Journal of Functional Analysis **175**, 17–51 (2000)
doi:10.1006/jfan.2000.3586, available online at http://www.idealibrary.com on IDEAL®

Holomorphic Discrete Models of Semisimple Lie Groups and their Symplectic Constructions[1]

Meng-Kiat Chuah

Department of Applied Mathematics, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, Taiwan
E-mail: chuah@math.nctu.edu.tw

Communicated by M. Vergne

Received February 5, 1999; accepted in revised form January 20, 2000

Let G be a connected real semisimple Lie group which contains a compact Cartan subgroup such that it has non-empty discrete series. A holomorphic discrete model of G is a unitary G-representation consisting of all its holomorphic discrete series with multiplicity one. We perform geometric quantization to a class of G-invariant pseudo-Kähler manifolds and construct a holomorphic discrete model. The construction of discrete series which are not holomorphic is also discussed. © 2000 Academic Press

Key Words: holomorphic discrete model; pseudo-Kähler

1. INTRODUCTION

Let G be a connected real Lie group. A unitary representation of G in which each irreducible representation occurs exactly once is called a *model*. This terminology is due to I. M. Gelfand and A. Zelevinski [6], who give several ingenious constructions of models for the classical compact groups. For compact semisimple Lie groups, the construction of a model is carried out in [4]. If G is not compact, then looking for its model is overly ambitious since the unitary dual of G is still unknown in general. However, suppose that G is semisimple and has a compact Cartan subgroup, so that it has a non-empty discrete series [10]. Following the spirit of the above definition of a model, we define the *holomorphic discrete model* as a unitary G-representation consisting of all the holomorphic discrete series with multiplicity one. The major purpose of this article is to use symplectic techniques to construct such an object. Namely, we start with certain classes of G-invariant pseudo-Kähler manifolds and use the machinery of geometric quantization [14] to convert the pseudo-Kähler structures into holomorphic Hermitian line bundles. The square-integrable holomorphic

[1] This work is supported in part by the National Science Council of Taiwan.

0022-1236/00 $35.00

sections of these line bundles are used to construct the desired holomorphic discrete model. More generally, the L^2-cohomology of these line bundles provides discrete series which may not be holomorphic. We can of course define the *discrete model* in a similar manner. Unfortunately these L^2-cohomologies do not provide a discrete model due to multiplicity problems. Nevertheless, by applying symplectic reduction [15] to the pseudo-Kähler manifolds we show that these L^2-cohomologies obey the principle "*quantization commutes with reduction*" [7].

We now describe our projects in more detail. Throughout this paper, G denotes a connected real semisimple Lie group which has a maximal compact Cartan subgroup. We also adopt the convention that the Lie algebra of a Lie group is always denoted by its lower case German letter; for instance, $\mathfrak{g}$ is the real Lie algebra of G.

Let $G^{\mathbf{C}}$ be the adjoint group of the complex Lie algebra $\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} \otimes \mathbf{C}$. Then G is a real form of $G^{\mathbf{C}}$. Let K be a maximal compact subgroup of G satisfying rank K = rank G and T a Cartan subgroup of both K and G. Thus $T \subset K \subset G$. The complex structure of $\mathfrak{g}^{\mathbf{C}}$ sends $\mathfrak{t}$ to an abelian subalgebra $\mathfrak{a}$. We obtain a complex Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + \mathfrak{a} \subset \mathfrak{g}^{\mathbf{C}}$ and a Cartan subgroup $H = TA \subset G^{\mathbf{C}}$. Consider the root system $\Delta \subset \mathfrak{h}^*$. Fix a system of positive roots Δ^+, which determines a unipotent subgroup N corresponding to the negative root spaces. Then $B = HN$ is a Borel subgroup of $G^{\mathbf{C}}$.

The positive roots Δ^+ are partitioned into compact roots Δ_{c}^+ and non-compact roots Δ_{n}^+. Namely, a root is said to be compact if its root space lies in $\mathfrak{k} \otimes \mathbf{C}$ and is said to be non-compact otherwise. Let Δ^{s} be the simple roots in Δ^+ and consider the compact simple roots $\Delta_{\mathrm{c}}^{\mathrm{s}} = \Delta_{\mathrm{c}}^+ \cap \Delta^{\mathrm{s}}$. Fix a subset $\sigma \subset \Delta_{\mathrm{c}}^{\mathrm{s}}$. We define $\mathfrak{t}_\sigma \subset \mathfrak{t}$ by

$$\mathfrak{t}_\sigma = \{x \in \mathfrak{t};\ (\alpha, x) = 0 \text{ for all } \alpha \in \sigma\}. \tag{1.1}$$

Since $\mathfrak{g}^{\mathbf{C}}$ is semisimple, its Killing form is non-degenerate. It identifies the subalgebras with their duals; for instance, we have $\mathfrak{t}_\sigma^* \subset \mathfrak{g}^*$. Whenever we pair two elements in the dual space to get a number, such as in (1.2) below, it is understood that this is always done via the Killing form.

Let $\bar{\sigma} \subset \Delta^+$ be the positive roots generated by σ. Define the σ-*regular* elements

$$(\mathfrak{t}_\sigma^*)_{\mathrm{reg}} = \{x \in \mathfrak{t}_\sigma^*;\ (\alpha, x) \neq 0 \text{ for all } \alpha \in \Delta^+ \backslash \bar{\sigma}\}. \tag{1.2}$$

So $(\mathfrak{t}_\sigma^*)_{\mathrm{reg}}$ is an open dense set in $\mathfrak{t}_\sigma^*$ whose connected components are cones. If σ is the empty set $\varnothing$, we clearly have $(\mathfrak{t}_\varnothing^*)_{\mathrm{reg}} = (\mathfrak{t}^*)_{\mathrm{reg}}$. By the complex structure, the above constructions lead to $(\mathfrak{a}_\sigma^*)_{\mathrm{reg}} \subset \mathfrak{a}_\sigma^*$. We shall always identify $(\mathfrak{t}_\sigma^*)_{\mathrm{reg}} \cong (\mathfrak{a}_\sigma^*)_{\mathrm{reg}}$. We define $\mathfrak{h}_\sigma = \mathfrak{t}_\sigma + \mathfrak{a}_\sigma \subset \mathfrak{h}$ and $H_\sigma = T_\sigma A_\sigma \subset H$ accordingly.

A subset $\sigma \subset \Delta_{\mathrm{c}}^{\mathrm{s}}$ determines a parabolic subgroup $P \subset G^{\mathbf{C}}$ via Langlands decomposition [13, p. 132]

$$B \subset P = M_\sigma A_\sigma N_\sigma; \qquad A_\sigma \subset A, \quad N_\sigma \subset N. \tag{1.3}$$

For example, $\sigma = \varnothing$ corresponds to $P = B = HN$.

Let ρ denote half the sum of all positive roots. We denote the discrete series of G by Harish-Chandra's notation $\Theta_{\lambda+\rho}$ for integral weights λ. Apply (1.3) and fix σ, P for now. Let $Y_\sigma \subset G^{\mathbf{C}}/P$ be the G-orbit containing the identity coset. It is a homogeneous G-space isomorphic to an elliptic orbit. Further, it is open in $G^{\mathbf{C}}/P$ and thus is a complex manifold. Let (P, P) be the commutator subgroup of P. We define $X_\sigma \subset G^{\mathbf{C}}/(P, P)$ by the natural fibration

$$\pi\colon G^{\mathbf{C}}/(P, P) \to G^{\mathbf{C}}/P, \qquad X_\sigma = \pi^{-1}(Y_\sigma). \tag{1.4}$$

By studying homogeneous line bundles over Y_σ corresponding to integral weights $\lambda \in \mathfrak{h}_\sigma^*$, we obtain discrete series of G [16]. However, each λ gives at most one discrete series. Therefore, we shall work on line bundles over X_σ instead to obtain the holomorphic discrete model.

Since Y_σ is an open G-orbit in $G^{\mathbf{C}}/P$, X_σ is an open G-space in $G^{\mathbf{C}}/(P, P)$. Thus X_σ is a complex manifold. Since H_σ normalizes (P, P), it acts on $G^{\mathbf{C}}/(P, P)$ on the right. Let G^σ be the centralizer of T_σ in G and let $G^\sigma_{\mathrm{ss}} \subset G^\sigma$ be its commutator subgroup. We shall see (Proposition 3.1) that $X_\sigma = (G/G^\sigma_{\mathrm{ss}})\, A_\sigma$. This implies that the right H_σ-action restricts to a right action on X_σ and that G-invariant functions on X_σ can be regarded as functions on A_σ.

A subscript of a Lie group shall always indicate invariance under the group action. For instance, $C^\infty_G(X_\sigma) = C^\infty(A_\sigma)$.

The exponential map is a diffeomorphism from $\mathfrak{a}_\sigma$ onto A_σ. Throughout this paper, we shall frequently make the identification

$$\mathfrak{a}_\sigma \cong A_\sigma, \qquad F \cdot \exp \in C^\infty(\mathfrak{a}_\sigma) \leftrightarrow F \in C^\infty(A_\sigma). \tag{1.5}$$

So if F is a G-invariant function on X_σ, we can identify it with a function on $\mathfrak{a}_\sigma$. We shall say that F is *strictly convex* if its Hessian matrix is positive definite everywhere and more generally that F is *non-degenerate* if its Hessian matrix is non-degenerate everywhere. The image of the gradient function $\frac{1}{2} F'\colon \mathfrak{a}_\sigma \to \mathfrak{a}_\sigma^*$ is denoted by U_F,

$$U_F = \tfrac{1}{2} F'(\mathfrak{a}_\sigma) \subset \mathfrak{a}_\sigma^*. \tag{1.6}$$

The inverse function theorem says that if F is non-degenerate then U_F is open. We shall see that $\frac{1}{2} F'$ is essentially the moment map of pseudo-Kähler form. By definition, a pseudo-Kähler form is a symplectic form of type (1, 1). It is weaker than a Kähler form only on the positive definite condition.

The starting point of this paper is classifying all the $(G \times T_\sigma)$-invariant pseudo-Kähler structures on X_σ. Since G is semisimple, the G-actions on these pseudo-Kähler forms are necessarily Hamiltonian [8, Theorem 26.1]. Their G-moment maps are denoted by

$$\Phi\colon X_\sigma \to \mathfrak{g}^*.$$

Since Φ is G-equivariant it is determined by its restriction on $A_\sigma \subset X_\sigma$.

Define the (possibly empty) chamber

$$\mathscr{C} = \{x \in \mathfrak{t}^*;\ (\Delta_{\mathrm{c}}^+, x) \geqslant 0,\ (\Delta_{\mathrm{n}}^+, x) < 0\}. \tag{1.7}$$

The complex structure also identifies $\mathscr{C}$ as a subset of $\mathfrak{a}^*$. Our theorems contain statements about $\mathscr{C}$. If $\mathscr{C} = \emptyset$ the results are still valid, and they simply say that the Kähler structures in question do not exist.

THEOREM 1. *Every $(G \times T_\sigma)$-invariant closed (1, 1)-form on X_σ is given by $\omega = \sqrt{-1}\,\partial\bar{\partial}F$, where $F \in C^\infty(A_\sigma)$. It is pseudo-Kähler if and only if F is non-degenerate and $U_F \subset (\mathfrak{a}_\sigma^*)_{\mathrm{reg}}$. It is Kähler if and only if F is strictly convex and $U_F \subset (\mathfrak{a}_\sigma^*)_{\mathrm{reg}} \cap \mathscr{C}$. The moment map satisfies $\Phi(a) = \frac{1}{2}F'(a) \in \mathfrak{a}_\sigma^* \cong \mathfrak{t}_\sigma^*$ for all $a \in A_\sigma$, so $U_F = \Phi(A_\sigma)$.*

Fix a $(G \times T_\sigma)$-invariant Kähler form ω on X_σ. By Theorem 1, ω is exact. So there exists a pre-quantum line bundle L [14] whose Chern class is $[\omega] = 0$. Further, L is equipped with a connection ∇ whose curvature is ω as well as an invariant Hermitian structure (,). A smooth section s on L is said to be holomorphic if $\nabla_v s = 0$ for all anti-holomorphic vector fields v. We shall show (Proposition 4.1) that X_σ has $(G \times A_\sigma)$-invariant measure μ_X, which is unique up to scalar. A section s is said to be square-integrable if the integral

$$\int_{X_\sigma} (s, s)\ \mu_X \tag{1.8}$$

converges. Consider the Hilbert space H_ω of all square-integrable holomorphic sections. The $(G \times T_\sigma)$-action on X_σ lifts to a unitary $(G \times T_\sigma)$-representation on H_ω. The next theorem describes the irreducible G-subrepresentations in H_ω.

Let $\lambda \in \mathfrak{h}_\sigma^*$ be an integral weight. We shall always write $\chi = e^\lambda$ for its character. Namely, $\chi\colon H_\sigma \to \mathbf{C}^\times$ is the multiplicative homomorphism satisfying

$$\chi(e^v) = \exp(\lambda, v), \qquad v \in \mathfrak{h}_\sigma. \tag{1.9}$$

If V is an H_σ-module we say that $v \in V$ *transforms by* λ if $h \cdot v = \chi(h)v$ for all $h \in H_\sigma$.

In the following theorem, $(H_\omega)_\lambda$ denotes the square-integrable holomorphic sections which transform by λ under the right T_σ-action. We also assume that $\lambda + \rho \in (\mathfrak{t}^*)_{\text{reg}}$, since this condition is needed for $\Theta_{\lambda+\rho}$ to exist.

THEOREM 2. *Let $\omega = \sqrt{-1}\,\partial\bar{\partial}F$ be a $(G \times T_\sigma)$-invariant Kähler form on X_σ. The holomorphic discrete series $\Theta_{\lambda+\rho}$ occurs in H_ω if and only if $\lambda \in U_F$. If so, it occurs with multiplicity one and is given by $(H_\omega)_\lambda$.*

In Section 5, we apply Theorems 1 and 2 to construct a holomorphic discrete model for G. Namely, we show that for a suitable choice of Kähler form ω_σ on X_σ, H_{ω_σ} contains every holomorphic discrete series $\Theta_{\lambda+\rho}$ in which $\lambda \in (\mathfrak{a}_\sigma^*)_{\text{reg}} \cap \mathscr{C}$ once. Consequently, if we vary σ over all subsets of $\Delta_{\text{c}}^{\text{s}}$, then $\bigoplus_\sigma H_{\omega_\sigma}$ becomes a holomorphic discrete model.

The square-integrable holomorphic sections H_ω can be generalized to the L^2-cohomology H_ω^q, defined below. From H_ω^q we obtain the discrete series which may not be holomorphic. But unlike the construction $\bigoplus_\sigma H_{\omega_\sigma}$ above, these H_ω^q do not form the discrete model when we vary σ, due to the multiplicity problem. Nevertheless, we shall study H_ω^q in the context of symplectic reduction.

Let $\omega = \sqrt{-1}\,\partial\bar{\partial}F$ be a $(G \times T_\sigma)$-invariant pseudo-Kähler form on X_σ, where F is strictly convex. Let L be the pre-quantum line bundle as before. We shall construct the L^2-cohomology H_ω^q as follows. We denote the Dolbeault $(0, q)$-forms with coefficients in L by $\Omega^{0,q}(X_\sigma, L)$. Define a G-invariant Hermitian structure on it and then integrate over μ_X to obtain an L^2-structure $\langle x, y\rangle^L$, $x, y \in \Omega^{0,q}(X_\sigma, L)$. We say that x is square-integrable if $\langle x, x\rangle^L < \infty$. Let $\bar{\partial}^*$ be the formal adjoint of $\bar{\partial}$ relative to this L^2-structure. The differential forms which are annihilated by $\bar{\partial}$ and $\bar{\partial}^*$ are said to be *harmonic*. Let H_ω^q be the square-integrable harmonic $(0, q)$-forms. Let $\lambda \in \mathfrak{t}_\sigma^*$ be an integral weight. The $(G \times T_\sigma)$-action lifts to a $G \times T_\sigma$-representation on H_ω^q and the right T_σ-action defines $(H_\omega^q)_\lambda$ as before. Let

$$l(\lambda) = \sharp\{\lambda \in \Delta_{\text{c}}^+ ; (\lambda + \rho, \alpha) < 0\} - \sharp\{\lambda \in \Delta_{\text{n}}^+ ; (\lambda + \rho, \alpha) > 0\}. \tag{1.10}$$

We now obtain the general discrete series from H_ω^q.

