λ_i in all the chains with length $\sigma_i^{\,l}$. i.e.,

$$H_{il}^{k} = \begin{bmatrix} h^{k}_{i(\eta_{i1}+\cdots+\eta_{i(l-1)}+1)} \\ h^{k}_{i(\eta_{i1}+\cdots+\eta_{i(l-1)}+2)} \\ \vdots \\ h^{k}_{i(\eta_{i1}+\cdots+\eta_{i(l-1)}+\eta_{il})} \end{bmatrix}, \qquad k = 1,\dots,\sigma_i^{\,l};\ l = 1,\dots,\phi_i \text{ and } \eta_{i0} = 0.$$

Then by (4) and (5), we have

$$H_{il}^{k}[A - \lambda_i E \quad B] = H_{il}^{k-1}[E \quad 0],$$
$$k = 1,\dots,\sigma_i^{\,l},\ H_{il}^{0} = 0,\ i = 1,\dots,\pi, \tag{6}$$

$$H_{\infty l}^{k}[E \quad B] = H_{\infty l}^{k-1}[A \quad 0], \qquad k = 1,\dots,\sigma_\infty^{\,l},\ H_{\infty l}^{0} = 0. \tag{7}$$

A series of matrices $H_{il}^1, H_{il}^2,\dots, H_{il}^{\sigma_i}$ group those chains of uncontrollable left generalized eigenvectors with the same length $\sigma_i^{\,l}$, $l = 1,\dots,\phi_i$.

Let

$$U_{il} = \begin{bmatrix} H_{i1}^{\sigma_i^1} \\ \vdots \\ H_{il}^{\sigma_i^l} \end{bmatrix}, \qquad l = 1,\dots,\phi_i.$$

U_{il} comprises all the last uncontrollable left generalized eigenvectors in the chains with length less than or equal to l. It can be seen that

$$U_{il} = \begin{bmatrix} U_{i(l-1)} \\ H_{il}^{\sigma_i^l} \end{bmatrix}.$$

The following example is given to show the meaning of the above notations.

EXAMPLE 2.1. Consider an uncontrollable system which has an uncontrollable finite eigenvalue λ_1, with $\theta_1^{\bar{c}} = 3$, $\rho_{11}^{\bar{c}} = 2$, $\rho_{11}^{\bar{c}} = 2$, $\rho_{12}^{\bar{c}} = 4$, $\rho_{13}^{\bar{c}} = 4$. The related notations about the uncontrollable generalized eigenvectors are showing in the diagram

$$\begin{array}{lllll l} \text{1st chain:} & h_{11}^1 & h_{11}^2 & & & \sigma_1^{\,1} = 2,\ \eta_{11} = 1, \\ \text{2nd chain:} & h_{12}^1 & h_{12}^2 & h_{12}^3 & h_{12}^4 & \multirow{2}{*}{$\left.\right\}\ \sigma_1^{\,2} = 4,\ \eta_{12} = 2.$} \\ \text{3rd chain:} & h_{13}^1 & h_{13}^2 & h_{13}^3 & h_{13}^4 & \end{array}$$

Since there are two different lengths 2 and 4, we have $\phi_1 = 2$. The related matrices are

$$H_{11}^1 = \begin{bmatrix} h_{11}^1 \end{bmatrix}, \qquad H_{11}^2 = \begin{bmatrix} h_{11}^2 \end{bmatrix}$$

$$H_{12}^1 = \begin{bmatrix} h_{12}^1 \\ h_{13}^1 \end{bmatrix}, \qquad H_{12}^2 = \begin{bmatrix} h_{12}^2 \\ h_{13}^2 \end{bmatrix}, \qquad H_{12}^3 = \begin{bmatrix} h_{12}^3 \\ h_{13}^3 \end{bmatrix}, \qquad H_{12}^4 = \begin{bmatrix} h_{12}^4 \\ h_{13}^4 \end{bmatrix},$$

and

$$U_{11}^2 = \begin{bmatrix} h_{11}^2 \end{bmatrix}, \qquad U_{12}^4 = \begin{bmatrix} h_{11}^2 \\ h_{12}^4 \\ h_{13}^4 \end{bmatrix}.$$

3. THE EIGENSTRUCTURE ASSIGNMENT PROBLEM

For considering the pure proportional and proportional-plus-derivative state feedback simultaneously, we use a feedback of the type

$$u(t) = K_1 x(t) - \gamma K_2 \dot{x}(t), \qquad K_1, K_2 \in R^{m\times n},\ \gamma \in R, \tag{8}$$

γ is an auxiliary number to distinguish different types of feedbacks. If $\gamma = 0$, (8) is a pure proportional state feedback (2). If $\gamma = 1$, (8) is a proportional-plus-derivative state feedback (3). When (8) is applied to (1), the resulting system becomes

$$(E + \gamma BK_2)\dot{x}(t) = (A + BK_1)x(t). \tag{9}$$

Eigenstructure Assignment Problem. For system (1) and feedback (8), the problem of eigenstructure assignment is to select appropriate state feedback gains K_1 and K_2 that will make the matrix pencil $((A + BK_1) - \lambda(E + \gamma BK_2))$ in (9) have admissible eigenvalues and right generalized eigenvectors.

Assume that the assigned eigenvalues in the resulting system (9) are λ_i, $i = 1, \dots, \mu, \infty$, where λ_∞ represents the infinite eigenvalue. The geometric multiplicity of λ_i is denoted by θ_i, and the lengths of those θ_i chains of generalized eigenvectors with λ_i are denoted by ρ_{ij}, $j = 1, \dots, \theta_i$. It is satisfied that $\sum_{i=1}^{\mu}\sum_{j=1}^{\theta_i}\rho_{ij} + \sum_{j=1}^{\theta_\infty}\rho_{\infty j} = n$. Note that the uncontrollable finite and infinite eigenvalues should be included in the assigned eigenvalues; i.e., if λ_i is an uncontrollable eigenvalue, it should hold that $\theta_i \geq \theta_i^{\bar{c}}$ and there are $\theta_i^{\bar{c}}$ chains whose lengths satisfy $\rho_{ij} \geq \rho_{ij}^{\bar{c}}$.

