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Discrete Mathematics 222 (2000) 89–100
DISCRETE MATHEMATICS
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# The optimal pebbling number of the complete $m$-ary tree☆

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Received 4 March 1997; revised 3 December 1999; accepted 13 December 1999

**Abstract**

In this paper, we find the optimal pebbling number of the complete $m$-ary tree. © 2000 Elsevier Science B.V. All rights reserved.

*MSC:* 05C05

*Keywords:* Optimal pebbling; Complete $m$-ary tree; Integer linear programming

## 1. Introduction

Throughout this paper, a *configuration* of a graph $G$ means a mapping from $V(G)$ into the set of non-negative integers $N\cup\{0\}$. Suppose $p$ pebbles are distributed onto the vertices of $G$; then we have the so-called distributing configuration (d.c.) $\delta$ where we let $\delta(v)$ be the number of pebbles distributed to $v\in V(G)$ and $\delta_H$ equals $\sum_{v\in V(H)}\delta(v)$ for each induced subgraph $H$ of $G$. Note that now $\delta_G=p$.

A pebbling move consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. If a d.c. $\delta$ lets us move at least one pebble to each vertex $v$ by applying pebbling moves repeatedly (if necessary), then $\delta$ is called a *pebbling* of $G$. The *optimal pebbling number* of $G$, $f'(G)$, is $\min\{\delta_G \mid \delta$ is a pebbling of $G\}$, and a d.c. $\delta$ is an *optimal pebbling* of $G$ if $\delta$ is a pebbling of $G$ such that $\delta_G=f'(G)$.

Note here that the *pebbling number* $f(G)$ of a graph $G$ is defined as the *minimum* number of pebbles $p$ such that any distributing configuration with $p$ pebbles is a pebbling of $G$. The problem of pebbling graphs was first proposed by Saks and Lagarias [1] as a tool for solving a number theoretical problems by Lemke and Kleitman [4]. In terms of pebbling, they expected the pebbling number of an $n$-cube to be $2^n$. Later,

☆ Supported in part by the National Science Council of the Republic of China (NSC-85-1221-M-009-010).
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PII: S0012-365X(00)00008-X

this problem was solved by Chung [1]. An alternative proof of the following theorem in number theory was thus obtained.

**Theorem 1.1** (Clarke et al. [2] and Lemke and Kleitman [4]). *For any given integers* $a_1, a_2, \ldots, a_d$ *there exists a nonempty subset* $X \subseteq \{1, 2, \ldots, d\}$ *such that* $d \mid \sum_{i \in X} a_i$ *and* $\sum_{i \in X} \gcd(a_i, d) \leqslant d$.

In [1], Chung also mentioned a conjecture by Graham: given two graphs $G$ and $H$, is the pebbling number of the Cartesian product of $G$ and $H$, $f(G \times H)$, no bigger than $f(G)f(H)$? So far, the problem remains unsolved in general. It is worth mentioning that Moews [5] showed that the inequality holds for trees $G$ and $H$.

On the other hand, the study of optimal pebbling number is equally interesting. First, an absolutely nontrivial result on paths was obtained by Pachter et al. [7].

**Theorem 1.2** (Pachter et al. [7]). *Let* $P$ *be a path of order* $3t + r$ *for* $0 \leqslant r \leqslant 2$, *i.e.*, $|V(P)| = 3t + r$. *Then* $f'(P) = 2t + r$.

Recently, Fu and Shiue [3] found $f'(T)$ for $T$ a caterpillar by way of a generalized pebbling on a path. Since the statement of the theorem is too long, we omit it here. Furthermore, in the same paper, they have proved an analog of the conjecture mentioned above.

**Theorem 1.3** (Fu and Shiue [3]). *For any graph* $G$ *and* $H$, $f'(G \times H) \leqslant f'(G)f'(H)$.

Clearly, Theorem 1.3 gives an upper bound for the optimal pebbling number of a product graph; for example, an $n$-cube $Q_n = Q_{n-1} \times K_2$. We note here that quite recently an upper bound for $f'(Q_n)$ has been obtained.

**Theorem 1.4** (Moews [6]). $f'(Q_n) = (\frac{4}{3})^{n + O(\log n)}$.

In this paper, we shall focus on the study of the optimal pebbling number of a complete $m$-ary tree, where a complete $m$-ary tree with height $h$, denoted by $T_h^m$, is an $m$-ary tree satisfying that $v$ has $m$ children for each vertex $v$ not in the $h$th level. In Section 2, we first obtain $f'(T)$ for a complete $m$-ary tree $T$ with $m \geqslant 3$. Then, in Section 3, we show that the optimal pebbling problem of the complete binary tree with height $h$, $T_h^2$, can be transformed to an instance of an integer linear programming problem(ILP) and we find an efficient algorithm to find the optimal solution for the instance of ILP which corresponds to the optimal pebbling number problem for the complete binary tree.