THEOREM 3. *Let $\omega = \sqrt{-1}\,\partial\bar{\partial}F$ be a $(G \times T_\sigma)$-invariant pseudo-Kähler form on X_σ with F strictly convex. The discrete series $\Theta_{\lambda+\rho}$ occurs in H_ω^q if and only if $\lambda \in U_F$, $\lambda + \rho \in (\mathfrak{t}^*)_{\text{reg}}$, and $q = l(\lambda)$. If this is so, it occurs with multiplicity one and is given by $(H_\omega^q)_\lambda$.*

Let ω be a $(G \times T_\sigma)$-invariant pseudo-Kähler form on X_σ. The right T_σ-action is Hamiltonian and we call its moment map

$$\Phi_r \colon X_\sigma \to \mathfrak{t}_\sigma^*$$

the right moment map. Suppose that $\lambda \in \mathfrak{t}_\sigma^*$ is in its image. We perform symplectic reduction [15] on it. This leads to the *reduced space* $R_\lambda = \Phi_r^{-1}(\lambda)/T_\sigma$ equipped with a symplectic form ω_λ, called the *reduced form*. The process

$$(X_\sigma, \omega, \lambda) \rightsquigarrow (R_\lambda, \omega_\lambda) \tag{1.11}$$

is known as symplectic reduction. We shall see (Proposition 7.2) that R_λ is a complex manifold and (Proposition 7.4) that ω_λ is a G-invariant pseudo-Kähler form on R_λ.

We want to study how ω and λ determine the reduced space in (1.11). For $i = 1, 2$, let $\lambda_i \in \mathfrak{t}_\sigma^*$ be in the image of the right moment maps of ω_i. We introduce the notions of

$$\lambda_1 \sim \lambda_2, \qquad (\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}, \qquad (\omega_1)_{\lambda_1} = (\omega_2)_{\lambda_2} \tag{1.12}$$

as follows. Regarding λ_i as elements of $\mathfrak{g}^*$, we write $\lambda_1 \sim \lambda_2$ if they lie in the same coadjoint G-orbit. For the reduced forms, we write $(\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}$ if there exists a G-equivariant symplectomorphism between them. In particular if this symplectomorphism can be made holomorphic and preserves the pseudo-Kähler structures we write $(\omega_1)_{\lambda_1} = (\omega_2)_{\lambda_2}$.

THEOREM 4. *The image of Φ_r lies inside $(\mathfrak{t}_\sigma^*)_{\text{reg}}$. The connected components in R_λ are mutually isomorphic pseudo-Kähler manifolds, each of which is a copy of Y_σ. They are Kähler if and only if $\lambda \in (\mathfrak{t}_\sigma^*)_{\text{reg}} \cap \mathscr{C}$. For $i = 1, 2$, suppose that R_{λ_i} has the same number of connected components. Then $(\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}$ if and only if $\lambda_1 \sim \lambda_2$, and $(\omega_1)_{\lambda_1} = (\omega_2)_{\lambda_2}$ if and only if $\lambda_1 = \lambda_2$.*

By this theorem, the reduction process is independent of ω and depends uniquely on λ. Assume for simplicity that R_λ is connected. For example, this happens when F is strictly convex or more generally when the gradient of F is injective. By Theorem 4, (1.11) simplifies to

$$(X_\sigma, \lambda) \rightsquigarrow (Y_\sigma, \omega_\lambda). \tag{1.13}$$

Further, the process $\lambda \rightsquigarrow \omega_\lambda$ is injective.

The following theorem classifies the G-invariant pseudo-Kähler forms on Y_σ and shows that (1.13) is actually a bijective correspondence. The G-action on Y_σ preserving a pseudo-Kähler form is Hamiltonian and we let $\psi \colon Y_\sigma \to \mathfrak{g}^*$ denote its moment map. Note that $Y_\sigma = G/G^\sigma$ (Proposition 3.1) and we write $e \in Y_\sigma$ for the identity coset eG^σ.

THEOREM 5. *The G-invariant pseudo-Kähler forms on Y_σ are not exact and are classified by the points in $(\mathfrak{t}_\sigma^*)_{\text{reg}}$ via $\psi(e) \in (\mathfrak{t}_\sigma^*)_{\text{reg}}$. All of them can be obtained by symplectic reduction from X_σ, and the one with $\psi(e) = \lambda$ is given by ω_λ.*

By Theorems 4 and 5, we see that the G-invariant Kähler forms on Y_σ are indexed by $(\mathfrak{t}_\sigma^*)_{\text{reg}} \cap \mathscr{C}$ (non-existant if $\mathscr{C} = \emptyset$). For example, take G to be compact and $\sigma = \emptyset$, $P = B$. In this case $(\mathfrak{t}_\sigma^*)_{\text{reg}} \cap \mathscr{C}$ is just the interior of the Weyl chamber in $\mathfrak{t}^*$. We recover the classic result of Borel [2] that the G-invariant Kähler structures on $Y_\sigma = G^{\mathbf{C}}/B$ are classified by the interior points of the Weyl chamber.

Let $\omega = \sqrt{-1}\,\partial\bar{\partial}F$ be a $(G \times T_\sigma)$-invariant pseudo-Kähler form on X_σ where F is strictly convex. We have quantized the left G-action and obtained a G-representation H_ω^q consisting of square-integrable harmonic forms. For integral weights $\lambda \in \mathfrak{t}_\sigma^*$ we obtain the subrepresentation $(H_\omega^q)_\lambda$ via the right T_σ-action. On the other hand, we can first perform symplectic reduction on the right T_σ-action and obtain $(R_\lambda, \omega_\lambda)$ and then quantize the G-action on $(R_\lambda, \omega_\lambda)$ to obtain a G-representation $H_{(\omega_\lambda)}^q$. We compare the G-representations $(H_\omega^q)_\lambda$ and $H_{(\omega_\lambda)}^q$ and show that *quantization commutes with reduction* [7].

THEOREM 6. *Let $\omega = \sqrt{-1}\,\partial\bar{\partial}F$ be a $(G \times T_\sigma)$-invariant pseudo-Kähler form on X_σ, with F strictly convex. Then $(H_\omega^q)_\lambda \cong H_{(\omega_\lambda)}^q$.*

In other words, quantizing the G-action followed by taking subrepresentation via the T_σ-action coincides with performing symplectic reduction via the T_σ-action followed by quantizing the G-action. Other results of this nature are summarized in [17].

We outline the structure of this paper as follows. In Section 2, we review some concepts of the Lie algebra and establish the common notations that will appear throughout the paper. In Section 3, we describe the spaces X_σ and Y_σ in terms of a torus and its centralizer. Also, we study the $(G \times T_\sigma)$-invariant pseudo-Kähler forms on X_σ and their moment maps, leading to the proof of Theorem 1. In Section 4, we show that X_σ has a $(G \times A_\sigma)$-invariant measure μ_X. Using the Hermitian structure on the line bundle L and the measure μ_X, we construct an L^2-structure on the sections of L and prove Theorem 2. In Section 5, we apply Theorems 1 and 2 to construct a holomorphic discrete model for G. In Section 6, we generalize Theorem 2 to Theorem 3: the Kähler structures, holomorphic discrete series, and square-integrable holomorphic sections H_ω are replaced by the pseudo-Kähler structures, discrete series, and L^2-cohomology H_ω^q. In Section 7, we perform symplectic reduction to the right T_σ-action on X_σ and prove Theorem 4. The reduced space is the flag domain Y_σ, and we study its pseudo-Kähler structures in Section 8. This leads to the proof of Theorem 5. Finally, in Section 9 we quantize the G-action on the reduced space and prove Theorem 6.

2. LIE ALGEBRAS

In this section, we review some aspects of the Lie algebra which will be used later. We also establish some common notations.

Recall that G is connected semisimple with compact Cartan subgroup and that it is a real form of $G^{\mathbf{C}}$. There exists a compact real form $U \subset G^{\mathbf{C}}$ such that $K = U \cap G$ is maximal compact in G. Let $\mathfrak{g}^{\mathbf{C}} = \mathfrak{h} + \sum_i \mathfrak{g}_{\pm i}$ be the root space decomposition, indexed over the positive roots $\alpha_i \in \Delta^+$. There exists $\xi_{\pm i} \in \mathfrak{g}_{\pm i}$ [11, p. 421] such that

$$\mathfrak{u} = \mathfrak{t} + \mathbf{R}(\xi_i - \xi_{-i}, \sqrt{-1}\,(\xi_i + \xi_{-i})). \tag{2.1}$$

For convenience, write

$$\varepsilon_i = \begin{cases} 1 & \text{if} \quad \alpha_i \in \Delta_{\mathrm{c}}^+, \\ \sqrt{-1} & \text{if} \quad \alpha_i \in \Delta_{\mathrm{n}}^+. \end{cases} \tag{2.2}$$

Then

$$\zeta_i = \varepsilon_i(\xi_i - \xi_{-i}), \qquad \gamma_i = \varepsilon_i \sqrt{-1}\,(\xi_i + \xi_{-i}) \in \mathfrak{g}. \tag{2.3}$$

From (2.3), it follows that for all $x \in \mathfrak{t}$,

$$[x, \zeta_i] = \sqrt{-1}\,\alpha_i(x)\,\gamma_i, \qquad [x, \gamma_i] = -\sqrt{-1}\,\alpha_i(x)\,\zeta_i. \tag{2.4}$$

The vectors in (2.1) can be normalized so that $[\xi_i - \xi_{-i}, \sqrt{-1}\,(\xi_i + \xi_{-i})] \in \mathfrak{t}$ is identified with the root $\alpha_i \in \mathfrak{t}^*$ by the Killing form. So by $\varepsilon_i^2 = \pm 1$ in (2.2), the Killing form identifies

$$[\zeta_i, \gamma_i] = \begin{cases} \alpha_i & \text{if} \quad \alpha_i \in \Delta_{\mathrm{c}}^+, \\ -\alpha_i & \text{if} \quad \alpha_i \in \Delta_{\mathrm{n}}^+. \end{cases} \tag{2.5}$$

Define V and V_i by

$$\mathfrak{g} = \mathfrak{t} + V, \qquad V = \sum_{\alpha_i \in \Delta^+} V_i, \qquad V_i = \mathbf{R}(\zeta_i, \gamma_i). \tag{2.6}$$

In fact, the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{q}$ is obtained by

$$\mathfrak{k} = \mathfrak{t} + \sum_{\alpha_i \in \Delta_{\mathrm{c}}^+} V_i, \qquad \mathfrak{q} = \sum_{\alpha_i \in \Delta_{\mathrm{n}}^+} V_i.$$

The Lie bracket between $\mathfrak{t}$ and V_i is given by (2.4). If we take the Lie bracket between V_i and V_j, (2.3) gives

$$x \in V_i, \qquad y \in V_j \Rightarrow [x, y] \in \begin{cases} \mathfrak{t} & \text{if } i = j, \\ V & \text{if } i \neq j. \end{cases} \tag{2.7}$$

Since $\mathfrak{g}$ is semisimple, its Killing form is non-degenerate. It identifies $\mathfrak{g}$ with $\mathfrak{g}^*$, so that $\mathfrak{t}^*$ and V_i^* can be regarded as subspaces of $\mathfrak{g}^*$. The subspaces $\{\mathfrak{t}, V_i\}_i$ are pairwise orthogonal under the Killing form, so

$$(\mathfrak{t}^*, V_i) = (V_i^*, \mathfrak{t}) = (V_i^*, V_j) = 0, \qquad i \neq j. \tag{2.8}$$

Consider the vectors

$$\zeta_i^*, \gamma_i^* \in V_i^* \subset V^* \subset \mathfrak{g}^* \tag{2.9}$$

dual to (2.3). The coadjoint representation $\mathrm{ad}^*\colon \mathfrak{g} \to \mathrm{End}\, \mathfrak{g}^*$ can be computed from the above identities. Namely, (2.4) says that for all $x \in \mathfrak{t}$,

$$\mathrm{ad}_x^* \zeta_i^* = \sqrt{-1}\, \alpha_i(x)\, \gamma_i^*, \qquad \mathrm{ad}_x^* \gamma_i^* = -\sqrt{-1}\, \alpha_i(x)\, \zeta_i^*. \tag{2.10}$$

For $\sigma \subset \Delta_c^s$, we have defined the subalgebras $\mathfrak{t}_\sigma$, $\mathfrak{g}^\sigma$, $\mathfrak{g}_{ss}^\sigma$ in Section 1. We now relate them to these V_i. Given $\alpha \in \Delta^+$, we write $(\alpha, \mathfrak{t}_\sigma) = 0$ if α annihilates $\mathfrak{t}_\sigma$. Otherwise, if $(\alpha, x) \neq 0$ for some $x \in \mathfrak{t}_\sigma$, we simply write $(\alpha, \mathfrak{t}_\sigma) \neq 0$. Let $\mathfrak{t}_\sigma^\perp \subset \mathfrak{t}$ be the complement of $\mathfrak{t}_\sigma$ in $\mathfrak{t}$, under the Killing form. Then

$$\mathfrak{g}^\sigma = \mathfrak{t} + \bigoplus_{(\alpha_i, \mathfrak{t}_\sigma) = 0} V_i, \qquad \mathfrak{g}_{ss}^\sigma = \mathfrak{t}_\sigma^\perp + \bigoplus_{(\alpha_i, \mathfrak{t}_\sigma) = 0} V_i, \tag{2.11}$$

and

$$\begin{aligned} \mathfrak{g} &= \mathfrak{t} + V \\ &= \mathfrak{t}_\sigma + \mathfrak{g}_{ss}^\sigma + \bigoplus_{(\alpha_i, \mathfrak{t}) \neq 0} V_i \\ &= \mathfrak{g}^\sigma + \bigoplus_{(\alpha_i, \mathfrak{t}_\sigma) \neq 0} V_i. \end{aligned}$$

We also introduce the concept of the relative Lie algebra cohomology. Consider in general a Lie group R with a closed subgroup S. Restricting the coadjoint representation to S, we get $\mathrm{Ad}^*\colon S \to \mathrm{Aut}\, \mathfrak{r}^*$. We extend this representation to the exterior algebras $\bigwedge^q \mathfrak{r}^*$, then differentiate to get $\mathrm{ad}^*\colon \mathfrak{s} \to \mathrm{End}(\bigwedge^q \mathfrak{r}^*)$. We define

$$\bigwedge^q (\mathfrak{r}, \mathfrak{s})^* = \left\{ \alpha \in \bigwedge^q \mathfrak{r}^*;\ \iota(v)\alpha = \mathrm{ad}_v^* \alpha = 0 \text{ for all } v \in \mathfrak{s} \right\}. \tag{2.12}$$

Here $\iota(v)\,\alpha \in \bigwedge^{q-1} \mathfrak{r}^*$ is the interior product. Then $\{\bigwedge^q(\mathfrak{r}, \mathfrak{s})^*\}_q$ can be identified with the R-invariant differential forms on R/S, and they become a chain complex under the de Rham operator d. We write $H^q(\mathfrak{r}, \mathfrak{s})$ for the resulting cohomology.