The right generalized eigenvectors of the resulting system (9) with λ_i are denoted by v_{ij}^k, $i = 1, \ldots, \mu, \infty$; $j = 1, \ldots, \theta_i$; $k = 1, \ldots, \rho_{ij}$. Then they satisfy that

$$((A + BK_1) - \lambda_i(E + \gamma BK_2))v_{ij}^k = (E + \gamma BK_2)v_{ij}^{k-1},$$
$$v_{ij}^0 = 0,\ k = 1, \ldots, \rho_{ij}, \quad (10)$$

$$(E + \gamma BK_2)v_{\infty j}^k = (A + BK_1)v_{\infty j}^{k-1}, \qquad v_{\infty j}^0 = 0,\ k = 1, \ldots, \rho_{\infty j}. \quad (11)$$

Let $w_{ij}^k = K_1 v_{ij}^k$ and $y_{ij}^k = K_2 v_{ij}^k$, $i = 1, \ldots, \mu, \infty$; $j = 1, \ldots, \theta_i$; $k = 1, \ldots, \rho_{ij}$. Then (10) and (11) can be rewritten as

$$(A + \lambda_i E)v_{ij}^k + Bw_{ij}^k - \lambda_i \gamma B y_{ij}^k = Ev_{ij}^{k-1} + \gamma B y_{ij}^{k-1},$$
$$v_{ij}^0 = 0,\ y_{ij}^0 = 0,\ k = 1, \ldots, \rho_{ij}, \quad (12)$$

$$Ev_{\infty j}^k + \gamma B y_{\infty j}^k - Bw_{\infty j}^{k-1} = Av_{\infty j}^{k-1}, \qquad v_{\infty j}^0 = 0,\ w_{\infty j}^0 = 0,\ k = 1, \ldots, \rho_{\infty j}. \quad (13)$$

The notations are defined as $V_{ij} = [v_{ij}^1 \quad \cdots \quad v_{ij}^{\rho_{ij}}]$, $V_i = [V_{i1} \quad \cdots \quad V_{i\theta_i}]$, $V_f = [V_1 \quad \cdots \quad V_\mu]$, $V_{\infty j} = [v_{\infty j}^1 \quad \cdots \quad v_{\infty j}^{\rho_{\infty j}}]$, $V_\infty = [V_{\infty 1} \quad \cdots \quad V_{\infty \theta_\infty}]$, $V = [V_f \quad V_\infty]$. The set of w_{ij}^k and y_{ij}^k is defined in a manner similar to the set of v_{ij}^k.

By a similar method as used by Kleion and Moore [3], the following theorem stating the existing condition of real K_1 and K_2 can be obtained.

THEOREM 3.1. *The assigned eigenvalues are given as above. Assume that they are symmetric with respect to the real axis. There exist feedback matrices* K_1, K_2, *of real number, such that* (10) *and* (11) *hold, if and only if the following three conditions are satisfied.*

(C1) *For each* i, j, *there exist a set of vectors* $v_{ij}^k, w_{ij}^k, y_{ij}^k$, $k = 1, \ldots, \rho_{ij}$, *satisfying* (12) *or* (13).

(C2) $[V_f \quad V_\infty]$ *is nonsingular.*

(C3) *If* $\lambda_{i_1} = \mathrm{conj}(\lambda_{i_2})$ ($\mathrm{conj}(x)$ *means the complex conjugate of* x) *then* $\theta_{i_1} = \theta_{i_2}$, $\rho_{i_1 j} = \rho_{i_2 j}$ *and* $v_{i_1 j}^k = \mathrm{conj}(v_{i_2 j}^k)$, $j = 1, \ldots, \theta_{i_1}$; $k = 1, \ldots, \rho_{ij}$.

For ensuring the uniqueness of the state response $x(t)$, the singular system should be regular [17, p. 6]. For considering the regularity of the resulting system, the following lemma is given.

LEMMA 3.1. *If* (C2) *is satisfied, i.e.,* $[V_f \quad V_\infty]$ *is nonsingular, then the resulting system* (9) *is regular if and only if* $[EV_f + \gamma BY_f \quad AV_\infty + BW_\infty]$ *is nonsingular.*

Proof. See Appendix A. ▮

So, in Theorem 3.1, if the resulting system is required to be regular, the following condition should be added:

(C4) $[EV_f + \gamma BY_f \quad AV_\infty + BW_\infty]$ is nonsingular.

Therefore, all possible solutions of the assignable right generalized eigenvectors v_{ij}^k are those satisfying (12), (13), and (C2), (C3), (C4). Equation (12) demonstrates the relation of the assignable right generalized eigenvectors with the finite eigenvalue and (13) demonstrates the relation of the assignable right generalized eigenvectors with the infinite eigenvalue. If the solutions of v_{ij}^k, w_{ij}^k, and y_{ij}^k in (12) and (13) satisfying (C2), (C3), and (C4) have been found, then $K_1V = W$ and $K_2V = Y$. By (C2), $\det V \neq 0$, the feedback gains can be obtained by $K_1 = WV^{-1}$ and $K_2 = YV^{-1}$.

Fahmy and O'Reilly [10], Duan [11], and Chen and Chang [12] have given the parametric solutions of (12) in the case $\gamma = 0$, i.e., the pure proportional state feedback, and the eigenvalues must be controllable. In all previous research, the infinite eigenvalue was never considered. So the parametric solutions of (13) have not been discussed.

In the following, we obtain the more general parametric solutions of v_{ij}^k, w_{ij}^k, and y_{ij}^k in (12) for controllable (or uncontrollable) finite eigenvalues in Section 4. The parametric solutions of $v_{\infty j}^k$, $w_{\infty j}^k$, and $y_{\infty j}^k$ in (13) for controllable (or uncontrollable) infinite eigenvalues are given in Section 5. With these parametric solutions, we can select the right generalized eigenvectors by choosing the free parameters that satisfy (C2), (C3), and (C4), and obtain the feedback gains by $K_1 = WV^{-1}$ and $K_2 = YV^{-1}$.

4. MAIN RESULTS FOR THE FINITE EIGENVALUES

The relation demonstrating the assignable right generalized eigenvectors with the finite eigenvalues is (12). Our main work in this section is to find the parametric solutions of v_{ij}^k, y_{ij}^k, and w_{ij}^k, $k = 1, \ldots, \rho_{ij}$, in (12), where the uncontrollability information of (E, A, B) stated in Section 2 is assumed.

By Lemma 2.2, $\operatorname{Rank}[A - \lambda_i E \quad B \quad -\lambda_i \gamma B] = \operatorname{Rank}[A - \lambda_i E \quad B] = r$, $r \leq n$, where r depends on the controllability of λ_i. So the following relation holds,

$$\begin{bmatrix} L_1^i \\ L_2^i \end{bmatrix} [A - \lambda_i E \quad B \quad -\lambda_i \gamma B] \begin{bmatrix} P_{11}^i & P_{12}^i & P_{13}^i \\ P_{21}^i & P_{22}^i & P_{23}^i \\ P_{31}^i & P_{32}^i & P_{33}^i \end{bmatrix} = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{14}$$

where $L_1^i \in C^{r\times n}$, $L_2^i \in C^{(n-r)\times n}$, $P_{11}^i \in C^{n\times r}$, $P_{21}^i, P_{31}^i \in C^{m\times r}$, $P_{12}^i \in C^{n\times(m+n-r)}$, $P_{22}^i, P_{32}^i \in C^{m\times(m+n-r)}$, $P_{13}^i \in C^{n\times m}$, $P_{23}^i, P_{33}^i \in C^{m\times m}$. I_r is the $r \times r$ identity matrix. The matrices

$$P^i = \begin{bmatrix} P_{11}^i & P_{12}^i & P_{13}^i \\ P_{21}^i & P_{22}^i & P_{23}^i \\ P_{31}^i & P_{32}^i & P_{33}^i \end{bmatrix} \qquad \text{and} \qquad L^i = \begin{bmatrix} L_1^i \\ L_2^i \end{bmatrix}$$

are nonsingular and they can be obtained by elementary column (or row) operations.