## 2. $m \geqslant 3$

Consider a d.c. $\delta$ of $G$ and let $H$ be a connected induced subgraph of $G$. Then $\delta|_{V(H)}(\delta|_H$ for short) is a d.c. of $H$ induced by $\delta$. In $H$, the maximum number of

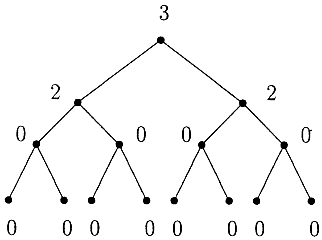


Fig. 1.

pebbles which can be moved to the vertex $v$ using pebbling moves in $\delta|_H$ is denoted by $\delta_H(v)$. Clearly, we have $\delta_H(v) \leqslant \delta_H$ for each $v \in V(H)$. Now, we have the first result.

**Theorem 2.1.** *$f'(T_h^m) = 2^h$ for each $m \geqslant 3$.*

**Proof.** Let $v_0$ be the vertex in the 0th level (root) of $T_h^m = T$, and $\delta$ be a d.c. such that $\delta(v_0) = 2^h$ and $\delta(v) = 0$ for each $v \in V(T) \setminus \{v_0\}$. Then it is obvious that $\delta$ is a pebbling of $T$, and thus $f'(T) \leqslant 2^h$. On the other hand, we will prove $f'(T) \geqslant 2^h$ by induction on $h$. This is trivial for $h=0$, so let $h > 1$. Assume that $f'(T_{h-1}^m) \geqslant 2^{h-1}$ and let $T - v_0$ contain components $T_1, T_2, \ldots, T_m$. Clearly, $f'(T_1) = f'(T_2) = \cdots = f'(T_m) = f'(T_{h-1}^m) \geqslant 2^{h-1}$. Now, let $\delta$ be an arbitrary pebbling of $T$ such that $\delta(v_0) = x_0$ and $\delta_{T_i} = x_i$, $i = 1, 2, \ldots, m$. Then, for $i = 1, 2, \ldots, m$,

$$f'(T_i) \leqslant \tfrac{1}{2} x_0 + x_i + \sum_{j \in \{1,2,\ldots,i-1,i+1,\ldots,m\}} \tfrac{1}{4} x_j.$$

This implies that

$$m f'(T_{h-1}^m) \leqslant \tfrac{m}{2} x_0 + \tfrac{m+3}{4} x_1 + \tfrac{m+3}{4} x_2 + \cdots + \tfrac{m+3}{4} x_m \leqslant \tfrac{m}{2} \delta_T, \qquad (*)$$

and we have $\delta_T \geqslant 2 f'(T_{h-1}^m) \geqslant 2^h$. Thus, $f'(T) \geqslant 2^h$ and this concludes the proof. □

We note finally that the proof fails when $m = 2$ because then $(m+3)/4 > m/2$. In fact, $f'(T_3^2) \leqslant 7 < 2^3$ (see Fig. 1).

## 3. $m = 2$

If a d.c. of a complete binary tree $T$ with height $h$ can be obtained by placing $x_i$ pebbles on each vertex in the $i$th level, $i = 0, 1, \ldots, h$, we denote it by $\langle x_0, x_1, \ldots, x_h \rangle_T$. (We may omit $T$ from this notation when it is clear which tree we are using.) In this section, we mainly prove that the optimal pebbling of a complete binary tree $T_h^2$ can be obtained using a d.c. of this type where $x_i$ is even for each $i > 0$; we will call such a d.c. *symmetric*. See Fig. 1 for an example.

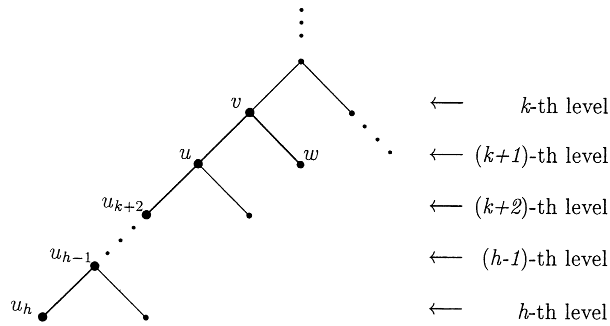


Fig. 2.

Since $f'(T_0^2)=1$ is easy to see, throughout this section, let $T$ be a complete binary tree with height $h\geqslant 1$. For each vertex $v\in V(T)$ in the $k$th level of $T$, the branch of $T$ including $v$ and all its descendants is denoted by $T_v$, and the subtree of $T$ obtained by removing $V(T_v)\backslash\{v\}$ is denoted by $\bar{T}_v$. If a d.c. $\delta$ has the property that $\delta|_{T_v}=\langle x_k,x_{k+1},\ldots,x_h\rangle_{T_v}$ where $x_i$ is even for each $i>k$, then we say $\delta$ is symmetric on $T_v$, and this implies that $\delta_{T_u}(u)=\sum_{j=i}^{h}x_j$ for each $u\in V(T_v)$ in the $i$th level of $T$, where $k\leqslant i\leqslant h$.

**Lemma 3.1.** *Let $\delta$ be a d.c. of $T$ and $v$ be a vertex in the $k$th level of $T$ which has two children $u$ and $w$, and let $\delta$ be symmetric on $T_u$ and $T_w$. If either $h>k+1$ or either $\delta(u)$ or $\delta(w)$ does not equal* 1, *then the following statements are equivalent*; *in any case*, (1) *implies* (2), *and* (2) *implies* (3).