3. PSEUDO-KÄHLER STRUCTURES

Fix a subset σ of the compact simple roots and it determines a parabolic subgroup P via (1.3). By (1.4), we define the domain $X_\sigma \subset G^{\mathbf{C}}/(P, P)$ as a fibration π over the flag domain Y_σ. The present section studies X_σ and its pseudo-Kähler structures and proves Theorem 1. Recall that $H = TA$ is a Cartan subgroup of $G^{\mathbf{C}}$. Let T_σ be the subtorus whose Lie algebra is the kernel of σ. It corresponds to a subgroup $A_\sigma \subset A$, where $H_\sigma = T_\sigma A_\sigma$ is complex. Let G^σ be the centralizer of T_σ in G and let G^σ_{ss} be its commutator subgroup. There is a natural action of $G \times H_\sigma$ on X_σ, a fact made clearer by the following description of X_σ.

PROPOSITION 3.1. *$X_\sigma = (G/G^\sigma_{ss})\, A_\sigma$ and $Y_\sigma = G/G^\sigma$ and the fiber of $X_\sigma \to Y_\sigma$ is H_σ.*

Proof. Recall that $\sigma \subset \Delta^s_c$. So (1.1) says that if $\alpha \in \Delta^+_n$, then $(\alpha, \mathfrak{t}_\sigma) \neq 0$. It follows that

$$T \subset G^\sigma \subset K, \tag{3.1}$$

and so G^σ is compact. Therefore, its commutator G^σ_{ss} is a compact semi-simple Lie group. Let $P = M_\sigma A_\sigma N_\sigma$ be the Langlands decomposition (1.3). Let $A^\perp_\sigma \subset A$ be the subgroup whose Lie algebra is the orthocomplement of $\mathfrak{a}_\sigma$ in $\mathfrak{a}$ (under the Killing form). We have the Iwasawa decomposition $(G^\sigma_{ss})^{\mathbf{C}} = G^\sigma_{ss} A^\perp_\sigma (M_\sigma \cap N)$. Hence

$$\begin{aligned} G^\sigma_{ss} A^\perp_\sigma N &= (G^\sigma_{ss})^{\mathbf{C}}\, N_\sigma \\ &= ((G^\sigma)^{\mathbf{C}}, (G^\sigma)^{\mathbf{C}})\, N_\sigma \\ &= (M_\sigma A_\sigma, M_\sigma A_\sigma)\, N_\sigma \\ &= (P, P). \end{aligned} \tag{3.2}$$

By (3.2), $X_\sigma = (G/G^\sigma_{ss})(A/A^\perp_\sigma) = (G/G^\sigma_{ss})\, A_\sigma$. The fiber of X_σ over Y_σ is the same as the fiber of P over (P, P), which is H_σ. So $Y_\sigma = G/G^\sigma$ and the proposition is proved. ∎

With this proposition, the right action of H_σ on X_σ is clear: A_σ acts by self-multiplication on $A_\sigma \subset (G/G^\sigma_{\mathrm{ss}})\, A_\sigma$, while T_σ acts on the right of G/G^σ_{ss} because it commutes with G^σ_{ss}.

Let ω be a $(G \times T_\sigma)$-invariant closed $(1, 1)$-form on X_σ. To solve $\omega = \sqrt{-1}\, \partial\bar{\partial} F$ for F we need the next two propositions. Here $H^q_G(X_\sigma)$ denotes the G-invariant de Rham cohomology of X_σ with real or complex coefficients.

PROPOSITION 3.2. $H^1_G(X_\sigma) = H^2_G(X_\sigma) = 0.$

Proof. By the previous proposition, $X_\sigma = (G/G^\sigma_{\mathrm{ss}})\, A_\sigma$. But A_σ is contractible. So it suffices to consider G-invariant forms on the space G/G^σ_{ss}, which can be identified with $\bigwedge^q(\mathfrak{g}, \mathfrak{g}^\sigma_{\mathrm{ss}})^*$ of (2.12). Therefore, we can prove the proposition by showing that $H^1(\mathfrak{g}, \mathfrak{g}^\sigma_{\mathrm{ss}}) = H^2(\mathfrak{g}, \mathfrak{g}^\sigma_{\mathrm{ss}}) = 0$.

Note that $\mathfrak{g}$ is semisimple. So the Whitehead lemma [8] says that its Lie algebra cohomology satisfies

$$H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0. \tag{3.3}$$

Consider a non-zero $\alpha \in \bigwedge^1(\mathfrak{g}, \mathfrak{g}^\sigma_{ss})^*$. We regard α as an element of $\bigwedge^1 \mathfrak{g}^*$. Then $d\alpha \neq 0$, because $H^1(\mathfrak{g}) = 0$ by (3.3). This proves that $H^1(\mathfrak{g}, \mathfrak{g}^\sigma_{\mathrm{ss}}) = 0$.

Next let $\omega \in \bigwedge^2(\mathfrak{g}, \mathfrak{g}^\sigma_{\mathrm{ss}})^*$ and suppose that $d\omega = 0$. Since $\omega \in \bigwedge^2 \mathfrak{g}^*$, (3.3) says that $\omega = d\beta$ for some $\beta \in \bigwedge^1 \mathfrak{g}^*$. To complete the proof, we need to show that $\beta \in \bigwedge^1(\mathfrak{g}, \mathfrak{g}^\sigma_{\mathrm{ss}})^*$. In other words, we check that for all $v \in \mathfrak{g}^\sigma_{\mathrm{ss}}$,

$$\langle \beta, v \rangle = \mathrm{ad}^*_v\, \beta = 0. \tag{3.4}$$

By (3.1), G^σ is compact, so G^σ_{ss} is semisimple. Hence up to linear combination, $v \in \mathfrak{g}^\sigma_{\mathrm{ss}}$ can be written as $v = [x, y]$ for $x, y \in \mathfrak{g}^\sigma_{\mathrm{ss}}$. Then

$$\begin{aligned} \langle \beta, v \rangle &= \langle \beta, [x, y] \rangle \\ &= d\beta(x, y) \\ &= \omega(x, y) \\ &= (\iota(x)\, \omega)(y). \end{aligned} \tag{3.5}$$

Since $\omega \in \bigwedge^2(\mathfrak{g}, \mathfrak{g}^\sigma_{\mathrm{ss}})^*$ and $x \in \mathfrak{g}^\sigma_{\mathrm{ss}}$, it follows that $\iota(x)\omega = 0$. Therefore, (3.5) vanishes.

For $x \in \mathfrak{g}^\sigma_{\mathrm{ss}}$ and $y \in \mathfrak{g}$, we apply the same argument as (3.5) and get

$$\langle \mathrm{ad}^*_x \beta, y \rangle = \langle \beta, [x, y] \rangle = (\iota(x)\omega)(y) = 0.$$

Hence $\mathrm{ad}^*_x \beta = 0$. This proves (3.4), and hence the proposition. ∎

As a side remark, we note that in the statement of Proposition 3.2 the subscript G is necessary for the vanishing of cohomology (unless of course

G is compact so that the G-invariant cohomology coincides with the usual cohomology): For non-compact G, the maximal compact subgroup K has a center Z of positive dimension. By (3.1), $G^\sigma_{\mathrm{ss}} \subset K$. Write the Cartan decomposition as $G = QK$, so that $G/G^\sigma_{\mathrm{ss}} = Q \times (K/G^\sigma_{\mathrm{ss}})$. Since Q is contractible,

$$H^q(X_\sigma) = H^q(G/G^\sigma_{ss}) = H^q(K/G^\sigma_{\mathrm{ss}}) = H^q(\mathfrak{k}, \mathfrak{g}^\sigma_{\mathrm{ss}}).$$

Since $\sigma \subset \Delta^{\mathrm{s}}_{\mathrm{c}}$, we get $\mathfrak{z} \subset \mathfrak{t}_\sigma$, so (2.11) implies that $\mathfrak{g}^\sigma_{\mathrm{ss}} \cap \mathfrak{z} = 0$. Hence for all $q \leqslant \dim \mathfrak{z}$, the non-zero elements of $\bigwedge^q \mathfrak{z}^* \subset \bigwedge^q(\mathfrak{k}, \mathfrak{g}^\sigma_{\mathrm{ss}})^*$ have non-trivial cohomology classes in $H^q(\mathfrak{k}, \mathfrak{g}^\sigma_{\mathrm{ss}})$.

The next proposition deals with a Dolbeault cohomology taken over $(G \times T_\sigma)$-invariant forms, as indicated by its subscript.

PROPOSITION 3.3. $H^{0,1}_{GT_\sigma}(X_\sigma) = 0.$

Proof. The vector space $(\mathfrak{g}/\mathfrak{g}^\sigma_{\mathrm{ss}}) + \mathfrak{a}_\sigma$ acquires the complex structure as a tangent space for X_σ. It contains $\mathfrak{h}_\sigma$ as a complex subspace, and so $\mathfrak{g}/(\mathfrak{g}^\sigma_{\mathrm{ss}} + \mathfrak{t}_\sigma) = \mathfrak{g}/\mathfrak{g}^\sigma$ is a complex vector space. Therefore, it makes sense to define $\bigwedge^{0,1}(\mathfrak{g}/\mathfrak{g}^\sigma)^*$. A basis for this space is

$$\{u_i = \zeta_i^* - \sqrt{-1}\, \gamma_i^*\}_{(\alpha_i, \mathfrak{t}_\sigma) \neq 0} \subset \bigwedge^{0,1} (\mathfrak{g}/\mathfrak{g}^\sigma)^*, \tag{3.6}$$

where $\zeta_i^*, \gamma_i^* \in \mathfrak{g}^*$ are the vectors in (2.9). By (2.10), for $x \in \mathfrak{t}$,

$$\mathrm{ad}_x^* u_i = \sqrt{-1}\, \alpha_i(x)\, u_i. \tag{3.7}$$

We apply Proposition 3.1 and express the G-invariant (0,1)-forms on X_σ in a manner similar to (2.12), namely

$$\Omega^{0,1}_G(X_\sigma) = \left\{ \sum_i f_i w_i \in C^\infty(A_\sigma) \otimes \left(\bigwedge^{0,1} (\mathfrak{g}/\mathfrak{g}^\sigma)^* \oplus \bigwedge^{0,1} \mathfrak{h}_\sigma{}^* \right); \right.$$
$$\left. \mathrm{ad}_x^* w_i = 0 \text{ for all } x \in \mathfrak{g}^\sigma \right\}. \tag{3.8}$$

Let $z \in \Omega^{0,1}_G(X_\sigma)$. We want to express z in terms of (3.8) but we omit the harmless linear combination $\sum_i$ for convenience. So $z = f(u+v)$ where $f \in C^\infty(A_\sigma)$, $u \in \bigwedge^{0,1}(\mathfrak{g}/\mathfrak{g}^\sigma)^*$, and $v \in \bigwedge^{0,1} \mathfrak{h}_\sigma^*$. Here u decomposes further to u_i of (3.6), indexed over $(\alpha_i, \mathfrak{t}_\sigma) \neq 0$. Let L and R be the left and right actions, and let $\chi_i = \exp \alpha_i$ be the character of α_i. For all $t \in T_\sigma$,

$$\begin{aligned} R_t^* u_i &= L_t^* R_t^* u_i && \text{by left invariance of } u_i \\ &= \mathrm{Ad}_t^* u_i \\ &= \chi_i(t)\, u_i && \text{by (3.7).} \end{aligned}$$

So each u_i transforms by α_i under the right T_σ-action. On the other hand, since H_σ is abelian, v is invariant under the right T_σ-action. We conclude that if $z = f(u+v)$ is $(G \times T_\sigma)$-invariant, then $u = 0$ and so $z = fv \in C^\infty(A_\sigma) \otimes \bigwedge^{0,1} \mathfrak{h}_\sigma^*$ with respect to (3.8). Indeed, since G^σ commutes with H_σ we get $\mathrm{ad}_x^* v = 0$ for all $x \in \mathfrak{g}^\sigma$, so fv satisfies the requirement for $\Omega_G^{0,1}(X_\sigma)$ in (3.8).

The subcomplex $C^\infty(A_\sigma) \otimes \bigwedge^{0,1} \mathfrak{h}_\sigma^*$ of (3.8) can be identified with the T_σ-invariant (0, 1)-forms on H_σ. Since H_σ is a Stein space, its (0, 1)-Dolbeault cohomology vanishes. Consequently, if $z \in C^\infty(A_\sigma) \otimes \bigwedge^{0,1} \mathfrak{h}_\sigma^*$ is $\bar{\partial}$-closed, then z has to be $\bar{\partial}$-exact. This proves the proposition. ∎

Let ω be a $(G \times T_\sigma)$-invariant closed (1, 1)-form on X_σ. We now apply Propositions 3.2 and 3.3 to obtain a potential function F for ω.

PROPOSITION 3.4. *Every* $(G \times T_\sigma)$*-invariant closed* (1, 1)*-form on* X_σ *can be written as* $\omega = \sqrt{-1}\, \partial\bar{\partial} F$.

Proof. Since ω is closed, Proposition 3.2 says that $\omega = d\beta$ for some real 1-form β. Since β is real, we write $\beta = \alpha + \bar{\alpha}$, where $\alpha \in \Omega_{GT_\sigma}^{0,1}(X_\sigma)$. Then, ω being a (1, 1)-form implies that $\bar{\partial}\alpha = 0$. By Proposition 3.3, $\alpha = \bar{\partial} f$ for some $f \in C_G^\infty(X_\sigma) = C^\infty(A_\sigma)$. Define a real-valued function $F = -\sqrt{-1}\,(f - \bar{f})$. Then

$$\begin{aligned} \omega = d\beta &= \partial f + \bar{\partial} \bar{f} \\ &= \partial\bar{\partial} f + \bar{\partial}\partial\bar{f} \\ &= \sqrt{-1}\, \partial\bar{\partial} F. \end{aligned}$$

This proves the proposition. ∎

We remark that in the above proposition right T_σ-invariance of ω is both necessary and sufficient for the existence of the potential function F. The necessity of right T_σ-invariance is proved in [3].

Given $\xi \in \mathfrak{g}$, we let $\xi^\#$ denote the infinitesimal vector field on X_σ obtained from the left G-action. If J is the almost complex structure and $J\xi \in \mathfrak{a}$ for $\xi \in \mathfrak{t}$, we let $(J\xi)^\# = J(\xi^\#)$. Let ω be a $(G \times T_\sigma)$-invariant (1, 1)-form on X_σ. By G-invariance, it suffices to study ω_a for $a \in A_\sigma \subset X_\sigma$. This is done in the next proposition.

The G-action preserving (X_σ, ω) is Hamiltonian, with moment map

$$\Phi \colon X_\sigma \to \mathfrak{g}^*. \tag{3.9}$$

Recall that $\mathfrak{g} = \mathfrak{t} + V$ and V splits into V_i in (2.6).

PROPOSITION 3.5. *Let* ω *be a* $(G \times T_\sigma)$*-invariant* (1, 1)*-form on* X_σ *and* $a \in A_\sigma$. *For* $i \neq j$, $\omega(\mathfrak{h}_\sigma{}^\#, V_i^\#)_a = \omega(V_i^\#, V_j^\#)_a = 0$.