If λ_i is a controllable eigenvalue, then $r = n$ and both L_i^2 and the uncontrollable left generalized eigenvector do not exist. On the other hand, if λ_i is an uncontrollable eigenvalue, then $r < n$, and both L_2^i and the uncontrollable left generalized eigenvector exist. In the later case, we can obtain the following lemma.

LEMMA 4.1. *If the finite eigenvalue λ_i is uncontrollable, $U_{il}E[P_{12}^i \quad P_{13}^i]$ is of full row rank, for any $l = 1, \ldots, \phi_i$.*

Proof. See Appendix B. ∎

By Lemma 4.1, if the finite eigenvalue λ_i is uncontrollable, then $U_{il}E[P_{12}^i \quad P_{13}^i]$ is of full row rank, so it can be transformed into $[I_{(\eta_{i1}+\cdots+\eta_{il})} \quad 0]$ by column operations, i.e.,

$$U_{il}E\begin{bmatrix} P_{12}^i & P_{13}^i \end{bmatrix} \begin{bmatrix} S_{11}^{il} & S_{12}^{il} \\ S_{21}^{il} & S_{22}^{il} \end{bmatrix} = \begin{bmatrix} I_{(\eta_{i1}+\cdots+\eta_{il})} & 0 \end{bmatrix}, \qquad l = 1, \ldots, \phi_i, \quad (15)$$

where $S_{11}^{il} \in C^{(m+n-r)\times(\eta_{i1}+\cdots+\eta_{il})}$, $S_{12}^{il} \in C^{(m+n-r)\times(\eta_{i(l+1)}+\cdots+\eta_{i\phi_i}+2m)}$, $S_{21}^{il} \in C^{m\times(\eta_{i1}+\cdots+\eta_{il})}$, $S_{22}^{il} \in C^{m\times(\eta_{i(l+1)}+\cdots+\eta_{i\phi_i}+2m)}$ and the matrix

$$\begin{bmatrix} S_{11}^{il} & S_{12}^{il} \\ S_{21}^{il} & S_{22}^{il} \end{bmatrix}$$

is nonsingular.

The following theorem for the parametric solutions of v_{ij}^k, y_{ij}^k, and w_{ij}^k, $k = 1, \ldots, \rho_{ij}$ in (12) can be given now:

THEOREM 4.1. (1) *If λ_i is a controllable eigenvalue, all possible solutions of* (12) *are*

$$\begin{cases} \begin{bmatrix} v_{ij}^k \\ w_{ij}^k \\ y_{ij}^k \end{bmatrix} = \begin{bmatrix} P_{11}^i \\ P_{21}^i \\ P_{31}^i \end{bmatrix} L_i^1 [E \quad \gamma B] \begin{bmatrix} v_{ij}^{k-1} \\ y_{ij}^{k-1} \end{bmatrix} + \begin{bmatrix} P_{12}^i & P_{13}^i \\ P_{22}^i & P_{23}^i \\ P_{32}^i & P_{33}^i \end{bmatrix} z_i^{k-1}, \\ v_{ij}^0 = 0, \qquad y_{ij}^0 = 0, \qquad k = 1, \dots, \rho_{ij}. \end{cases} \tag{16}$$

(2) *If λ_i is an uncontrollable eigenvalue, then the following holds*:
(a) *If $\rho_{ij} \le \sigma_i^1$, all possible solutions of* (12) *are*

$$\begin{cases} \begin{bmatrix} v_{ij}^k \\ w_{ij}^k \\ y_{ij}^k \end{bmatrix} = \begin{bmatrix} P_{11}^i \\ P_{21}^i \\ P_{31}^i \end{bmatrix} L_i^1 [E \quad \gamma B] \begin{bmatrix} v_{ij}^{k-1} \\ y_{ij}^{k-1} \end{bmatrix} + \begin{bmatrix} P_{12}^i & P_{13}^i \\ P_{22}^i & P_{23}^i \\ P_{32}^i & P_{33}^i \end{bmatrix} z_i^{k-1}, \\ v_{ij}^0 = 0, \qquad y_{ij}^0 = 0, \qquad k = 1, \dots, \rho_{ij}. \end{cases} \tag{17}$$

(b) *If $\sigma_i^b < \rho_{ij} \le \sigma_i^{b+1}$ for some b and $\sigma_i^{\phi_i+1} = \infty$, let $c_1 = 1$, $e_1 = (\rho_{ij} - \sigma_i^b)$, $c_l = (\rho_{ij} - \sigma_i^{b-l+2}) + 1$, and $e_l = (\rho_{ij} - \sigma_i^{b-l+1})$, $l = 2, \dots, b$, all possible solutions of* (12) *can be represented as*

$$\begin{cases} \begin{bmatrix} v_{ij}^k \\ w_{ij}^k \\ y_{ij}^k \end{bmatrix} = \left(\begin{bmatrix} P_{11}^i \\ P_{21}^i \\ P_{31}^i \end{bmatrix} - \begin{bmatrix} P_{12}^i & P_{13}^i \\ P_{22}^i & P_{23}^i \\ P_{32}^i & P_{33}^i \end{bmatrix} \begin{bmatrix} S_{11}^{i(b-l+1)} \\ S_{21}^{i(b-l+1)} \end{bmatrix} \right) U_{i(b-l+1)} E P_{11}^i L_i^1 \\ \qquad [E \quad \gamma B] \begin{bmatrix} v_{ij}^{k-1} \\ v_{ij}^{k-1} \end{bmatrix} + \begin{bmatrix} P_{12}^i & P_{13}^i \\ P_{22}^i & P_{23}^i \\ P_{32}^i & P_{33}^i \end{bmatrix} \begin{bmatrix} S_{12}^{i(b-l+1)} \\ S_{22}^{i(b-l+1)} \end{bmatrix} z_i^{k-1}, \\ \qquad v_{ij}^0 = 0, \qquad y_{ij}^0 = 0, \qquad k = c_l, \dots, e_l, \qquad l = 1, \dots, b, \end{cases} \tag{18}$$

$$\begin{cases} \begin{bmatrix} v_{ij}^k \\ w_{ij}^k \\ y_{ij}^k \end{bmatrix} = \begin{bmatrix} P_{11}^i \\ P_{21}^i \\ P_{31}^i \end{bmatrix} L_i^1 [E \quad \gamma B] \begin{bmatrix} v_{ij}^{k-1} \\ y_{ij}^{k-1} \end{bmatrix} + \begin{bmatrix} P_{12}^i & P_{13}^i \\ P_{22}^i & P_{23}^i \\ P_{32}^i & P_{33}^i \end{bmatrix} z_i^{k-1}, \\ k = \left(\rho_{ij} - \sigma_i^1 \right) + 1, \dots, \rho_{ij}. \end{cases} \tag{19}$$

z_i^{k-1} is a column vector with appropriate dimension, representing the free parameters.

Proof. See Appendix C. ▮

Note that if $\gamma = 0$, P_{31}^i, P_{32}^i, and P_{33}^i in (14) are any matrices with appropriate dimensions which make P^i nonsingular. So by Theorem 4.1, y_{ij}^k may have many solutions. This reflects the fact that if $\gamma = 0$, y_{ij} in (12) is undetermined. In this case, the pure proportional state feedback is used. So only v_{ij}^k and w_{ij}^k are considered, and the value of y_{ij}^k is not important. Also, it can be seen in Theorem 4.1 that if $\gamma = 0$, the value of v_{ij}^k and w_{ij}^k are not affected by the value of y_{ij}^{k-1}.