Proof. The complex structure of X_σ sends $\mathfrak{t}_\sigma^\#$ to $\mathfrak{a}_\sigma^\#$. Since ω is of type $(1, 1)$, it suffices to check that for all $a \in A_\sigma$,

$$\omega(\mathfrak{t}_\sigma^\#, V_i^\#)_a = \omega(V_i^\#, V_j^\#)_a = 0, \qquad i \neq j. \tag{3.10}$$

By the Killing form, $V_i^* \subset \mathfrak{g}^*$ satisfies (2.8). Let $\theta_i \colon \mathfrak{g}^* \to V_i^*$ be the corresponding projection. We now prove the first part of (3.10). Let $x \in \mathfrak{t}_\sigma$ and $y \in V_i$. For all $t \in T_\sigma$,

$$\omega(x^\#, y^\#)_{ta} = (\Phi(ta), [x, y]) = (\Phi(a), \mathrm{Ad}_t[x, y]) \tag{3.11}$$

because Φ is G-equivariant. According to (2.4), $[x, y] \in V_i$ and Ad_t acts on the two-dimensional V_i by rotation. So $\mathrm{Ad}_t[x, y] \in V_i$. Then (3.11) becomes

$$\omega(x^\#, y^\#)_{ta} = (\theta_i \Phi(a), \mathrm{Ad}_t[x, y]) \tag{3.12}$$

due to (2.8). On the other hand, since H_σ is abelian,

$$\omega(x^\#, y^\#)_{ta} = \omega(x^\#, y^\#)_{at} = \{R_t^*(\omega(x^\#, y^\#))\}_a, \tag{3.13}$$

where R denotes right action. Since ω, $x^\#$, $y^\#$ are all right T_σ-invariant, the function $\omega(x^\#, y^\#) \in C^\infty(X_\sigma)$ is right T_σ-invariant too. So (3.13) becomes

$$\begin{aligned} \omega(x^\#, y^\#)_{ta} &= \{R_t^*(\omega(x^\#, y^\#))\}_a \\ &= \omega(x^\#, y^\#)_a && \text{by right } T_\sigma\text{-invariance} \\ &= (\Phi(a), [x, y]) \\ &= (\theta_i \Phi(a), [x, y]) && \text{by (2.8) and } [x, y] \in V_i. \end{aligned} \tag{3.14}$$

By (3.12) and (3.14),

$$(\theta_i \Phi(a), \mathrm{Ad}_t[x, y]) = (\theta_i \Phi(a), [x, y]). \tag{3.15}$$

This equation is valid for all $t \in T_\sigma$, $x \in \mathfrak{t}_\sigma$, and $y \in V_i$. Fix x, y with $0 \neq [x, y] \in V_i$. As t varies in T_σ, $\mathrm{Ad}_t[x, y]$ traces out a circle in the two-dimensional V_i. So for (3.15) to be valid, $\theta_i \Phi(a) \in V_i^*$ has to be 0. Then (3.15) vanishes, and this implies the vanishing of (3.11), (3.12), (3.13), and (3.14). This proves the first part of (3.10).

Since $\theta_i \Phi(a) = 0$ for all i, by (2.8) we get $\Phi(a) \in \mathfrak{t}^*$. Let $\mathfrak{t}_\sigma^\perp$ be the orthocomplement of $\mathfrak{t}_\sigma$ in $\mathfrak{t}$, via the Killing form. By (2.11), $T_\sigma^\perp \subset G_{\mathrm{ss}}^\sigma$. So the left action of $T_\sigma^\perp$ fixes a and the statement $\Phi(a) \in \mathfrak{t}^*$ can be sharpened to

$$\Phi(a) \in \mathfrak{t}_\sigma^*. \tag{3.16}$$

We next show the second part of (3.10). Let $x \in V_i$ and $y \in V_j$, with $i \neq j$. Then

$$\begin{aligned} \omega(x^\#, y^\#)_a &= (\Phi(a), [x, y]) \in (\mathfrak{t}_\sigma^*, V) && \text{by (2.7), (3.16)} \\ &= 0 && \text{by (2.8).} \end{aligned}$$

This proves the second part of (3.10). The proposition follows. ∎

We next study the moment map (3.9). We shall describe it by the potential function F. Since Φ is G-equivariant and $X_\sigma = (G/G_{ss}^\sigma)\, A_\sigma$, Φ is determined entirely by its restriction to $A_\sigma \subset X_\sigma$. By (3.16), we can write

$$\Phi\colon A_\sigma \to \mathfrak{t}_\sigma^*. \tag{3.17}$$

On the other hand, the gradient of the potential function $F \in C^\infty(A_\sigma)$ is

$$F'\colon A_\sigma \to \mathfrak{a}_\sigma^* \cong \mathfrak{t}_\sigma^*, \tag{3.18}$$

where $\mathfrak{a}_\sigma \cong \mathfrak{t}_\sigma$ by the complex structure. The maps (3.17) and (3.18) are related by

Proposition 3.6. *For* $a \in A_\sigma$, $\Phi(a) = \frac{1}{2} F'(a)$.

Proof. Let $\iota\colon H_\sigma \hookrightarrow X_\sigma$ be the natural imbedding. By (3.16), the moment maps of ω and $\iota^*\omega$ coincide when restricted to H_σ. So we may consider Φ as the moment map for the T_σ-invariant form $\iota^*\omega$. To compute Φ, we introduce coordinates

$$\begin{aligned} H_\sigma &= T_\sigma \times A_\sigma = (\mathbf{R}^r/\mathbf{Z}^r) \times \mathbf{R}^r \\ &= \{z_i = [x_i] + \sqrt{-1}\, y_i;\ i = 1, \ldots, r\}. \end{aligned} \tag{3.19}$$

By T_σ-invariance $F(z) = F(y)$, and we get

$$\begin{aligned} \iota^*\omega &= \sqrt{-1}\, \partial\bar\partial F \\ &= \sqrt{-1}\, \frac{\partial}{\partial z_i} \frac{\partial}{\partial \bar z_j} F\, dz_i \wedge d\bar z_j \\ &= \frac{1}{2} \frac{\partial^2 F}{\partial y_i\, \partial y_j}\, dx_i \wedge dy_j. \end{aligned} \tag{3.20}$$

The T_σ-invariant 1-form $\beta = -\frac{1}{2}(\partial F/\partial y_i)\, dx_i$ satisfies $\iota^*\omega = d\beta$. Write $v = (v_i) \in \mathfrak{t}_\sigma \cong \mathbf{R}^r$. For all $y \in A_\sigma$ and $v \in \mathfrak{t}_\sigma$,

$$
\begin{aligned}
(\Phi(y), v) &= -(\beta, v^{\#})_y \qquad \text{by [1, Theorem 4.2.10]} \\
&= \left(\frac{1}{2}\frac{\partial F}{\partial y_i}\, dx_i, v_j \frac{\partial}{\partial x_j}\right)_y \\
&= \frac{1}{2}\frac{\partial F}{\partial y_i}(y)\, v_i \\
&= \left(\frac{1}{2} F'(y), v\right).
\end{aligned}
$$

This proves the proposition. ∎

Proof of Theorem 1. Let ω be a $(G \times T_\sigma)$-invariant closed $(1, 1)$-form on X_σ. By Proposition 3.4 we get $\omega = \sqrt{-1}\, \partial\bar{\partial} F$, and the moment map $\Phi = \frac{1}{2} F'$ is given by Proposition 3.6. The Kähler and pseudo-Kähler conditions on ω remain to be studied. By G-invariance, it suffices to consider ω_a for $a \in A_\sigma$. In view of Proposition 3.5, we can study ω_a on $\mathfrak{h}_\sigma^{\#}$ and each $V_i^{\#}$ separately.

We first consider ω_a on $\mathfrak{h}_\sigma^{\#}$. Let ι be the imbedding of Proposition 3.6. Then ω is Kähler or pseudo-Kähler on $(\mathfrak{h}_\sigma)_a^{\#}$ exactly when $\iota^*\omega$ is Kähler or pseudo-Kähler. By the Hessian matrix $(\partial^2 F/\partial y_i\, \partial y_j)_{ij}$ from (3.20), $\iota^*\omega$ is Kähler or pseudo-Kähler exactly when F is strictly convex or non-degenerate.

We next restrict ω_a to $V_i^{\#}$. Here i is indexed according to $\alpha_i \in \Delta^+$. If $(\alpha_i, \mathfrak{t}_\sigma) = 0$, then (2.11) says that $V_i \subset \mathfrak{g}_{ss}^\sigma$, so $(V_i^{\#})_a = 0$ because G_{ss}^σ fixes a. Therefore, it suffices to consider $(\alpha_i, \mathfrak{t}_\sigma) \neq 0$. By (2.5) and Proposition 3.6, the vectors $\zeta_i, \gamma_i \in V_i$ satisfy

$$
\omega(\zeta_i^{\#}, \gamma_i^{\#})_a = (\Phi(a), [\zeta_i, \gamma_i]) = \pm(\tfrac{1}{2} F'(a), \alpha_i). \tag{3.21}
$$

Recall the definition of $(\mathfrak{t}_\sigma^*)_{\text{reg}} \cong (\mathfrak{a}_\sigma^*)_{\text{reg}}$ from (1.2). The last expression of (3.21) is non-zero if and only if $\frac{1}{2} F'(a)$ is not perpendicular to α_i or equivalently $\frac{1}{2} F'(a) \in (\mathfrak{a}_\sigma^*)_{\text{reg}}$. We determine the sign $\pm$ by (2.5). If $\alpha_i \in \Delta_{\text{c}}^+$, then (3.21) is positive if and only if $(\frac{1}{2} F'(a), \alpha_i) > 0$. If $\alpha_i \in \Delta_{\text{n}}^+$, then (3.21) is positive if and only if $(\frac{1}{2} F'(a), \alpha_i) < 0$. So (3.21) is positive for all $\alpha_i \in \Delta^+$ exactly when $\frac{1}{2} F'(a) \in (\mathfrak{a}_\sigma^*)_{\text{reg}} \cap \mathscr{C}$. This proves Theorem 1. ∎

4. GEOMETRIC QUANTIZATION

Let ω be a $(G \times T_\sigma)$-invariant Kähler form on X_σ. The purpose of this section is to apply the standard scheme of geometric quantization [14] to (X_σ, ω), and prove Theorem 2.

We need a nice invariant measure on X_σ to perform integration later. This is given by the next proposition.

PROPOSITION 4.1. *There exists a $(G \times A_\sigma)$-invariant measure on X_σ which is unique up to scalar.*

Proof. Recall that $X_\sigma = (G/G_{ss}^\sigma)\, A_\sigma$. Since G and G_{ss}^σ are semisimple, they are in particular unimodular, so G/G_{ss}^σ has a G-invariant volume form [12, p. 89]. Taking its product with the Haar measure on A_σ creates a $(G \times A_\sigma)$-invariant measure on X_σ. Further, since $G \times A_\sigma$ acts transitively on X_σ, such a measure has to be unique up to scalar. ∎

By Theorem 1, ω has a strictly convex potential function F. As explained in the Introduction, ω corresponds to a pre-quantum line bundle L with compatible connection ∇ and Hermitian structure $(\, , \,)$. The $(G \times T_\sigma)$-action on X_σ lifts to a $(G \times T_\sigma)$-representation on the holomorphic sections on L.

PROPOSITION 4.2. *There exists a $(G \times T_\sigma)$-invariant non-vanishing holomorphic section s_0 on L satisfying $(s_0, s_0)_a = e^{-F(a)}$ for all $a \in A_\sigma \subset X_\sigma$.*

Proof. Since the Chern class of L is $[\omega] = 0$, we can pick a $(G \times T_\sigma)$-invariant non-vanishing smooth section s. Let $\alpha = \sqrt{-1}\, \frac{\nabla s}{s}$, so that by the definition of the curvature form $d\alpha = \omega$. Since s is G-invariant, so are ∇s and α. Let $\gamma = -\sqrt{-1}\, \partial F$, so that $d\gamma = \bar{\partial}\gamma = \omega$. Hence α and γ are complex G-invariant 1-forms satisfying $d(\gamma - \alpha) = 0$. By Proposition 3.2, there exists a G-invariant complex-valued function $f \in C^\infty(A_\sigma)$ such that

$$\gamma - \alpha = df.$$

Let $s_0 = e^{-\sqrt{-1} f} s$. Then

$$\begin{aligned} \nabla s_0 &= \nabla(e^{-\sqrt{-1} f} s) \\ &= e^{-\sqrt{-1} f}\, df\, s + e^{-\sqrt{-1} f}\, \nabla s \\ &= -\sqrt{-1}\, df\, s_0 - \sqrt{-1}\, \alpha\, s_0 \\ &= -\sqrt{-1}\, \gamma\, s_0. \end{aligned} \tag{4.1}$$

It follows from $s_0 = e^{-\sqrt{-1} f} s$ that s_0 is non-vanishing and $(G \times T_\sigma)$-invariant. Since γ is a $(1, 0)$-form, it follows from $\nabla s_0 = -\sqrt{-1}\, \gamma s_0$ that ∇s_0 annihilates anti-holomorphic vector fields. So s_0 is holomorphic.

Since s_0 is G-invariant, (s_0, s_0) is determined by its value on $A_\sigma \subset X_\sigma$. This is found by differentiating along the A_σ direction. We write $z = [x] + \sqrt{-1}\, y$ as in (3.19). Since $\gamma = -\sqrt{-1}\, \partial F$, (4.1) gives

$$\nabla s_0 = \frac{\sqrt{-1}}{2} \frac{\partial F}{\partial y}\, dz\, s_0. \tag{4.2}$$

We evaluate (s_0, s_0) against $\frac{\partial}{\partial y}$ and get

$$\begin{aligned}\frac{\partial}{\partial y}(s_0, s_0) &= (\nabla_{\partial/\partial y} s_0, s_0) + (s_0, \nabla_{\partial/\partial y} s_0)\\ &= (\sqrt{-1}\,\nabla_{\partial/\partial x} s_0, s_0) + (s_0, \sqrt{-1}\,\nabla_{\partial/\partial x} s_0) \quad \text{since } s \text{ is holomorphic}\\ &= \left(-\frac{1}{2}\frac{\partial F}{\partial y} s_0, s_0\right) + \left(s_0, -\frac{1}{2}\frac{\partial F}{\partial y} s_0\right) \qquad \text{by (4.2)}\\ &= -\frac{\partial F}{\partial y}(s_0, s_0).\end{aligned}$$

Replacing s_0 by a constant multiple if necessary, it follows that $(s_0, s_0) = e^{-F}$. Hence we have the proposition. ▮

Let μ_X be the $(G \times A_\sigma)$-invariant volume form on X_σ given by Proposition 4.1. We define an L^2-structure on the holomorphic sections s on L via (1.8). The holomorphic sections which converge under this L^2-structure are said to be square-integrable and are denoted by H_ω. Since the L^2-structure is $(G \times T_\sigma)$-invariant, H_ω is a unitary $(G \times T_\sigma)$-representation. For an integral weight $\lambda \in \mathfrak{t}_\sigma^*$, let $H(L)_\lambda$ be the holomorphic sections which transform by λ under the right T_σ-action. We want to evaluate the conditions on λ in which $H(L)_\lambda \subset H_\omega$ and prove Theorem 2.

We think of the integral weight λ as an element of $\mathfrak{h}_\sigma^*$, $\mathfrak{t}_\sigma^*$, or $\mathfrak{a}_\sigma^*$. Thus $\lambda \in \mathfrak{h}_\sigma^*$ defines a homogeneous line bundle over Y_σ,

$$L_\lambda \to Y_\sigma.$$

Let $C^\infty(Y_\sigma, L_\lambda)$ be the smooth sections on L_λ. There is a natural assignment

$$s \in C^\infty(Y_\sigma, L_\lambda) \mapsto f_s \in C^\infty(X_\sigma). \tag{4.3}$$

Here f_s transforms by $\lambda \in \mathfrak{h}_\sigma^*$ under the right H_σ-action. Recall from (1.9) that $\chi = e^\lambda$ is a multiplicative homomorphism from H_σ to $\mathbf{C}^\times$. Its restriction to A_σ is

$$\chi\colon A_\sigma \to \mathbf{R}^+. \tag{4.4}$$

The right action R of $a \in A_\sigma$ gives

$$R_a^*(f_s \bar{f}_t) = (\chi_a f_s)(\overline{\chi(a) f_t}) = \chi(a)^2 f_s \bar{f}_t, \tag{4.5}$$

for all $s, t \in C^\infty(Y_\sigma, L_\lambda)$.