5. MAIN RESULTS FOR THE INFINITE EIGENVALUES

The relation demonstrating the assignable right generalized eigenvectors with the infinite eigenvalues is (13). Our main work in this section is to find the parametric solutions of $v_{\infty j}^k$, $y_{\infty j}^k$ and $w_{\infty j}^k$, $k = 1, \ldots, \rho_{\infty j}$, in (13) where the uncontrollability condition of (E, A, B) stated in Section 2 is assumed.

1. *The Solution for $k = 1$*

When $k = 1$, (13) becomes $Ev_{\infty j}^1 + \gamma B y_{\infty j}^1 = 0$. We can find matrices Q_1 and Q_2 satisfying the relation

$$[E \quad \gamma B]\begin{bmatrix} Q_1^\infty \\ Q_2^\infty \end{bmatrix} = 0, \tag{20}$$

where

$$\begin{bmatrix} Q_1^\infty \\ Q_2^\infty \end{bmatrix}$$

is of full column rank and its columns span the null space of $[E \quad \gamma B]$. Then all possible solutions of $v_{\infty j}^1$ and $y_{\infty j}^1$ are

$$\begin{bmatrix} v_{\infty j}^1 \\ y_{\infty j}^1 \end{bmatrix} = \begin{bmatrix} Q_1^\infty \\ Q_2^\infty \end{bmatrix} z_{\infty j}^0, \tag{21}$$

where $z_{\infty j}^0$ is a free parameter.

2. *The Solutions for $k \geq 2$*

By Lemma 2.2, $\text{Rank}[E \quad \gamma B \quad -B] = \text{Rank}[E \quad B] = r$, $r \leq n$, where r depends on the controllability of the infinite eigenvalue. So the following relation exists,

$$\begin{bmatrix} L_1^\infty \\ L_2^\infty \end{bmatrix} [E \quad \gamma B \quad -B] \begin{bmatrix} P_{11}^\infty & P_{12}^\infty & P_{13}^\infty \\ P_{21}^\infty & P_{22}^\infty & P_{23}^\infty \\ P_{31}^\infty & P_{32}^\infty & P_{33}^\infty \end{bmatrix} = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

where $L_1^\infty \in R^{r \times n}$, $L_2^\infty \in R^{(n-r)\times n}$, $P_{11}^\infty \in R^{n \times r}$, $P_{21}^\infty, P_{31}^\infty \in R^{m \times r}$, $P_{12}^\infty \in R^{n \times (m+n-r)}$, $P_{22}^\infty, P_{32}^\infty \in R^{m \times (m+n-r)}$, $P_{13}^\infty \in R^{n \times m}$, $P_{23}^\infty, P_{33}^\infty \in R^{m \times m}$. The matrices

$$P^\infty = \begin{bmatrix} P_{11}^\infty & P_{12}^\infty & P_{13}^\infty \\ P_{21}^\infty & P_{22}^\infty & P_{23}^\infty \\ P_{31}^\infty & P_{32}^\infty & P_{33}^\infty \end{bmatrix} \quad \text{and} \quad L^\infty = \begin{bmatrix} L_1^\infty \\ L_2^\infty \end{bmatrix} \tag{23}$$

are nonsingular.

If the infinite eigenvalue is uncontrollable, the following lemma similar to those for the finite eigenvalue can be obtained.

LEMMA 5.1. *For any* $l = 1, \dots, \phi_\infty$; $U_{\infty l} A [P_{12}^\infty \quad P_{13}^\infty]$ *is of full row rank.*

By Lemma 5.1, $U_{\infty l} A [P_{12}^\infty \quad P_{13}^\infty]$ is of full row rank, so it can be transformed by column operations into $[I_{(\eta_{\infty 1} + \cdots + \eta_{\infty l})} \quad 0]$, i.e.,

$$U_{\infty l} A [P_{12}^\infty \quad P_{13}^\infty] \begin{bmatrix} S_{11}^{\infty l} & S_{12}^{\infty l} \\ S_{21}^{\infty l} & S_{22}^{\infty l} \end{bmatrix} = \begin{bmatrix} I_{(\eta_{\infty 1} + \cdots + \eta_{\infty l})} & 0 \end{bmatrix}, \qquad l = 1, \dots, \phi_\infty, \tag{24}$$

where $S_{11}^{\infty l} \in R^{(m+n-r)\times(\eta_{\infty 1} + \cdots + \eta_{\infty l})}$, $S_{12}^{\infty l} \in R^{(m+n-r)\times(\eta_{\infty(l+1)} + \cdots + \eta_{\infty\phi_\infty} + 2m)}$, $S_{21}^{\infty l} \in R^{m \times (\eta_{\infty 1} + \cdots + \eta_{\infty l})}$, $S_{22}^{\infty l} \in R^{m \times (\eta_{\infty(l+1)} + \cdots + \eta_{\infty\phi_\infty} + 2m)}$, and the matrix

$$\begin{bmatrix} S_{11}^{\infty l} & S_{12}^{\infty l} \\ S_{21}^{\infty l} & S_{22}^{\infty l} \end{bmatrix}$$

is nonsingular.

By a similar method for finite eigenvalues, the following theorem concerning the parametric solutions with the infinite eigenvalue when $k \geq 2$ can be obtained.

THEOREM 5.1. (1) *If the infinite eigenvalue is controllable, all possible solutions of* (13) *are*

$$\begin{cases} \begin{bmatrix} v_{\infty j}^{k} \\ y_{\infty j}^{k} \\ w_{\infty j}^{k-1} \end{bmatrix} = \begin{bmatrix} P_{11}^{\infty} \\ P_{21}^{\infty} \\ P_{31}^{\infty} \end{bmatrix} L_{\infty}^{1} A v_{\infty j}^{k-1} + \begin{bmatrix} P_{12}^{\infty} & P_{13}^{\infty} \\ P_{22}^{\infty} & P_{23}^{\infty} \\ P_{32}^{\infty} & P_{33}^{\infty} \end{bmatrix} z_{\infty}^{k-1}, \\ v_{\infty j}^{1} \text{ and } y_{\infty j}^{1} \text{ are given in } (21), \qquad k = 2, \dots, \rho_{\infty j}, \text{ and } w_{\infty j}^{\rho_{\infty j}} \text{ is free}. \end{cases} \tag{25}$$

(2) *If the infinite eigenvalue is uncontrollable, the following holds*:
(a) *If* $\rho_{\infty j} \leq \sigma_{\infty}^{1}$, *all possible solutions of* (13) *are*

$$\begin{cases} \begin{bmatrix} v_{\infty j}^{k} \\ y_{\infty j}^{k} \\ w_{\infty j}^{k-1} \end{bmatrix} = \begin{bmatrix} P_{11}^{\infty} \\ P_{21}^{\infty} \\ P_{31}^{\infty} \end{bmatrix} L_{\infty}^{1} A v_{\infty j}^{k-1} + \begin{bmatrix} P_{12}^{\infty} & P_{13}^{\infty} \\ P_{22}^{\infty} & P_{23}^{\infty} \\ P_{32}^{\infty} & P_{33}^{\infty} \end{bmatrix} z_{\infty}^{k-1}, \\ v_{\infty j}^{1} \text{ and } y_{\infty j}^{1} \text{ are given in } (21), \qquad k = 2, \dots, \rho_{\infty j}, \text{ and } w_{\infty j}^{\rho_{\infty j}} \text{ is free}. \end{cases} \tag{26}$$