Define a G-invariant function

$$\chi_A \in C_G^\infty(X_\sigma)$$

simply by extending (4.4) to X_σ by G-invariance. The function χ_A is non-vanishing, so it makes sense to consider its negative power $\chi_A^{-2} \in C_G^\infty(X_\sigma)$.

From π of (1.4), we get $\pi^*: C^\infty(Y_\sigma) \to C^\infty(X_\sigma)$.

PROPOSITION 4.3. *A G-invariant Hermitian structure* $(\ ,\ )^{L_\lambda}$ *on* $C^\infty(Y_\sigma, L_\lambda)$ *is given by* $\pi^*(s, t)^{L_\lambda} = f_s \bar{f}_t \chi_A^{-2}$. *It is unique up to scalars.*

Proof. We claim that $f_s \bar{f}_t \chi_A^{-2}$ is right H_σ-invariant. By (4.5), it is right A_σ-invariant. So it only remains to check the right T_σ-action.

Since χ_A^{-2} is defined to be G-invariant, it is necessarily right T_σ-invariant. So it suffices to consider $f_s \bar{f}_t$. Note that $\chi: T_\sigma \to S^1 \subset \mathbf{C}^\times$. For all $\theta \in T_\sigma$,

$$R_\theta^*(f_s \bar{f}_t) = (\chi(\theta)\, f_s)(\overline{\chi(\theta)\, f_t}) = \chi(\theta)\, \overline{\chi(\theta)}\, f_s \bar{f}_t = f_s \bar{f}_t.$$

This shows that $f_s \bar{f}_t$ is right T_σ-invariant and so is $f_s \bar{f}_t \chi_A^{-2}$. We conclude that $f_s \bar{f}_t \chi_A^{-2}$ is right H_σ-invariant, as claimed.

This means that $f_s \bar{f}_t \chi_A^{-2}$ is in the image of π^*. Since π^* is injective, let $(s, t)^{L_\lambda}$ be the unique function in which $\pi^*(s, t)^{L_\lambda} = f_s \bar{f}_t \chi_A^{-2}$. This defines a G-invariant Hermitian structure on $C^\infty(Y_\sigma, L_\lambda)$. Since G acts transitively on Y_σ, such a Hermitian structure is unique up to scalars. This proves the proposition. ∎

Recall that in (1.6) we define $U_F \subset \mathfrak{a}_\sigma^*$ to be the image of $\frac{1}{2} F'$. The following proposition will be helpful in proving Theorem 2.

PROPOSITION 4.4. *The integral* $\int_{A_\sigma} \chi(a)^2\, e^{-F(a)}\, da$ *converges if and only if* $\lambda \in U_F$.

Proof. We change the variables by the diffeomorphism

$$e: \mathfrak{a}_\sigma \to A_\sigma, \qquad e^v = a.$$

By (1.9), $\chi(a)^2 = \exp(2\lambda, v)$ whenever $e^v = a$. Let dV be the Lebesgue measure of $\mathfrak{a}_\sigma$. We identify $F(a)$ with $F(v)$ via (1.5). Then

$$\int_{A_\sigma} \chi(a)^2\, e^{-F(a)}\, da = \int_{\mathfrak{a}_\sigma} \exp((2\lambda, v) - F(v))\, dV. \tag{4.6}$$

Since F is strictly convex, [5, Appendix] says that the RHS of (4.6) converges if and only if $\lambda \in U_F$. Hence we have the proposition. ∎

Proof of Theorem 2. Consider the natural fibrations

$$\pi_1: X_\sigma \to G/G_{ss}^\sigma, \qquad \pi_2: G/G_{ss}^\sigma \to G/G^\sigma = Y_\sigma. \tag{4.7}$$

Thus we have that (1.4) is $\pi = \pi_2 \cdot \pi_1$.

Let $s \in H(L)_\lambda$. We want to determine if s is square-integrable with respect to (1.8). By Proposition 4.2, we write $s = fs_0$, where $f \in H(X_\sigma)_\lambda$. The $(G \times A_\sigma)$-invariant measure of Proposition 4.1 is of the form $\mu_X = dg\, da$, where dg is the G-invariant measure on G/G^σ_{ss} and da is the Haar measure on A_σ. We have

$$\int_{X_\sigma} (s, s)\, \mu_X = \int_{ga \in (G/G^\sigma_{ss})\, A_\sigma} (fs_0, fs_0)_{ga}\, dg\, da$$

$$= \int_{ga \in (G/G^\sigma_{ss})\, A_\sigma} (f\bar{f})_{ga}\, e^{-F(a)}\, dg\, da. \tag{4.8}$$

Since f is holomorphic and transforms by $\lambda \in \mathfrak{t}^*_\sigma$ under the right T_σ-action, it necessarily transforms by the complexified $\lambda \in \mathfrak{h}^*_\sigma$ under the right H_σ-action. So f is in the image of (4.3), i.e., $f = f_u$ for some holomorphic section u on L_λ. By Proposition 4.3, Eq. (4.8) becomes

$$\int_{X_\sigma} (s, s)\, \mu_X = \int_{ga \in (G/G^\sigma_{ss})\, A_\sigma} \pi^*(u, u)^{L_\lambda}_g\, \chi_A(a)^2\, e^{-F(a)}\, dg\, da$$

$$= \int_{G/G^\sigma_{ss}} \pi_2^*(u, u)^{L_\lambda}_g\, dg \int_{A_\sigma} \chi(a)^2\, e^{-F(a)}\, da, \tag{4.9}$$

where π_2 is the fibration in (4.7). The flag domain Y_σ has a G-invariant measure μ_Y [9]. The fiber of π_2 is T_σ, which is compact. So by [12, Proposition 1.13, p. 95],

$$\int_{G/G^\sigma_{ss}} \pi_2^*(u, u)^{L_\lambda}_g\, dg = \int_{y \in Y_\sigma} (u, u)^{L_\lambda}_y\, \mu_Y. \tag{4.10}$$

By (4.8), (4.9), and (4.10),

$$\int_{X_\sigma} (s, s)\, \mu_X = \int_{y \in Y_\sigma} (u, u)^{L_\lambda}_y\, \mu_Y \int_{A_\sigma} \chi(a)^2\, e^{-F(a)}\, da. \tag{4.11}$$

We may assume that $\lambda + \rho \in (\mathfrak{t}^*)_{\text{reg}}$, for otherwise there is no discrete series in $\Theta_{\lambda+\rho}$. If $\lambda \notin U_F$, then Proposition 4.4 says that $\int_{A_\sigma} \chi(a)^2\, e^{-F(a)}\, da$ diverges and so Eq. (4.11) diverges.

Conversely, suppose that $\lambda \in U_F$. By Proposition 4.4, $\int_{A_\sigma} \chi(a)^2\, e^{-F(a)}\, da$ converges. Also, Theorem 1 says that $\lambda \in U_F \subset (\mathfrak{a}^*_\sigma)_{\text{reg}} \cap \mathscr{C}$. By [9], the holomorphic sections u on L_λ converge in (4.11) and form the discrete series $\Theta_{\lambda+\rho}$. Theorem 2 follows. ∎

5. HOLOMORPHIC DISCRETE MODEL

A unitary G-representation is said to be a holomorphic discrete model if it contains every holomorphic discrete series exactly once. We now apply Theorems 1 and 2 to construct such a representation.

Let W be a real vector space of dimension r and let $\lambda_1, ..., \lambda_r$ be a basis of its dual W^*. The following proposition defines a function $F \in C^\infty(W)$ and studies its gradient F': $W \to W^*$.

PROPOSITION 5.1. *Let $F(y) = \sum_1^r \exp(\lambda_k, y)$, for $y \in W$. Then $F'(0) = \sum_1^r \lambda_k$, and the image of F' is all positive linear combinations of $\lambda_1, ..., \lambda_r$. Also, F is strictly convex.*

Proof. The basis $\{\lambda_k\}$ identifies W with $\mathbf{R}^r$ and introduces coordinates $y_1, ..., y_r$ on W. Under such coordinates, F becomes $F(y) = \sum_1^r \exp y_k$.

Its gradient is $F'(y) = (\partial F/\partial y_i)_i = (\exp y_i)_i$. Since $\exp y_i > 0$ the image of F' is $(\mathbf{R}^+)^r$, which is identified with the positive linear combinations of $\lambda_1, ..., \lambda_r$. In particular, $F'(0) = (\exp y_i)_i|_{y=0} = (1, ..., 1)$, which is identified with the element $\sum_1^r \lambda_k \in W^*$.

The Hessian matrix of F is $((\partial^2/\partial y_i \, \partial y_j) \sum_1^r \exp y_k)_{ij}$, and this is a diagonal matrix with entries $(\exp y_1, ..., \exp y_r)$. It is a positive definite matrix, so F is strictly convex.

Hence we have the proposition. ∎

Let $\sigma \subset \Delta_{\mathrm{c}}^{\mathrm{s}}$. Recall that $(\mathfrak{t}_\sigma^*)_{\mathrm{reg}} \cong (\mathfrak{a}_\sigma^*)_{\mathrm{reg}}$ and that $\mathscr{C}$ are defined in (1.2) and (1.7). We assume that $\mathscr{C} \neq \varnothing$ in this section. Let $\lambda_1, ..., \lambda_r \in \mathfrak{a}^*$ be the fundamental weights for the cone $(\mathfrak{a}_\sigma^*)_{\mathrm{reg}} \cap \mathscr{C}$, namely

$$(\mathfrak{a}_\sigma^*)_{\mathrm{reg}} \cap \mathscr{C} = \left\{ \sum_1^r c_i \lambda_i ; \, c_i > 0 \right\}.$$

We define $F_\sigma \in C^\infty(\mathfrak{a}_\sigma)$ by $F_\sigma(y) = \sum_1^r \exp(\lambda_i, y)$. By Proposition 5.1, the image of F'_σ is $(\mathfrak{a}_\sigma^*)_{\mathrm{reg}} \cap \mathscr{C}$ and F_σ is strictly convex. By (1.5), we identify it with $F_\sigma \in C^\infty(A_\sigma)$. By Theorem 1, $\omega_\sigma = \sqrt{-1}\, \partial \bar{\partial} F_\sigma$ is Kähler. By Theorem 2, the representation H_{ω_σ} contains every holomorphic discrete series $\Theta_{\lambda+\rho}$ in which $\lambda \in U_F = (\mathfrak{a}_\sigma^*)_{\mathrm{reg}} \cap \mathscr{C}$ once.

The holomorphic discrete series $\Theta_{\lambda+\rho}$ are parametrized by integral weights λ in which $\lambda + \rho \in (\mathfrak{t}^*)_{\mathrm{reg}}$ and λ lies in the region

$$\{x \in \mathfrak{a}^*; \, (\Delta_{\mathrm{c}}^+, x) \geqslant 0, \, (\Delta_{\mathrm{n}}^+, x + \rho) < 0\}.$$

This region is contained in $\mathscr{C}$. Observe that $\{(\mathfrak{a}_\sigma^*)_{\mathrm{reg}} \cap \mathscr{C}\}_{\sigma \subset \Delta_{\mathrm{c}}^{\mathrm{s}}}$ is a partition of $\mathscr{C}$. Therefore, as σ varies over all subsets of $\Delta_{\mathrm{c}}^{\mathrm{s}}$,

$$\bigoplus_\sigma H_{\omega_\sigma}$$

is a holomorphic discrete model of G.

6. L^2-COHOMOLOGY

The space of holomorphic sections on the pre-quantum line bundle has allowed us to construct the holomorphic discrete series of G. In this section, we consider the L^2-cohomology in order to obtain other discrete series and prove Theorem 3. We fix σ, P via (1.3) and define X_σ, Y_σ as before.

Since $\mathfrak{g}^{\mathbf{C}}$ is semisimple, its Killing form is non-degenerate. Recall that we let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}^{\mathbf{C}}$ in (2.1). So the Killing form is negative definite on $\mathfrak{u}$ and positive definite on $\sqrt{-1}\,\mathfrak{u}$. By making suitable sign changes on the Killing form, we obtain a positive definite inner product $(-, -)$ on $\mathfrak{g}^{\mathbf{C}}$. The subspaces V_i of (2.6) satisfy

$$(\mathfrak{h}_\sigma, \mathfrak{h}_\sigma^\perp) = (\mathfrak{h}_\sigma, V_i) = (\mathfrak{h}_\sigma^\perp, V_i) = (V_i, V_j) = 0 \tag{6.1}$$

for $i \neq j$. The parabolic subalgebra $\mathfrak{p}$ and its commutator $[\mathfrak{p}, \mathfrak{p}]$ are built by piecing together the various subspaces of $\mathfrak{g}^{\mathbf{C}}$ which appear in (6.1). Therefore, the orthogonality conditions in Eq. (6.1) imply that it descends to inner products on $\mathfrak{g}^{\mathbf{C}}/[\mathfrak{p}, \mathfrak{p}]$ and $\mathfrak{g}^{\mathbf{C}}/\mathfrak{p}$. Taking their duals, we get inner products on $(\mathfrak{g}^{\mathbf{C}}/[\mathfrak{p}, \mathfrak{p}])^*$ and $(\mathfrak{g}^{\mathbf{C}}/\mathfrak{p})^*$. These dual spaces can be identified with the cotangent spaces of X_σ and Y_σ respectively at the identity cosets e. We now have the inner products

$$(\ ,\ ): T_e^* X_\sigma \times T_e^* X_\sigma \to \mathbf{C} \tag{6.2}$$

and

$$(\ ,\ ): T_e^* Y_\sigma \times T_e^* Y_\sigma \to \mathbf{C}. \tag{6.3}$$

The Killing form is adjoint invariant. This means that if we translate (6.2) to all of X_σ by the $(G \times A_\sigma)$-action, we get a $(G \times A_\sigma)$-invariant Hermitian structure on the cotangent bundle

$$(\ ,\ )^X: T^* X_\sigma \times T^* X_\sigma \to \mathbf{C}. \tag{6.4}$$

Similarly, (6.3) leads to a G-invariant Hermitian structure on the cotangent bundle

$$(\, , \,)^Y \colon T^* Y_\sigma \times T^* Y_\sigma \to \mathbf{C}. \tag{6.5}$$

Recall that π denotes the fibration of X_σ over Y_σ. We get an injection between the Dolbeault $(0, q)$-forms,

$$\pi^* \colon \Omega^{0,q}(Y_\sigma) \to \Omega^{0,q}(X_\sigma). \tag{6.6}$$

These Dolbeault differential forms inherit Hermitian structures from (6.4) and (6.5). We still use the same notation and write

$$(\, , \,)^X \colon \Omega^{0,q}(X_\sigma) \times \Omega^{0,q}(X_\sigma) \to C^\infty(X_\sigma) \tag{6.7}$$

as well as

$$(\, , \,)^Y \colon \Omega^{0,q}(Y_\sigma) \times \Omega^{0,q}(Y_\sigma) \to C^\infty(Y_\sigma). \tag{6.8}$$

They are related by the injection (6.6).

PROPOSITION 6.1. *For all* $\alpha, \beta \in \Omega^{0,q}(Y_\sigma)$, $(\pi^*\alpha, \pi^*\beta)^X = \pi^*(\alpha, \beta)^Y$.

Proof. Consider the natural map $\pi \colon \mathfrak{g}^{\mathbf{C}}/[\mathfrak{p}, \mathfrak{p}] \to \mathfrak{g}^{\mathbf{C}}/\mathfrak{p}$. The inner products (6.2) and (6.3) induced by the Killing form satisfy

$$(u, v) = (\pi^* u, \pi^* v) \tag{6.9}$$

for all $u, v \in (\mathfrak{g}^{\mathbf{C}}/\mathfrak{p})^*$.

Since $\Omega^{0,q}(Y_\sigma)$ is obtained from the linear combinations and exterior powers of $\Omega^{0,1}(Y_\sigma)$, it suffices to prove the proposition for $q = 1$. So let $\alpha, \beta \in \Omega^{0,1}(Y_\sigma)$. Then (6.9) says that at e,

$$(\alpha, \beta)^Y_e = (\alpha_e, \beta_e) = ((\pi^*\alpha)_e, (\pi^*\beta)_e) = (\pi^*\alpha, \pi^*\beta)^X_e.$$

By G-equivariance of π this suffices, since checking $(\alpha, \beta)^Y \in C^\infty(Y_\sigma)$ at another point $L_g(e) \in Y_\sigma$ is the same as checking $(L_g^*\alpha, L_g^*\beta)^Y \in C^\infty(Y_\sigma)$ at e. The proposition follows. ∎

Let $\omega = \sqrt{-1}\, \partial\bar{\partial} F$ be a $(G \times T_\sigma)$-invariant pseudo-Kähler form on X_σ with F strictly convex. As before, it leads to a pre-quantum line bundle L over X_σ. Fix an integral weight $\lambda \in \mathfrak{t}_\sigma^*$. Let $\Omega^{0,q}_\lambda(X_\sigma)$ be the $(0, q)$-forms on X_σ which transform by λ under the right T_σ-action. We similarly define $\Omega^{0,q}_\lambda(X_\sigma, L)$ for coefficients in L. This space acquires a Hermitian structure $(\, , \,)^L$ by taking the product of (6.7) and the Hermitian structure of L. The holomorphic section s_0 of Proposition 4.2 leads to a $(G \times T_\sigma)$-equivariant trivialization

$$\Omega_\lambda^{0,q}(X_\sigma) \to \Omega_\lambda^{0,q}(X_\sigma, L); \qquad \alpha \mapsto \alpha \otimes s_0. \tag{6.10}$$

Extend $\lambda \in \mathfrak{t}_\sigma^*$ to $\mathfrak{h}_\sigma^*$ by complex linearity, and let L_λ be the homogeneous line bundle over Y_σ corresponding to λ. We let $\Omega^{0,q}(Y_\sigma, L_\lambda)$ be the $(0, q)$-forms on Y_σ with coefficients in L_λ. Taking the product of Hermitian structures of Proposition 4.3 and (6.8), we obtain a Hermitian structure on $\Omega^{0,q}(Y_\sigma, L_\lambda)$ by

$$(\alpha \otimes s, \beta \otimes t)^{L_\lambda} = (\alpha, \beta)^Y (s, t)^{L_\lambda}. \tag{6.11}$$

In fact, this is the Hermitian structure discussed in [16].

Recall from Section 4 that μ_X and μ_Y are respectively the $(G \times A_\sigma)$-invariant measure on X_σ and the G-invariant measure on Y_σ. By integrating $(\ ,\ )^L \mu_X$ we obtain an L^2-structure $\langle\ ,\ \rangle^L$ on $\Omega_\lambda^{0,q}(X_\sigma, L)$. Similarly, by integrating $(\ ,\ )^{L_\lambda} \mu_Y$, we obtain an L^2-structure $\langle\ ,\ \rangle^{L_\lambda}$ on $\Omega^{0,q}(Y_\sigma, L_\lambda)$.

By (4.3), a smooth section s on L_λ can be naturally identified with $f_s \in \Omega_\lambda^{0,0}(X_\sigma)$. We use (6.6) and (6.10) to define

$$\pi_\lambda^*: \Omega^{0,q}(Y_\sigma, L_\lambda) \to \Omega_\lambda^{0,q}(X_\sigma, L); \qquad \pi_\lambda^*(\alpha \otimes s) = f_s \pi^* \alpha \otimes s_0. \tag{6.12}$$

For the rest of this section, we use π_λ^* to compare the L^2-structures $\langle\ ,\ \rangle^L$ and $\langle\ ,\ \rangle^{L_\lambda}$.

PROPOSITION 6.2. *Suppose that $\phi, \varphi \in C_G^\infty(X_\sigma) = C^\infty(A_\sigma)$ and that $x, y \in \Omega^{0,q}(Y_\sigma, L_\lambda)$ are square-integrable. Then $\langle \phi \pi_\lambda^* x, \varphi \pi_\lambda^* y \rangle^L = \int_{A_\sigma} \phi \bar{\varphi} \chi^2 \times e^{-F}\, da \,\langle x, y \rangle^{L_\lambda}$.*

Proof. Write $x = \alpha \otimes s$, $y = \beta \otimes t \in \Omega^{0,q}(Y_\sigma, L_\lambda)$. Then

$$\begin{aligned}
&\langle \phi \pi_\lambda^*(\alpha \otimes s), \varphi \pi_\lambda^*(\beta \otimes t) \rangle^L \\
&\quad = \int_{X_\sigma} \phi \bar{\varphi} f_s \bar{f}_t (\pi^* \alpha, \pi^* \beta)^X (s_0, s_0)\, \mu_X && \text{by (6.12)} \\
&\quad = \int_{X_\sigma} \phi \bar{\varphi} f_s \bar{f}_t \pi^* (\alpha, \beta)^Y e^{-F} \mu_X && \text{by Propositions 4.2 and 6.1} \\
&\quad = \int_{X_\sigma} \phi \bar{\varphi} \pi^* (s, t)^{L_\lambda} \chi_A^2 \pi^* (\alpha, \beta)^Y e^{-F} \mu_X && \text{by Proposition 4.3} \\
&\quad = \int_{X_\sigma} \phi \bar{\varphi} \pi^* (\alpha \otimes s, \beta \otimes t)^{L_\lambda} \chi_A^2 e^{-F} \mu_X && \text{by (6.11).}
\end{aligned} \tag{6.13}$$

Recall the fibrations π_1, π_2 of (4.7) satisfying $\pi = \pi_2 \cdot \pi_1$. Write $\mu_X = dg\, da$. The last expression of (6.13) becomes

$$\int_{G/G^\sigma_{ss}} \pi_2^*(x, y)^{L_\lambda}\, dg \int_{A_\sigma} \phi\bar{\varphi}\chi^2 e^{-F}\, da. \tag{6.14}$$

Each fiber of π_2 is a copy of T_σ. Since T_σ is compact, $\pi_2^*(x, y)^{L_\lambda}\, dg$ is integrable over each fiber of π_2. So by [12, Proposition 1.13, p. 95],

$$\int_{G/G^\sigma_{ss}} \pi_2^*(x, y)^{L_\lambda}\, dg = \int_{G/G^\sigma} (x, y)^{L_\lambda}\, \mu_Y = \langle x, y\rangle^{L_\lambda}. \tag{6.15}$$

The proposition follows from (6.13), (6.14), and (6.15). ∎

Observe that if we set $\phi = \varphi \equiv 1$ in the above proposition, then

$$\langle \pi_\lambda^* x, \pi_\lambda^* y\rangle^L = \int_{A_\sigma} \chi^2 e^{-F}\, da\, \langle x, y\rangle^{L_\lambda} \tag{6.16}$$

for all square-integrable $x, y \in \Omega^{0,q}(Y_\sigma, L_\lambda)$. Proposition 4.4 says that $\int_{A_\sigma} \chi^2 e^{-F}\, da < \infty$ if and only if $\lambda \in U_F$. So when this happens, π_λ^* preserves square-integrability.

Let $I_\lambda^q \subset \Omega_\lambda^{0,q}(X_\sigma, L)$ denote the image of π_λ^*. Since $\Omega_\lambda^{0,q}(X_\sigma, L)$ consists of differential forms which transform by $\lambda \in \mathfrak{t}_\sigma^*$ under the right T_σ-action, we can write

$$\Omega_\lambda^{0,q}(X_\sigma, L) = \bigoplus_{r+s=q} \left(I_\lambda^r \otimes \left(C^\infty(A_\sigma) \otimes \bigwedge^{0,s} \mathfrak{h}_\sigma^*\right)\right). \tag{6.17}$$

Here $\{I_\lambda^q\}_q$ and $\{C^\infty(A_\sigma) \otimes \wedge^{0,q} \mathfrak{h}_\sigma^*\}_q$ are both subcomplexes under $\bar\partial$, so (6.17) is a tensor product of chain complexes. The subcomplex $\{C^\infty(A_\sigma) \otimes \wedge^{0,q} \mathfrak{h}_\sigma^*\}_q$ can be identified with the T_σ-invariant Dolbeault differential forms on H_σ. Since H_σ is a Stein space, this subcomplex has trivial cohomology

$$H^q\left(\left\{C^\infty(A_\sigma) \otimes \bigwedge^{0,s} \mathfrak{h}_\sigma^*\right\}_s\right) = \begin{cases} \mathbf{C} & q = 0, \\ 0 & q \geqslant 1. \end{cases} \tag{6.18}$$

It also follows from (6.1) that whenever $(r, s) \neq (t, u)$,

$$\left\langle I_\lambda^r \otimes C^\infty(A_\sigma) \otimes \bigwedge^{0,s} \mathfrak{h}_\sigma^*, I_\lambda^t \otimes C^\infty(A_\sigma) \otimes \bigwedge^{0,u} \mathfrak{h}_\sigma^* \right\rangle^L = 0. \tag{6.19}$$

Against the L^2-structure $\langle\ ,\ \rangle^L$ on $\Omega_\lambda^{0,*}(X_\sigma, L)$, we define the formal adjoint $\bar\partial^*$ of $\bar\partial$. Namely, $\langle \bar\partial x, y\rangle^L = \langle x, \bar\partial^* y\rangle^L$ for all square-integrable

$x, y \in \Omega_\lambda^{0,*}(X_\sigma, L)$. Similarly, let $\bar\partial^*$ denote the formal adjoint of $\bar\partial$ relative to the L^2-structure $\langle\ ,\ \rangle^{L_\lambda}$ on $\Omega^{0,*}(Y_\sigma, L_\lambda)$.

PROPOSITION 6.3. $\pi_\lambda^* \bar\partial^* = \bar\partial^* \pi_\lambda^*$.

Proof. To prove this proposition, we need to show that for arbitrary square-integrable $\alpha \in \Omega_\lambda^{0,q}(X_\sigma, L)$ and $\beta \in \Omega^{0,q+1}(Y_\sigma, L_\lambda)$,

$$\langle \alpha, \pi_\lambda^* \bar\partial^* \beta \rangle^L = \langle \alpha, \bar\partial^* \pi_\lambda^* \beta \rangle^L. \tag{6.20}$$

We apply (6.17) to α and write $\alpha = \sum_{r+s=q} (\pi_\lambda^* x_r) \otimes y_s$ up to linear combination. The same indices r, s apply to all $\sum$ below. Note that $\pi_\lambda^* x_q \in I_\lambda^q$ and $y_0 \in C^\infty(A_\sigma)$. Thus the LHS of (6.20) becomes

$$\begin{aligned}\left\langle \sum (\pi_\lambda^* x_r) \otimes y_s, \pi_\lambda^* \bar\partial^* \beta \right\rangle^L &= \langle (\pi_\lambda^* x_q) \otimes y_0, \pi_\lambda^* \bar\partial^* \beta \rangle^L \qquad \text{by (6.19)}\\ &= \int_{A_\sigma} y_0 \chi^2 e^{-F}\, da\, \langle x_q, \bar\partial^* \beta \rangle^{L_\lambda} \text{ by Proposition 6.2}\\ &= \int_{A_\sigma} y_0 \chi^2 e^{-F}\, da\, \langle \bar\partial x_q, \beta \rangle^{L_\lambda}. \end{aligned}\tag{6.21}$$

On the other hand, the RHS of (6.20) becomes

$$\begin{aligned}\left\langle \sum (\pi_\lambda^* x_r) \otimes y_s, \bar\partial^* \pi_\lambda^* \beta \right\rangle^L &= \left\langle \bar\partial \sum (\pi_\lambda^* x_r) \otimes y_s, \pi_\lambda^* \beta \right\rangle^L\\ &= \left\langle \sum (\bar\partial \pi_\lambda^* x_r) \otimes y_s + (-1)^r (\pi_\lambda^* x_r) \otimes \bar\partial y_s, \pi_\lambda^* \beta \right\rangle^L. \end{aligned}\tag{6.22}$$

Since $\{C^\infty(A_\sigma) \otimes \wedge^{0,q} \mathfrak{h}_\sigma^*\}_q$ is a subcomplex it contains $\bar\partial y_s$, and therefore (6.19) says that $\langle (\pi_\lambda^* x_r) \otimes \bar\partial y_s, \pi_\lambda^* \beta \rangle^L = 0$. Then (6.22) becomes

$$\begin{aligned}\left\langle \sum (\bar\partial \pi_\lambda^* x_r) \otimes y_s, \pi_\lambda^* \beta \right\rangle^L &= \langle (\bar\partial \pi_\lambda^* x_q) \otimes y_0, \pi_\lambda^* \beta \rangle^L \qquad \text{by (6.19)}\\ &= \langle (\pi_\lambda^* \bar\partial x_q) \otimes y_0, \pi_\lambda^* \beta \rangle^L\\ &= \int_{A_\sigma} y_0 \chi^2 e^{-F}\, da\, \langle \bar\partial x_q, \beta \rangle^{L_\lambda} \quad \text{by Proposition 6.2.} \end{aligned}\tag{6.23}$$

Thus (6.20) follows from (6.21), (6.22), and (6.23). Hence we have the proposition. ∎

We have defined $\bar{\partial}$ and $\bar{\partial}^*$ on the square-integrable differential forms in $\Omega^{0,q}(X_\sigma, L)$ and $\Omega^{0,q}(Y_\sigma, L_\lambda)$. The differential forms which are annihilated by $\bar{\partial}$ and $\bar{\partial}^*$ are known as the harmonic forms. The Hilbert space of square-integrable harmonic forms are denoted by $H^q_\omega \subset \Omega^{0,q}(X_\sigma, L)$ and $H^q(L_\lambda) \subset \Omega^{0,q}(Y_\sigma, L_\lambda)$ and are called the L^2-cohomology. The next proposition considers $(H^q_\omega)_\lambda = H^q_\omega \cap \Omega^{0,q}_\lambda(X_\sigma, L)$. Recall that π^*_λ is defined in (6.12).

PROPOSITION 6.4. *$(H^q_\omega)_\lambda$ lies in the image of π^*_λ.*

Proof. Let I^q_λ denote the image of π^*_λ as before, and let $E^q = (H^q_\omega)_\lambda \cap I^q_\lambda$. So E^q is a closed subspace in the Hilbert space $(H^q_\omega)_\lambda$. This gives the direct sum and orthogonal projection

$$(H^q_\omega)_\lambda = E^q \oplus E^q_\perp, \qquad \theta\colon (H^q_\omega)_\lambda \to E^q,$$

where $E^q_\perp$ is the orthogonal complement of E^q in $(H^q_\omega)_\lambda$. Our goal is obviously to show that $E^q_\perp = 0$, so that $(H^q_\omega)_\lambda \subset I^q_\lambda$.

Since $\theta\bar{\partial} = \bar{\partial}\theta = 0$ on $(H^q_\omega)_\lambda$, the projection θ defines a map θ_* on the Dolbeault cohomology classes. Apply the Kunneth theorem to (6.17) and (6.18). It says that the natural inclusion $\iota\colon I^q_\lambda \hookrightarrow \Omega^{0,q}_\lambda(X_\sigma, L)$ leads to an isomorphism ι_* in Dolbeault cohomology. Note that $\theta \cdot \iota$ is the identity map on E^q. So θ_* and ι_* are inverses of each other. In particular, for all $\alpha \in (H^q_\omega)_\lambda$, α and $\theta\alpha$ are cohomologous.