(b) *If* $\sigma_{\infty}^{b} < \rho_{\infty j} \leq \sigma_{\infty}^{b+1}$, $\sigma_{\infty}^{\phi_{\infty}+1} = \infty$, *let* $c_1 = 2$, $e_1 = (\rho_{\infty j} - \sigma_{\infty}^{b})$ *and* $c_l = (\rho_{\infty j} - \sigma_{\infty}^{b-l+2}) + 1$, $e_l = (\rho_{\infty j} - \sigma_{\infty}^{b-l+1})$, $l = 2, \dots, b$, *and all possible solutions of* (13) *can be represented as*

$$\begin{cases} \begin{bmatrix} v_{\infty j}^{k} \\ y_{\infty j}^{k} \\ w_{\infty j}^{k-1} \end{bmatrix} = \left(\begin{bmatrix} P_{11}^{\infty} \\ P_{21}^{\infty} \\ P_{31}^{\infty} \end{bmatrix} - \begin{bmatrix} P_{12}^{\infty} & P_{13}^{\infty} \\ P_{22}^{\infty} & P_{23}^{\infty} \\ P_{32}^{\infty} & P_{33}^{\infty} \end{bmatrix} \begin{bmatrix} S_{11}^{\infty(b-l+1)} \\ S_{21}^{\infty(b-l+1)} \end{bmatrix} U_{\infty(b-l+1)} \right) \\ \qquad \cdot A P_{11}^{\infty} L_{\infty}^{1} A v_{\infty j}^{k-1} + \begin{bmatrix} P_{12}^{\infty} & P_{13}^{\infty} \\ P_{22}^{\infty} & P_{23}^{\infty} \\ P_{32}^{\infty} & P_{33}^{\infty} \end{bmatrix} \begin{bmatrix} S_{12}^{\infty(b-l+1)} \\ S_{22}^{\infty(b-l+1)} \end{bmatrix} z_{\infty}^{k-1}, \\ v_{\infty j}^{1} \text{ and } y_{\infty j}^{1} \text{ are given in } (21), \qquad k = c_l, \dots, e_l;\ l = 1, \dots, b, \end{cases} \tag{27}$$

$$\begin{cases} \begin{bmatrix} v_{\infty j}^{k} \\ y_{\infty j}^{k} \\ w_{\infty j}^{k-1} \end{bmatrix} = \begin{bmatrix} P_{11}^{\infty} \\ P_{21}^{\infty} \\ P_{31}^{\infty} \end{bmatrix} L_{\infty}^{1} A v_{\infty j}^{k-1} + \begin{bmatrix} P_{12}^{\infty} & P_{13}^{\infty} \\ P_{22}^{\infty} & P_{23}^{\infty} \\ P_{32}^{\infty} & P_{33}^{\infty} \end{bmatrix} z_{\infty}^{k-1}, \\ k = \left(\rho_{\infty j} - \sigma_{\infty}^{1} \right) + 1, \dots, \rho_{\infty j}, \qquad \text{and } w_{\infty j}^{\rho_{\infty j}} \text{ is free}. \end{cases} \tag{28}$$

z_{i}^{k-1} *is a column vector with appropriate dimension, representing the free parameters*.

6. EXAMPLE

Consider the system

$$E = \begin{bmatrix} -1 & 2 & -1 & 1 & 2 \\ 1 & -1 & 0 & 0 & 1 \\ -1 & 2 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & 1 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & -2 & 1 & -2 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 2 & -3 & 1 & -1 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 2 & -2 & 1 & -1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This system has uncontrollable finite eigenvalue -1, controllable finite eigenvalue 1, and uncontrollable infinite eigenvalue. For the uncontrollable finite eigenvalue $\lambda_1 = -1$, $\theta_1^{\bar{c}} = 1$, $\rho_{11}^{\bar{c}} = 2$, and

$$U_{11} = H_{11}^2 = h_{11}^2 = [0.5 \quad 0 \quad 0 \quad 0.5 \quad 0.5].$$

For the uncontrollable infinite eigenvalue, $\theta_\infty^{\bar{c}} = 1$, $\rho_{\infty 1}^{\bar{c}} = 2$, and

$$U_{\infty 1} = H_{\infty 1}^2 = h_{\infty 1}^2 = [0 \quad 0 \quad 0 \quad -1 \quad 0].$$

Since the system contains the uncontrollable finite and infinite eigenvalue, pure proportional or proportional-plus-derivative state feedback cannot be designed by the previous research about the parametric solutions of eigenstructure assignment. However, they can be designed by our solutions.

The eigenvalue 1 is unstable. We want to use the pure proportional-plus-derivative state feedback to stabilize the system. We prepare to move the unstable finite eigenvalue 1 to the stable finite eigenvalue -1. Then the resulting system would have infinite eigenvalue where $\theta_\infty = 1$, $\rho_{\infty 1} = 2$, and finite eigenvalue $\lambda_1 = -1$ where $\theta_1 = 2$, $\rho_{11} = 2$, and $\rho_{12} = 1$.

According to our solutions, the assignable right generalized eigenvectors are as follows.

(1) Since $\rho_{11}, \rho_{12} \leq \rho_{11}^{\bar{c}}$, the assignable right generalized eigenvectors for the uncontrollable finite eigenvalue $\lambda_1 = -1$ are

$$\begin{bmatrix} v_{1j}^k \\ w_{1j}^k \\ y_{1j}^k \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & w - 0.5 \\ 0.75 & -1 & 0.5 & -0.5 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1.5 & -2 & 0 & 0 & 0 & 0.5 \\ 0.25 & 0 & -0.5 & 0.5 & 1 & -0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{1j}^{k-1} \\ y_{1j}^{k-1} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -0.5 & -0.5 \\ 0 & -0.25 & -0.25 \\ 1 & 0 & 0 \\ 0 & -0.5 & -0.5 \\ 0 & 0.25 & 0.25 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z_{1j}^{k-1},$$

$$k = 1, \dots, \rho_{1j}, \; j = 1, 2.$$

(2) Since $\rho_{\infty 1} \leq \rho_{\infty 1}^{\bar{c}}$, the assignable right generalized eigenvectors are

$$\begin{bmatrix} v_{\infty 1}^1 \\ y_{\infty 1}^1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & -0.5 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0.5 \\ 0 & 1 \end{bmatrix} z_{\infty 1}^0,$$

$$\begin{bmatrix} v_{\infty 1}^2 \\ y_{\infty 1}^2 \\ w_{\infty 1}^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1.5 & -0.5 & 0 & -0.5 & 0.5 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.5 & 0.5 & 0 & -0.5 & -0.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} v_{\infty 1}^1$$

$$+ \begin{bmatrix} 0 & -1 & 1 \\ 0 & -0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z_{\infty 1}^1.$$