Pick $\alpha \in (H^q_\omega)_\lambda$. Namely, α is square-integrable and $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$. Write

$$\alpha = \xi + \eta \in E^q \oplus E^q_\perp = (H^q_\omega)_\lambda.$$

Since α and $\theta\alpha = \xi$ define the same Dolbeault cohomology class, there exists a β such that $\bar{\partial}\beta = \alpha - \theta\alpha = \eta$. Since $\bar{\partial}^*\alpha = 0$,

$$0 = \langle \beta, \bar{\partial}^*\alpha \rangle^L = \langle \bar{\partial}\beta, \alpha \rangle^L = \langle \eta, \xi + \eta \rangle^L = \langle \eta, \eta \rangle^L.$$

Thus $\eta = 0$. This means that $E^q_\perp = 0$ and the proposition follows. ∎

The next proposition relates the L^2-cohomology spaces $(H^q_\omega)_\lambda$ and $H^q(L_\lambda)$. Recall that U_F is the image of $\frac{1}{2}F'$.

PROPOSITION 6.5. *If $\lambda \in U_F$ then π^*_λ defines an isomorphism $(H^q_\omega)_\lambda \cong H^q(L_\lambda)$. If $\lambda \notin U_F$ then $(H^q_\omega)_\lambda = 0$.*

Proof. Suppose first that $\lambda \in U_F$. By Proposition 4.4, $\int_{A_\sigma} \chi(a)^2\, e^{-F(a)}\, da < \infty$. Since π^*_λ is injective, it suffices to prove that

$$\pi^*_\lambda(H^q(L_\lambda)) = (H^q_\omega)_\lambda. \tag{6.24}$$

Clearly π_λ^* commutes with $\bar{\partial}$, and Proposition 6.3 says that it commutes with $\bar{\partial}^*$ too. Further, by (6.16), π_λ^* preserves square-integrability. These observations lead to the $\subset$ part of (6.24).

It remains to prove the $\supset$ part of (6.24). Let $\alpha \in (H_\omega^q)_\lambda$. By Proposition 6.4, $\alpha = \pi_\lambda^* \beta$. The harmonic property of α and the injectivity of π_λ^* imply that β is harmonic. Further, (6.16) says that the square-integrability of α implies the square-integrability of β. Hence $\beta \in H^q(L_\lambda)$, which implies the $\supset$ part of (6.24). This proves the proposition for $\lambda \in U_F$.

Next suppose that $\lambda \notin U_F$. Let $\alpha \in \Omega_\lambda^{0,q}(X_\sigma, L)$. If $\alpha \in H_\omega^q$, then Proposition 6.4 says that α is in the image of π_λ^*. Consequently, Proposition 4.4 and (6.16) say that α is not square-integrable unless $\alpha \equiv 0$. We conclude that $(H_\omega^q)_\lambda = 0$. Hence we have the proposition. ∎

Proof of Theorem 3. By Proposition 6.5, it suffices to consider $(H_\omega^q)_\lambda$ for $\lambda \in U_F$. Assuming this, Theorem 3 follows directly from Proposition 6.5 and the well-known results [16] on $H^q(L_\lambda)$. ∎

7. SYMPLECTIC REDUCTION

Let $\omega = \sqrt{-1}\, \partial\bar{\partial} F$ be a $(G \times T_\sigma)$-invariant pseudo-Kähler form on X_σ. In this section, we perform symplectic reduction [15] to the right T_σ-action. The moment map for this action is denoted

$$\Phi_r \colon X_\sigma \to \mathfrak{t}_\sigma^*$$

and is called the right moment map. Recall that $(\mathfrak{t}_\sigma^*)_{\mathrm{reg}} \cong (\mathfrak{a}_\sigma^*)_{\mathrm{reg}}$ is defined in (1.2).

PROPOSITION 7.1. *For all* $ga \in (G/G_{\mathrm{ss}}^\sigma)\, A_\sigma = X_\sigma$, $\Phi_r(ga) = \frac{1}{2} F'(a) \in (\mathfrak{t}_\sigma^*)_{\mathrm{reg}}$.

Proof. Since the right T_σ-action commutes with the G-action, it is clear that Φ_r is G-invariant. So it suffices to consider $\Phi_r(a)$ for $a \in A_\sigma$.

Let $v \in \mathfrak{t}_\sigma$, and let $v^\#$ and v^r denote the infinitesimal vector fields on X_σ corresponding to the left and right actions respectively. Since $T_\sigma A_\sigma$ is abelian, $v_a^\# = v_a^r$ for all $a \in A_\sigma$. Let β be the real $G \times T_\sigma$-invariant 1-form satisfying $d\beta = \omega$. Then

$$\begin{aligned} (\Phi_r(a), v) &= -(\beta, v^r)_a && \text{by [1, Theorem 4.2.10]} \\ &= -(\beta, v^\#)_a \\ &= (\Phi(a), v) \\ &= (\tfrac{1}{2} F'(a), v) && \text{by Proposition 3.6.} \end{aligned}$$

Finally, Theorem 1 says that the image of $\frac{1}{2}F'$ lies in $(\mathfrak{t}_\sigma^*)_{\text{reg}}$. Hence we have the proposition. ▮

Let $\lambda \in (\mathfrak{t}_\sigma^*)_{\text{reg}}$ be in the image of Φ_r. We consider the reduced space $R_\lambda = \Phi_r^{-1}(\lambda)/T_\sigma$.

Proposition 7.2. *Each connected component of $\Phi_r^{-1}(\lambda)/T_\sigma$ is a copy of the flag domain Y_σ.*

Proof. Since ω is pseudo-Kähler, Theorem 1 says that F is non-degenerate. By the inverse function theorem, F' is a local diffeomorphism. So there exists a discrete set $\Gamma \subset A_\sigma$ such that $(\frac{1}{2}F')^{-1}(\lambda) = \Gamma$. By Proposition 7.1, $\Phi_r^{-1}(\lambda) = (G/G_{\text{ss}}^\sigma)\,\Gamma \subset (G/G_{\text{ss}}^\sigma)\,A_\sigma$. Consequently,

$$\Phi_r^{-1}(\lambda)/T_\sigma = (G/G^\sigma)\,\Gamma. \tag{7.1}$$

A typical connected component of this space is of the form $(G/G^\sigma)a$, $a \in \Gamma$. Hence we have the proposition. ▮

Consider the inclusion

$$\iota\colon \Phi_r^{-1}(\lambda) \to X_\sigma \tag{7.2}$$

and the fibration

$$\rho\colon \Phi_r^{-1}(\lambda) \to R_\lambda. \tag{7.3}$$

The reduced form ω_λ is defined to be the unique symplectic form on R_λ such that $\rho^*\omega_\lambda = \iota^*\omega$. Since ι and ρ commute with the G-action, it is clear that ω_λ is G-invariant. Let

$$\psi\colon R_\lambda \to \mathfrak{g}^*$$

be the moment map of the G-action preserving ω_λ.

By (7.1), write a typical element of R_λ as ga. If g is the identity coset eG^σ, we write $a = ga$ for simplicity.

Proposition 7.3. $\psi(a) = \lambda \in (\mathfrak{t}_\sigma^*)_{\text{reg}}$.

Proof. Pick $x \in \mathfrak{g}$. By abuse of notation, let $x^\#$ be the infinitesimal vector field for the G-action on X_σ, $\Phi_r^{-1}(\lambda)$ or R_λ, depending on the context. Also, let a denote the appropriate element in any of these three spaces. Since (7.2) and (7.3) commute with the G-action,

$$\iota(a) = a, \qquad \rho(a) = a, \qquad \iota_*(x_a^\#) = x_a^\#, \qquad \rho_*(x_a^\#) = x_a^\#.$$

Since $\mathfrak{g}$ is semisimple, up to linear combination $x=[u,v]$. Then

$$(\psi(a), x)=(\psi(a), [u,v])=\omega_\lambda(u^\#, v^\#)_a=\rho^*\omega_\lambda(u^\#, v^\#)_a=\iota^*\omega(u^\#, v^\#)_a$$
$$=\omega(u^\#, v^\#)_a=(\Phi(a), [u,v])=(\lambda, [u,v])=(\lambda, x). \tag{7.4}$$

So $\psi(a)=\lambda$ and the proposition follows. ▮

Since Y_σ is an open set of $G^{\mathbf{C}}/P$, it is a complex manifold. Consequently the reduced space R_λ is complex. Recall that $\mathscr{C}$ is defined in (1.7).

PROPOSITION 7.4. *The reduced form ω_λ is a G-invariant pseudo-Kähler form on R_λ. In particular, it is Kähler if and only if $\lambda \in (\mathfrak{t}_\sigma^*)_{\text{reg}} \cap \mathscr{C}$.*

Proof. The G-invariance of ω_λ follows from the discussions in (7.2) and (7.3). So its pseudo-Kähler and Kähler properties remain to be checked.

Consider the elements $\zeta_i, \gamma_i \in \mathfrak{g}$ from (2.3), indexed by the positive roots α_i. Here $\{\zeta_i, \gamma_i\}_{(\alpha_i, \mathfrak{t}_\sigma)\neq 0}$ can be regarded as a basis of $\mathfrak{g}/\mathfrak{g}^\sigma$. The almost complex structure inherited from $G^{\mathbf{C}}/P$ sends ζ_i to γ_i and γ_i to $-\zeta_i$. Substituting $u=\zeta_i$ and $v=\gamma_i$ in (7.4), we get

$$\omega_\lambda(\zeta_i^\#, \gamma_i^\#)_a=\omega(\zeta_i^\#, \gamma_i^\#)_a. \tag{7.5}$$

Since ω is pseudo-Kähler, it follows from (7.5) that ω_λ is pseudo-Kähler too.

In fact, ω_λ is Kähler if and only if (7.5) is positive for all $(\alpha_i, \mathfrak{t}_\sigma)\neq 0$. Following the argument in (7.4), we see from (2.5) that

$$\omega_\lambda(\zeta_i^\#, \gamma_i^\#)_a=\pm(\lambda, \alpha_i). \tag{7.6}$$

Here the sign $\pm$ is positive when α_i is compact and negative when α_i is non-compact. So (7.6) is positive for all $(\alpha_i, \mathfrak{t}_\sigma)\neq 0$ if and only if $\lambda \in (\mathfrak{t}_\sigma^*)_{\text{reg}} \cap \mathscr{C}$, and this is the equivalent condition for ω_λ to be Kähler. ▮

For $i=1,2$, consider the reduced spaces $(R_{\lambda_i}, (\omega_i)_{\lambda_i})$, with moment maps $\psi_i\colon R_{\lambda_i} \to \mathfrak{g}^*$. By the previous proposition, these reduced spaces are pseudo-Kähler. So we can compare them under the notions of $\sim$ and $=$ introduced in (1.12).

PROPOSITION 7.5. *Suppose that R_{λ_i} have the same number of connected components. Then $(\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}$ if and only if $\lambda_1 \sim \lambda_2$, and $(\omega_1)_{\lambda_1}=(\omega_2)_{\lambda_2}$ if and only if $\lambda_1=\lambda_2$.*

Proof. Suppose that this proposition has been proved for all connected reduced spaces. Let R_λ be a reduced space, possibly non-connected. For $i=1,2$, let $Y_\sigma a_i$ be connected components of R_λ. By Proposition 7.3, their moment maps satisfy $\psi_i(a_i) = \lambda$. So by the present proposition for connected reduced spaces, $Y_\sigma a_1$ and $Y_\sigma a_2$ are isomorphic pseudo-Kähler manifolds. We conclude that all connected components of R_λ are isomorphic to one another, and so the present proposition holds for non-connected reduced spaces too.

From this observation, we only have to prove the proposition for connected reduced spaces. So assume that R_{λ_i} are connected for $i = 1, 2$. Write $R_{\lambda_i} = (G/G^\sigma)\, a_i$ for some $a_i \in A_\sigma$.

Suppose that $\lambda_1 \sim \lambda_2$. Thus there is a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ which contains λ_1 and λ_2. By Proposition 7.3, $\psi_i(a_i) = \lambda_i$. By Theorem 1 and Proposition 7.1, $\lambda_i \in (\mathfrak{t}_\sigma^*)_{\text{reg}} \subset \mathfrak{t}^*$, so the isotropy subgroup of λ_i in G is G^σ. Hence $\mathcal{O} = G/G^\sigma$. So ψ_i is a diffeomorphism from $(G/G^\sigma)\, a_i$ onto the elliptic orbit $\mathcal{O}$. In fact, since ψ_i is G-equivariant, it identifies $(\omega_i)_{\lambda_i}$ with the Kirillov–Kostant symplectic form $\omega_{\mathbf{KK}}$ on $\mathcal{O}$. We conclude that $(\omega_1)_{\lambda_1} \sim \omega_{\mathbf{KK}} \sim (\omega_2)_{\lambda_2}$.

Conversely, if $(\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}$, then ψ_i have the same image $\mathcal{O}$. By Proposition 7.3, $\psi_i(a_i) = \lambda_i \in \mathcal{O}$, so $\lambda_1 \sim \lambda_2$.

The last part of this proposition remains to be proved, where $\sim$ is replaced with $=$. Suppose that $\lambda_1 = \lambda_2$. By (7.4), for all $u, v \in \mathfrak{g}$,

$$(\omega_1)_{\lambda_1}(u^\#, v^\#)_{a_1} = (\lambda_i, [u, v]) = (\omega_2)_{\lambda_2}(u^\#, v^\#)_{a_2}. \tag{7.7}$$

Consider the G-equivariant biholomorphic map

$$\kappa\colon Y_\sigma a_1 \to Y_\sigma a_2, \qquad \kappa(ga_1) = ga_2. \tag{7.8}$$

By (7.7), $\kappa^*(\omega_2)_{\lambda_2}$ and $(\omega_1)_{\lambda_1}$ agree on a_1. By G-invariance, they agree everywhere. So κ preserves the pseudo-Kähler structures and $(\omega_1)_{\lambda_1} = (\omega_2)_{\lambda_2}$.

Conversely, suppose that $\lambda_1 \neq \lambda_2$. If λ_i are in different coadjoint G-orbits, then the first part of the proposition says that $(\omega_i)_{\lambda_i}$ are not symplectomorphic, so in particular $(\omega_1)_{\lambda_1} \neq (\omega_2)_{\lambda_2}$. Hence we may assume that λ_i are in the same orbit. Each connected component of $(\mathfrak{t}_\sigma^*)_{\text{reg}} \subset \mathfrak{g}^*$ intersects a G-orbit at most once. From $\lambda_i \in (\mathfrak{t}_\sigma^*)_{\text{reg}}$, $\lambda_1 \neq \lambda_2$ and $\lambda_1 \sim \lambda_2$ we conclude that λ_i are in different connected components of $(\mathfrak{t}_\sigma^*)_{\text{reg}}$. The holomorphic map (7.8) fails to preserve the pseudo-Kähler structures because (7.6) shows that there is a sign problem arising from (2.5). Other symplectomorphisms between $(\omega_i)_{\lambda_i}$ have to permute the connected components of $(\mathfrak{t}_\sigma^*)_{\text{reg}}$, so they cannot be holomorphic. We conclude that $(\omega_1)_{\lambda_1} \neq (\omega_2)_{\lambda_2}$. This proves the proposition. ∎

Proof of Theorem 4. The theorem follows directly from Propositions 7.1 through 7.5. ∎

8. FLAG DOMAINS

In this section, we study the G-invariant pseudo-Kähler structures on the flag domain $Y_\sigma = G/G^\sigma$ and their relations to symplectic reduction (1.11). These will lead to Theorem 5.

If G is compact, then Y_σ is compact and obviously the pseudo-Kähler forms on Y_σ are not exact. But when G is not compact some work is needed to show that the G-invariant pseudo-Kähler forms on Y_σ are not exact. This is done by the next proposition.