Suppose the free parameters are chosen as

$$z_{11}^{0} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \qquad z_{11}^{1} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \qquad z_{12}^{0} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \qquad z_{\infty 1}^{0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$z_{\infty 1}^{1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad w_{\infty 1 = 1}^{2}.$$

Then the assigned right generalized eigenvectors are

$$V = \begin{bmatrix} -2 & -2 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ -2 & -2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}, \qquad W = [2\ \ 2\ \ 1\ \ 1\ \ 1],$$

$$Y = [2\ \ 2\ \ -1\ \ 0\ \ 0],$$

and the feedback gain is

$$K_1 = [\,-0.5 \quad -2.5 \quad 1 \quad 0 \quad -1.5\,],$$
$$K_2 = [\,-2.5 \quad -0.5 \quad -1 \quad 1 \quad -1.5\,].$$

Then the resulting system is

$$\begin{bmatrix} 1.5 & 2.5 & 0 & 0 & 3.5 \\ 1 & -1 & 0 & 0 & 1 \\ -1 & 2 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ -1.5 & 0.5 & -2 & 2 & -0.5 \end{bmatrix} \dot{x}(t)$$
$$= \begin{bmatrix} 1.5 & 0.5 & 0 & -2 & 1.5 \\ -1 & 2 & 0 & 0 & 0 \\ 2 & -3 & 1 & -1 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 1.5 & -4.5 & 2 & -1 & -1.5 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t).$$

This system contains the infinite eigenvalue, the finite eigenvalue -1, and the assigned right generalized eigenvectors V.

7. CONCLUSION

The parametric solutions of the right generalized eigenvectors for finite controllable (or uncontrollable) eigenvalues and controllable (or uncontrollable) infinite eigenvalues, when the system is applied by pure proportional or proportional-plus-derivative state feedback, are given. By these results, the parametric solution of eigenstructure can be used to design the state feedback of controllable (or uncontrollable) systems. A condition for detecting the regularity of the resulting system is also given, which is explicitly represented in terms of the possible solutions. After the uncontrollability information is obtained, only elementary column and row operations are needed to construct the solutions.

APPENDIX

A. Proof of Lemma 3.1. Let

$$J_{ij} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_i \end{bmatrix}, \qquad J_i = \begin{bmatrix} J_{i1} & 0 & \cdots & 0 \\ 0 & J_{i2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & J_{i\theta_i} \end{bmatrix},$$

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & J_\mu \end{bmatrix},$$

$$N_j = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} N_1 & 0 & \cdots & 0 \\ 0 & N_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & N_{\theta_\infty} \end{bmatrix},$$

where $J_{ij} \in R^{\rho_{ij} \times \rho_{ij}}$ is in Jordan form with eigenvalue λ_i and $N_j \in R^{\rho_{\infty j} \times \rho_{\infty j}}$ is in Jordan form with eigenvalue 0. Then it can be shown that

$$\begin{aligned} &\big((A + BK_1) - \bar{\lambda}(E + \gamma BK_2)\big)\begin{bmatrix} V_f & V_\infty \end{bmatrix} \\ &\quad = \begin{bmatrix} EV_f + \gamma BY_f & AV_\infty + BW_\infty \end{bmatrix} \begin{bmatrix} J - \bar{\lambda} I & 0 \\ 0 & I - \bar{\lambda} N \end{bmatrix} \end{aligned} \tag{29}$$

for all $\bar{\lambda} \in C$.

Necessity. If $((A + BK_1) - \lambda(E + \gamma BK_2))$ is regular, a number $\bar{\lambda} \in C$ can be found such that $\det((A + BK_1) - \bar{\lambda}(E + \gamma BK_2)) \neq 0$, i.e., $((A + BK_1) - \bar{\lambda}(E + \gamma BK_2))$ is nonsingular. By (C1), $[V_f \quad V_\infty]$ is also nonsingular, so the left hand side of (29) is nonsingular. Therefore the right hand side of (29) is also nonsingular. This implies that $[EV_f + \gamma BY_f \quad AV_\infty + BW_\infty]$ is nonsingular.

Sufficiency. Because J is in Jordan form with eigenvalues λ_i, $i = 1, \dots, \mu$, and N is in Jordan form with eigenvalue 0, if $\bar{\lambda} \neq 0$ and $\bar{\lambda} \neq \lambda_i$, $i = 1, \dots, \mu$, then $\det(\operatorname{diag}((J - \lambda_p I), (I - \lambda_p \mathbf{N}))) \neq 0$. If $[EV_f + \gamma BY_f \quad AV_\infty + BW_\infty]$ is nonsingular, the right hand side of (29) is nonsingular. So the left hand side of (29) is also nonsingular. This implies that $((A + BK_1) - \lambda_p(E + \gamma BK_2))$ is nonsingular, so $((A + BK_1) - \lambda(E + \gamma BK_2))$ is regular. ∎

B. Proof of Lemma 4.1. Before proving Lemma 4.1, two lemmas are first given. They can be derived by the property of generalized eigenvectors.

LEMMA B.1. *h_i is a linear combination of some uncontrollable left generalized eigenvectors of grade smaller than or equal to d, $d \leq \max_j \rho_{ij}^{\bar{c}}$, with the same eigenvalue λ_i, and the coefficients of the uncontrollable left generalized eigenvectors of grade d are not all zero if and only if h_i is an uncontrollable left generalized eigenvector of grade d with eigenvalue λ_i.*

LEMMA B.2. *If h_i satisfies the same conditions which are given in Lemma* B.1 *and all the uncontrollable left generalized eigenvectors of grade d with nonzero coefficients have no next uncontrollable left generalized eigenvectors, then h_i also has no next left uncontrollable generalized eigenvector.*

Proof of Lemma 4.1. The mathematical induction is used to prove this lemma. First, the case $l = 1$ is proved. It can be seen that $U_{i1} = H_{i1}^{\sigma_i^1}$. If $H_{i1}^{\sigma_i^1} E[P_{12}^i \quad P_{13}^i]$ is not full rank, a nonzero vector $f \in R^{1 \times f_{i1}}$ satisfies $f H_{i1}^{\sigma_i^1} E[P_{12}^i \quad P_{13}^i] = 0$. Let $h = f H_{i1}^{\sigma_i^1}$. Because $f \neq 0$, by Lemmas B.1 and B.2, h is an uncontrollable left generalized eigenvector of grade σ_i^1 and has no next uncontrollable generalized eigenvector.

Because $f H_{i1}^{\sigma_i^1} E[P_{12}^i \quad P_{13}^i] = 0$. There is a row vector $\bar{h}$ satisfying $\bar{h} \operatorname{diag}(I_{r \times r} \quad 0) = hE[P_{11}^i \quad P_{12}^i \quad P_{13}^i]$. If both sides of this relation are left multiplied by $P^{i^{-1}}$, then $\bar{h} L_i [A - \lambda_i E - \lambda_i B \quad B] = h[E \quad 0]$. It implies that $\bar{h} L_i [A - \lambda_i E \quad B] = h[E \quad 0_{n \times m}]$. So $\bar{h} L_i$ is the next uncontrollable left generalized eigenvector of h. This is a contradiction. So $U_{i1} E[P_{12}^i \quad P_{13}^i]$ is full row rank.

Assume that $U_{i(d-1)} E[P_{12}^i \quad P_{13}^i]$ is full row rank where $d \leq \phi_i$. If $U_{id} E[P_{12}^i \quad P_{13}^i]$ is not full rank, then there is a row vector $f \in R^{1 \times (\eta_{i1} + \cdots + \eta_{id})}$ satisfying $f U_{id} E[P_{12}^i \quad P_{13}^i] = 0$. Let $f = [f_1 \quad f_2]$, where $f_1 \in R^{1 \times (\eta_{i1} + \cdots + \eta_{i(d-1)})}$ and $f_2 \in R^{1 \times \eta_{id}}$. Then $f U_{id} E[P_{12}^i \quad P_{13}^i] =$

$(f_1 U_{i(d-1)} + f_2 H_{id}^{\sigma_i^d}) E[P_{12}^i \quad P_{13}^i] = 0$ and $f_2 \neq 0$. Let $h = (f_1 U_{i(d-1)} + f_2 H_{id}^{\sigma_i^d})$. By Lemmas B.1 and B.2, h is an uncontrollable left generalized eigenvector of grade $\sigma_i{}^d$ and has no next uncontrollable left generalized eigenvector.

Because $fU_{id} E[P_{12}^i \quad P_{13}^i] = 0$, there is a row vector $\bar{h}$ satisfying $\bar{h}\,\mathrm{diag}(I_{r\times r} \quad 0) = hE[P_{11}^i \quad P_{12}^i \quad P_{13}^i]$. If both sides of this relation are right multiplied by $P^{i^{-1}}$, then $\bar{h}L_i[A - \lambda_i E - \lambda_i B \quad B] = h[E \quad 0]$. It implies that $\bar{h}L_i[A - \lambda_i E \quad B] = h[E \quad 0]$. So $\bar{h}L_i$ is the next uncontrollable left generalized eigenvector of h. This is a contradiction. So $U_{id}E[P_{12}^i \; P_{13}^i]$ is also of full row rank. ∎

C. Proof of Theorem 4.1. *Necessity*. A variable transformation is adopted as

$$\begin{bmatrix} v_{ij}^k \\ w_{ij}^k \\ y_{ij}^k \end{bmatrix} = \begin{bmatrix} P_{11}^i & P_{12}^i & P_{13}^i \\ P_{21}^i & P_{22}^i & P_{23}^i \\ P_{31}^i & P_{32}^i & P_{33}^i \end{bmatrix} \begin{bmatrix} \tilde{v}_{ij}^k \\ \tilde{w}_{ij}^k \\ \tilde{y}_{ij}^k \end{bmatrix}, \qquad k = 0, \dots, \rho_{ij}, \tag{30}$$

where $\tilde{v}_{ij}^k \in C^{r\times 1}$, $\tilde{w}_{ij}^k \in C^{(m+n-r)\times 1}$, $\tilde{y}_{ij}^k \in C^{m\times 1}$.

If both sides of (12) are left multiplied by L_i and we substitute (30) into (12), then

$$\begin{aligned} \tilde{v}_{ij}^k = {} & L_1^i\big(EP_{11}^i + \gamma BP_{31}^i\big)\tilde{v}_{ij}^{k-1} + L_1^i\big(EP_{12}^i + \gamma BP_{32}^i\big)\tilde{w}_{ij}^{k-1} \\ & + L_1^i\big(EP_{13}^i + \gamma BP_{33}^i\big)\tilde{y}_{ij}^{k-1}, \end{aligned} \tag{31}$$

$$\begin{aligned} 0 = {} & L_2^i\big(EP_{11}^i + \gamma BP_{31}^i\big)\tilde{v}_{ij}^{k-1} + L_2^i\big(EP_{12}^i + \gamma BP_{32}^i\big)\tilde{w}_{ij}^{k-1} \\ & + L_2^i\big(EP_{13}^i + \gamma BP_{33}^i\big)\tilde{y}_{ij}^{k-1}, \end{aligned} \tag{32}$$

$$\begin{aligned} P_{11}^i\tilde{v}_{ij}^0 + P_{12}^i\tilde{w}_{ij}^0 + P_{13}^i\tilde{y}_{ij}^0 = 0, \qquad & P_{31}^i\tilde{v}_{ij}^0 + P_{32}^i\tilde{w}_{ij}^0 + P_{33}^i\tilde{y}_{ij}^0 = 0, \\ & k = 1, \dots, \rho_{ij}. \end{aligned} \tag{33}$$

Since (30) is invertible and L_i is nonsingular, (31), (32), and (33) in the domain of $\tilde{v}_{ij}^k$, $\tilde{w}_{ij}^k$, and $\tilde{y}_{ij}^k$, represent the equivalent algebraic relation of (12) in the domain of $v_{ij}^k, w_{ij}^k, y_{ij}^k$. Equation (31) is a dynamic constraint of $\tilde{v}_{ij}^k$, $\tilde{w}_{ij}^k$, and $\tilde{y}_{ij}^k$. Equation (32) is a static constraint of $\tilde{v}_{ij}^k$, $\tilde{w}_{ij}^k$, and $\tilde{y}_{ij}^k$. Equation (33) is the initial condition.

Taking the inverse transform of (30) into the right hand side of (31), we can obtain that (31) is equivalent to the relation

$$\tilde{v}_{ij}^k = L_1^i\big(Ev_{ij}^{k-1} + \gamma By_{ij}^{k-1}\big), \qquad k = 1, \dots, \rho_{ij}. \tag{34}$$

(1) If λ_i is controllable, L_2^i does not exist, nor does (32). According to (31) or (34), $\tilde{v}_{ij}^k$ is dependent on the variables of its previous step, i.e., the $(k-1)$st step, and

$$\begin{bmatrix} \tilde{w}_{ij}^k \\ \tilde{y}_{ij}^k \end{bmatrix}$$

is independent on the variables of its previous step. So we have

$$\begin{bmatrix} \tilde{w}_{ij}^k \\ \tilde{y}_{ij}^k \end{bmatrix} = z_{ij}^{k-1}, \tag{35}$$

where z_{ij}^{k-1} is a free parameter vector. Substituting (34) and (35) into (30), (16) is obtained.