Proposition 8.1. *A G-invariant pseudo-Kähler form on Y_σ cannot be exact.*

Proof. Recall from (3.1) that $\mathfrak{g}^\sigma \subset \mathfrak{k}$, and define the relative exterior algebra $\bigwedge^1(\mathfrak{k}, \mathfrak{g}^\sigma)^*$ from (2.12). We first claim that

$$\bigwedge^1 (\mathfrak{k}, \mathfrak{g}^\sigma)^* = 0. \tag{8.1}$$

Pick $\beta \in \bigwedge^1(\mathfrak{k}, \mathfrak{g}^\sigma)^*$. Since $\mathfrak{k}$ is compact, $\mathfrak{k} = \mathfrak{k}_{ss} + \mathfrak{z}$, where $\mathfrak{k}_{ss}$ and $\mathfrak{z}$ are respectively the semisimple commutator subalgebra and the center of $\mathfrak{k}$. From $\beta \in \bigwedge^1(\mathfrak{k}, \mathfrak{g}^\sigma)^*$ and $\mathfrak{z} \subset \mathfrak{t} \subset \mathfrak{g}^\sigma$, it is necessary that $\beta \in \mathfrak{k}_{ss}^* \subset \mathfrak{k}^*$. Consider $\mathfrak{k}_{ss} = (\mathfrak{k}_{ss} \cap \mathfrak{t}) + (\mathfrak{k}_{ss} \cap V)$, where V is the space from (2.6). Since $(\mathfrak{k}_{ss} \cap \mathfrak{t}) \subset \mathfrak{g}^\sigma$, we get $\beta \in (\mathfrak{k}_{ss} \cap V)^*$.

Suppose that $\beta \neq 0$. Since $\mathfrak{k}_{ss}$ is semisimple, there exists x in its Cartan subalgebra $\mathfrak{k}_{ss} \cap \mathfrak{t}$ such that $\mathrm{ad}_x^* \beta \neq 0$. Since $x \in (\mathfrak{k}_{ss} \cap \mathfrak{t}) \subset \mathfrak{g}^\sigma$, the condition $\mathrm{ad}_x^* \beta \neq 0$ contradicts $\beta \in \bigwedge^1(\mathfrak{k}, \mathfrak{g}^\sigma)^*$. So β has to vanish and (8.1) follows.

Consider the restriction map $\iota^*: \Omega^q(G/G^\sigma) \to \Omega^q(K/G^\sigma)$. Let ω be a G-invariant pseudo-Kähler form on G/G^σ. Also, let ζ_i, γ_i be the vectors from (2.6) with $\alpha_i \in \Delta_c^+$ and $(\alpha_i, \mathfrak{t}_\sigma) \neq 0$. It is clear from earlier discussions that $(\iota^*\omega)(\zeta_i^\sharp, \gamma_i^\sharp) \neq 0$, so $\iota^*\omega \neq 0$. Since ι^* commutes with the K-action, $\iota^*\omega$ is K-invariant. Consequently,

$$0 \neq \iota^*\omega \in \bigwedge^2 (\mathfrak{k}, \mathfrak{g}^\sigma)^*. \tag{8.2}$$

If ω is exact, then so is $\iota^*\omega$. But this is impossible due to (8.1) and (8.2). The proposition follows. ∎

As before, we let $\psi: Y_\sigma \to \mathfrak{g}$ be the moment map and let $e \in Y_\sigma$ be the identity coset. Recall the notion of $\sim$ in (1.12).

Proposition 8.2. *The G-invariant pseudo-Kähler forms on Y_σ are classified by $(\mathfrak{t}_\sigma^*)_{\mathrm{reg}}$ via $\psi(e) \in (\mathfrak{t}_\sigma^*)_{\mathrm{reg}}$. If we ignore the complex structures, then the*

G-invariant symplectic forms on Y_σ are classified by $(\mathfrak{t}_\sigma^*)_{\mathrm{reg}}/\sim$ up to G-symplectomorphisms.

Proof. Let ω be a G-invariant pseudo-Kähler form on Y_σ with moment map ψ. We want to show that $\psi(e) \in (\mathfrak{t}_\sigma^*)_{\mathrm{reg}}$.

Choose $u \in \mathfrak{t}$ and $v \in V_i$ from (2.6) such that $0 \neq [u, v] \in V_i$. For all $g \in G^\sigma$, $ge = e \in Y_\sigma$. Since ψ is G-equivariant,

$$(\psi(e), [u, v]) = (\psi(ge), [u, v]) = (\psi(e), \mathrm{Ad}_g[u, v]). \tag{8.3}$$

Let g vary in $T \subset G^\sigma$, so that $\mathrm{Ad}_g[u, v] \in \mathrm{Ad}_T V_i = V_i$. Let $\theta_i \colon \mathfrak{g}^* \to V_i^*$ be the projection with respect to (2.8). Since $[u, v]$ and $\mathrm{Ad}_g[u, v]$ are in V_i, (8.3) can be written as

$$(\theta_i \psi(e), [u, v]) = (\theta_i \psi(e), \mathrm{Ad}_g[u, v]). \tag{8.4}$$

But as g varies in T, $\mathrm{Ad}_g[u, v]$ traces out a circle in the two-dimensional space V_i. So for (8.4) to hold we need $0 = \theta_i \psi(e) \in V_i^*$. This happens for all V_i, so $\psi(e) \in \mathfrak{t}^*$.

Pick $u \in V_i$, $v \in V_j$ where $i \neq j$. By (2.7), $[u, v] \in V$. Since $\psi(e) \in \mathfrak{t}^*$, it follows that

$$\omega(u^\#, v^\#)_e = (\psi(e), [u, v]) \in (\mathfrak{t}^*, V) = 0.$$

We conclude that

$$\omega(V_i^\#, V_j^\#)_e = 0, \qquad i \neq j. \tag{8.5}$$

We still have to go from $\psi(e) \in \mathfrak{t}^*$ to the sharper $\psi(e) \in (\mathfrak{t}_\sigma^*)_{\mathrm{reg}}$. Consider ζ_i, γ_i from (2.3). By (2.5),

$$\omega(\zeta_i^\#, \gamma_i^\#)_e = (\psi(e), [\zeta_i, \gamma_i]) = \pm(\psi(e), \alpha_i). \tag{8.6}$$

Consider $(\alpha_i, \mathfrak{t}_\sigma) = 0$ so that $V_i \subset \mathfrak{g}^\sigma$ by (2.11). Since G^σ fixes $e \in Y_\sigma$, $(\zeta_i^\#)_e = (\gamma_i^\#)_e = 0$. So (8.6) vanishes whenever $(\alpha_i, \mathfrak{t}_\sigma) = 0$, which implies that $\psi(e) \in \mathfrak{t}_\sigma^*$.

On the other hand, consider $(\alpha_i, \mathfrak{t}_\sigma) \neq 0$. By (2.11), $V_i \not\subset \mathfrak{g}^\sigma$. Then $(\zeta_i^\#)_e$, $(\gamma_i^\#)_e \neq 0$. Since ω is non-degenerate, by (8.5) $\omega(\zeta_i^\#, \gamma_i^\#)_e \neq 0$. So by (8.6) $(\psi(e), \alpha_i) \neq 0$ whenever $(\alpha_i, \mathfrak{t}_\sigma) \neq 0$. We conclude that $\psi(e) \in (\mathfrak{t}_\sigma^*)_{\mathrm{reg}}$.

We have proved that the moment map of a G-invariant pseudo-Kähler form on Y_σ satisfies $\psi(e) \in (\mathfrak{t}_\sigma^*)_{\mathrm{reg}}$. The rest of the proposition on the classifications of the pseudo-Kähler and symplectic forms follows essentially from arguments in Proposition 7.5. Hence we have the proof. ∎

We now show that every G-invariant pseudo-Kähler form on Y_σ can be obtained via symplectic reduction from X_σ. In view of Theorem 4, the reduction process (1.11) simplifies to (1.13).

PROPOSITION 8.3. *Every G-invariant pseudo-Kähler form on Y_σ can be obtained via symplectic reduction* (1.13).

Proof. Let Ω be a G-invariant pseudo-Kähler form on Y_σ with moment map satisfying $\psi(e)=\lambda$. By Proposition 8.2, $\lambda\in(\mathfrak{t}_\sigma^*)_{\text{reg}}$. Note that $(\mathfrak{t}_\sigma^*)_{\text{reg}}$ consists of connected components which are open cones. Let D be the connected component containing λ. Let $\lambda_1, ..., \lambda_r \in \mathfrak{t}^*$ be on the edges of D so that D consists of positive linear combinations of $\lambda_1, ..., \lambda_r$. We can normalize them so that $\lambda=\sum_1^r \lambda_k$. Define $F\in C^\infty(\mathfrak{a}_\sigma)$ by $F(y)=2\sum_1^r \times \exp(\lambda_k, y)$. By (1.5), we identify it with $F\in C^\infty(A_\sigma)$. By Proposition 5.1 F is strictly convex and the image of F' is D. So by Theorem 1 $\omega=\sqrt{-1}\,\partial\bar{\partial}F$ is pseudo-Kähler. Also, Propositions 5.1 and 7.1 say that $\Phi_r(e)=\frac{1}{2}F'(0)=\lambda$. Since F is strictly convex F' is injective, so the reduced space R_λ is connected. By Proposition 7.2 $R_\lambda=Y_\sigma$. By Proposition 7.3 the moment map of $(R_\lambda, \omega_\lambda)$ sends e to λ. Since the moment maps of Ω and ω_λ agree on e, Proposition 8.2 says that $\Omega=\omega_\lambda$. Hence we have the proposition. ▮

Proof of Theorem 5. By Propositions 8.1 and 8.2 the G-invariant pseudo-Kähler forms on Y_σ are not exact and are classified by $\psi(e)\in(\mathfrak{t}_\sigma^*)_{\text{reg}}$. By Propositions 7.3 and 8.3 all of them can be obtained by symplectic reduction and the one with $\psi(e)=\lambda$ is obtained from (X_σ, λ). ▮

9. QUANTIZATION COMMUTES WITH REDUCTION

The main purpose of this section is to prove Theorem 6. For convenience, the integral weights in $\mathfrak{t}_\sigma^*$ are denoted by $\mathbf{Z}(\mathfrak{t}_\sigma^*)$. Let ω be a $(G\times T_\sigma)$-invariant pseudo-Kähler form on X_σ. By Theorem 1 $\omega=\sqrt{-1}\,\partial\bar{\partial}F$. In this section we assume that F is strictly convex. Let $\lambda\in\mathbf{Z}(\mathfrak{t}_\sigma^*)$. In Theorem 3, we prove that

$$(H_\omega^q)_\lambda=\begin{cases}\Theta_{\lambda+\rho} & \text{if } \lambda\in U_F\cap\mathbf{Z}(\mathfrak{t}_\sigma^*),\ \lambda+\rho\in(\mathfrak{t}^*)_{\text{reg}}, \text{ and } q=l(\lambda),\\ 0 & \text{otherwise.}\end{cases} \tag{9.1}$$

By Theorem 1 and Proposition 7.1, we know that U_F is also the image of the right moment map Φ_r. Define $R_\lambda=\Phi_r^{-1}(\lambda)/T_\sigma$ as in Section 7 and let ω_λ be the reduced form on R_λ. We want to quantize [14] the G-action on $(R_\lambda, \omega_\lambda)$ and construct the G-representation $H^q_{(\omega_\lambda)}$. Clearly we need $\lambda\in U_F$, for otherwise $R_\lambda=\varnothing$. For $\lambda\in U_F$, we know that $(\frac{1}{2}F')^{-1}(\lambda)$ has exactly one element because F being strictly convex implies that $\frac{1}{2}F'$ is injective. So by Proposition 7.2 $R_\lambda = Y_\sigma$. The next step is to quantize the G-action on the reduced space $(Y_\sigma, \omega_\lambda)$. By Proposition 8.1, $0 \neq [\omega_\lambda] \in H^2(Y_\sigma, \mathbf{R})$. We can find a line bundle $L_\lambda \to Y_\sigma$ with Chern class $[\omega_\lambda]$ if and only if $[\omega_\lambda] \in H^2(Y_\sigma, \mathbf{Z})$ or equivalently $\lambda \in \mathbf{Z}(\mathfrak{t}_\sigma^*)$. We have thus shown that $\lambda \in U_F \cap \mathbf{Z}(\mathfrak{t}_\sigma^*)$ is necessary for quantization. Assuming this, $L_\lambda \to Y_\sigma$ is the homogeneous line bundle corresponding to the character $e^\lambda \colon H_\sigma \to \mathbf{C}^\times$. The harmonic forms on Y_σ with coefficients in L_λ are denoted by $H^q_{(\omega_\lambda)}$. It is a unitary G-representation and we conclude from [16] that

$$H^q_{(\omega_\lambda)} = \begin{cases} \Theta_{\lambda+\rho} & \text{if } \lambda \in U_F \cap \mathbf{Z}(\mathfrak{t}_\sigma^*),\ \lambda+\rho \in (\mathfrak{t}^*)_{\text{reg}}, \text{ and } q = l(\lambda), \\ 0 & \text{otherwise.} \end{cases} \tag{9.2}$$

Proof of Theorem 6. This follows directly from (9.1) and (9.2). ∎

REFERENCES

1. R. Abraham and J. Marsden, "Foundations of Mechanics," 2nd. ed., Addison–Wesley, Reading, MA, 1985.
2. A. Borel, Kählerian coset spaces of semisimple Lie groups, *Proc. Nat. Acad. Sci. U.S.A.* **40** (1954), 1147–1151.
3. M. K. Chuah, K-invariant Kaehler structures on $K_{\mathbf{C}}/N$ and the associated line bundles, *Proc. Amer. Math. Soc.* **124** (1996), 3481–3491.
4. M. K. Chuah, Kaehler structures on $K_{\mathbf{C}}/(P, P)$, *Trans. Amer. Math. Soc.* **349** (1997), 3373–3390.
5. M. K. Chuah and V. Guillemin, Kaehler structures on $K_{\mathbf{C}}/N$, *Contemp. Math.* **154** (1993), 181–195.
6. I. M. Gelfand and A. Zelevinski, Models of representations of classical groups and their hidden symmetries, *Funct. Anal. Appl.* **18** (1984), 183–198.
7. V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, *Invent. Math.* **67** (1982), 515–538.
8. V. Guillemin and S. Sternberg, "Symplectic Techniques in Physics," Cambridge Univ. Press, Cambridge, UK, 1984.
9. Harish-Chandra, Representations of semisimple Lie groups, VI, *Amer. J. Math.* **78** (1956), 564–628.
10. Harish-Chandra, Discrete series for semisimple Lie groups. II. Explicit determination of the characters, *Acta. Math.* **116** (1966), 1–111.
11. S. Helgason, "Differential Geometry, Lie Groups, and Symmetric Spaces," Academic Press, New York, 1978.
12. S. Helgason, "Groups and Geometric Analysis," Academic Press, New York, 1984.
13. A. Knapp, "Representation Theory of Semisimple Groups," Princeton Univ. Press, Princeton, NJ, 1986.
14. B. Kostant, "Quantization and Unitary Representations," Lecture Notes in Mathematics, Vol. 170, pp. 87–208, Springer-Verlag, New York/Berlin, 1970.
15. J. Marsden and A. Weinstein, Reduction of symplectic manifolds with symmetry, *Rep. Math. Phys.* **5** (1974), 121–130.
16. W. Schmid, L^2-cohomology and the discrete series, *Ann. of Math.* **103** (1976), 375–394.
17. R. Sjamaar, Symplectic reduction and Riemann–Roch formulas for multiplicities, *Bull. Amer. Math. Soc.* **33** (1996), 327–338.