(2) If λ_i is uncontrollable, L_2^i exists and so does (32). By (14), the row space of L_2^i is the orthogonal complement of the column space of $[A - \lambda_i E \quad B \quad -\lambda_i \gamma^2 B]$. By (6), the space generated by the row vectors of all the matrices H_{il}^1, $l = 1, \ldots, \phi_i$ is the orthogonal complement of the column space of $[A - \lambda_i E \quad B]$. Therefore, the space is the same as the row space of L_2^i. Also, it can be seen that $H_{il}^1 B = 0$. Therefore, (32) is equivalent to the relation

$$\begin{bmatrix} H_{i1}^1 \\ \vdots \\ H_{i\phi_i}^1 \end{bmatrix} E\left(P_{11}^i \tilde{v}_{ij}^{k-1} + P_{12}^i \tilde{w}_{ij}^{k-1} + P_{13}^i \tilde{y}_{ij}^{k-1}\right) = 0, \qquad k = 1, \ldots, \rho_{ij}. \tag{36}$$

By (6), we can obtain that $H_{il}^k[A - \lambda_i E \quad B \quad -\lambda_i \gamma B] = H_{il}^{k-1}[E \quad 0 \quad 0]$, $k = 1, \ldots, \sigma_i^l$, $H_{il}^0 = 0$, $i = 1, \ldots, \pi$. If both sides of this relation are right multiplied by

$$\begin{bmatrix} v_{ij}^t \\ w_{ij}^t \\ y_{ij}^t \end{bmatrix},$$

then $H_{il}^k E v_{ij}^{t-1} = H_{il}^{k-1} E v_{ij}^t$. Substituting (30) into this relation, the following can be obtained,

$$H_{il}^1 E\Big(P_{11}^i \tilde{v}_{ij}^{t-1} + P_{12}^i \tilde{w}_{ij}^{t-1} + P_{13}^i \tilde{y}_{ij}^{t-1}\Big) = \begin{cases} H_{il}^{\sigma_i^l} E\Big(P_{11}^i \tilde{v}_{ij}^{(t-\sigma_i^l)} + P_{12}^i \tilde{w}_{ij}^{(t-\sigma_i^l)} + P_{13}^i \tilde{y}_{ij}^{(t-\sigma_i^l)}\Big), \\ \quad \text{if } t > {\sigma_i}^l, \\ H_{il}^t E\Big(P_{11}^i \tilde{v}_{ij}^0 + P_{12}^i \tilde{w}_{ij}^0 + P_{13}^i \tilde{y}_{ij}^0\Big) \equiv 0, \\ \quad \text{if } t \le {\sigma_i}^l, \end{cases} \qquad l = 1, \dots, \phi_i. \tag{37}$$

(a) If $\rho_{ij} \le {\sigma_i}^1$, by (36) and (37), all constraints in (32) are trivial. Substituting (34) and (35) into (30), (17) is obtained.

(b) If ${\sigma_i}^b < \rho_{ij} \le {\sigma_i}^{b+1}$ for some b and ${\sigma_i}^{\phi_i} = \infty$, by (36) and (37), (32) is equivalent to the relation

$$0 = U_{i(b-l+1)} E\Big(P_{11}^i \tilde{v}_{ij}^k + P_{12}^i \tilde{w}_{ij}^k + P_{13}^i \tilde{y}_{ij}^k\Big), \quad \text{where } k = c_l, \dots, e_l,\ l = 1, \dots, b. \tag{38}$$

If we view

$$\begin{bmatrix} \tilde{w}_{ij}^k \\ \tilde{y}_{ij}^k \end{bmatrix}$$

in (38) as unknown variables and $\tilde{v}_{ij}^k$ as known variables, then by (15),

$$-\left(\begin{bmatrix} S_{11}^{i(b-l+1)} \\ S_{21}^{i(b-l+1)} \end{bmatrix} U_{i(b-l+1)} E P_{11}^i \tilde{v}_{ij}^k\right) \text{ is a particular solution of } \begin{bmatrix} \tilde{w}_{ij}^k \\ \tilde{y}_{ij}^k \end{bmatrix},$$

and

$$\begin{bmatrix} S_{12}^{i(b-l+1)} \\ S_{22}^{i(b-l+1)} \end{bmatrix} z_{ij}^{k-1}$$

generates all the $(m + \eta_{i(l+1)} + \cdots + \eta_{i\phi_i})$-dimension homogeneous solutions of

$$\begin{bmatrix} \tilde{w}_{ij}^k \\ \tilde{y}_{ij}^k \end{bmatrix}$$

by the free parameter z_{ij}^{k-1} where $z_{ij}^{k-1} \in R^{(m+\eta_{i(l+1)}+\cdots+\eta_{i\phi_i})}$ is a free column vector. So (38) is equivalent to

$$\begin{bmatrix} \tilde{w}_{ij}^k \\ \tilde{y}_{ij}^k \end{bmatrix} = -\begin{bmatrix} S_{11}^{i(b-l+1)} \\ S_{21}^{i(b-l+1)} \end{bmatrix} U_{i(b-l+1)} E P_{11}^i \tilde{v}_{ij}^k + \begin{bmatrix} S_{12}^{i(b-l+1)} \\ S_{22}^{i(b-l+1)} \end{bmatrix} z_{ij}^{k-1}. \tag{39}$$

For $k = 1, \ldots, (\rho_{ij} - \sigma_i^1)$, replacing $\tilde{v}_{ij}^k$ of (39) by (34) and substituting (34) and (39) into (30), (18) is obtained.

For $k = (\rho_{ij} - \sigma_i^l) + 1, \ldots, \rho_{ij}$, by (38),

$$\begin{bmatrix} \tilde{w}_{ij}^k \\ \tilde{y}_{ij}^k \end{bmatrix} \text{ is free.}$$

Substituting (34) and (35) into (30), (19) is obtained.

Sufficiency. The variable

$$\begin{bmatrix} \tilde{v}_{ij}^k \\ \tilde{w}_{ij}^k \\ \tilde{y}_{ij}^k \end{bmatrix}$$

is introduced by (30).

(1) If λ_i is controllable, $r = n$. Since the transformation is invertible, by (16) and (30), (34), (35), and (33) can be obtained. If both sides of (34) are left multiplied by L_i^{-1} and the inverse transformation of (30) is adopted, then by (14), (12) is obtained.

(2) If λ_i is uncontrollable, we consider the following cases:

(a) Since $\rho_{ij} \le \sigma_i^1$ and the transformation is invertible, by (17) and (30), (34), (35), and (33) can be obtained. Also, (34) implies (31). By (36) and (37), (35) implies (32). Equations (31) and (32) can be rewritten as

$$\begin{bmatrix} I_{r\times r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}_{ij}^k \\ \tilde{w}_{ij}^k \\ \tilde{y}_{ij}^k \end{bmatrix} = \begin{bmatrix} L_i^1 \\ L_i^2 \end{bmatrix} \Big(\big(E P_{11}^i + \gamma B P_{31}^i \big) \tilde{v}_{ij}^{k-1}$$

$$+ \big(E P_{12}^i + \gamma B P_{32}^i \big) \tilde{w}_{ij}^{k-1} + \big(E P_{13}^i + \gamma B P_{33}^i \big) \tilde{y}_{ij}^{k-1} \Big),$$

$$k = 1, \ldots, \rho_{ij}. \tag{40}$$

If both sides of (40) are left multiplied by L_i^{-1} and the inverse transformation of (30) is adopted, then by (14), (12) is obtained.

(b) Since $\sigma_i^b < \rho_{ij} \leq \sigma_i^{b+1}$ for some b and the transformation is invertible, (34), (38), and (33) can be obtained by (18), (19), and (30). Also, (34) implies (31). By (36) and (37), (38) implies (32). Equations (31) and (32) can be rewritten as (40). Hence, (12) is obtained. ∎

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