(1) $\delta_T(v')\geqslant 1$ *for each* $v'\in V(T_v)$;

(2) $\delta_{T_u}-\frac{1}{3}\delta_{T_u}(u)+\frac{1}{3}\delta_{\bar{T}_v}(v)+\frac{1}{6}\delta_{T_w}(w)\geqslant\frac{1}{3}\times 2^{h-k}$ *and* $\delta_{T_w}-\frac{1}{3}\delta_{T_w}(w)+\frac{1}{3}\delta_{\bar{T}_v}(v)+\frac{1}{6}\delta_{T_u}(u)\geqslant\frac{1}{3}\times 2^{h-k}$;

(3) $\delta_T(v_h)\geqslant 1$ *for each* $v_h\in V(T_v)$ *in the* $h$*th level of* $T$.

**Proof.** Let $\delta|_{T_u}=\langle y_{k+1},y_{k+2},\ldots,y_h\rangle_{T_u}$, $u_h\in V(T_u)$ be an arbitrary vertex in the $h$th level of $T$ and $P=w,v,u,u_{k+2},\ldots,u_h$ be the path connecting $w$ and $u_h$. Clearly, $u_i$ is in the $i$th level of $T$, $i=k+2,k+3,\ldots,h$; see Fig. 2. We first move pebbles from $V(T)\setminus V(P)$ to the vertices of $P$ by applying pebbling moves such that the number of pebbles in $P$ is as large as possible. Then there exists a d.c. $\delta'$ of $P$ and $m_{k-1},\ldots,m_h$ which are defined by $\delta'(w)=m_{k-1}=\delta_{T_w}(w)$, $\delta'(v)=m_k=\delta_{\bar{T}_v}(v)$, $\delta'(u)=m_{k+1}=y_{k+1}+\frac{1}{2}\sum_{j=k+2}^{h}y_j$, $\delta'(u_h)=m_h=y_h$ and $\delta'(u_i)=m_i=y_i+\frac{1}{2}\sum_{j=i+1}^{h}y_j$ for $i=k+2,k+3,\ldots,h-1$. Note that $\delta_T(u_h)=\delta'_P(u_h)$. Now, we are ready to prove our implications.

(1) ⇒ (2): Clearly,

$$\delta_T(u_h)=m_h+\lfloor\tfrac{1}{2}(m_{h-1}+\lfloor\tfrac{1}{2}(m_{h-2}+\lfloor\tfrac{1}{2}(\ldots(m_k+\lfloor\tfrac{1}{2}m_{k-1}\rfloor)\ldots)\rfloor)\rfloor\geqslant 1.$$

By taking away all the floors, we have

$$\sum_{i=k-1}^{h} (\tfrac{1}{2})^{h-i} m_i$$
$$= (\tfrac{1}{2})^{h-(k-1)} \delta_{T_w}(w) + (\tfrac{1}{2})^{h-k} \delta_{\bar{T}_v}(v) + \sum_{i=k+1}^{h-1} \left( (\tfrac{1}{2})^{h-i} \left( y_i + \frac{1}{2} \sum_{j=i+1}^{h} y_j \right) \right) + y_h$$
$$= (\tfrac{1}{2})^{h-(k-1)} \delta_{T_w}(w) + (\tfrac{1}{2})^{h-k} \delta_{\bar{T}_v}(v) + \sum_{i=k+1}^{h} (\tfrac{1}{2})^{h-i} y_i + \sum_{i=k+1}^{h-1} (\tfrac{1}{2})^{h-i+1} \sum_{j=i+1}^{h} y_j$$
$$= (\tfrac{1}{2})^{h-(k-1)} \delta_{T_w}(w) + (\tfrac{1}{2})^{h-k} \delta_{\bar{T}_v}(v) + \sum_{i=k+1}^{h} \left( (\tfrac{1}{2})^{h-i} + \sum_{j=h-i+2}^{h-k} (\tfrac{1}{2})^{j} \right) y_i$$
$$= (\tfrac{1}{2})^{h-(k-1)} \delta_{T_w}(w) + (\tfrac{1}{2})^{h-k} \delta_{\bar{T}_v}(v) + \sum_{i=k+1}^{h} (3 \times (\tfrac{1}{2})^{h-i+1} - (\tfrac{1}{2})^{h-k}) y_i \geqslant 1.$$

This is equivalent to saying that

$$\sum_{i=k+1}^{h} (2^{i-(k+1)} - \tfrac{1}{3}) y_i + \tfrac{1}{3} \delta_{\bar{T}_v}(v) + \tfrac{1}{6} \delta_{T_w}(w)$$
$$= \delta_{T_u} - \tfrac{1}{3} \delta_{T_u}(u) + \tfrac{1}{3} \delta_{\bar{T}_v}(v) + \tfrac{1}{6} \delta_{T_w}(w) \geqslant \tfrac{1}{3} \times 2^{h-k}.$$

Similarly, we obtain $\delta_{T_w} - \frac{1}{3}\delta_{T_w}(w) + \frac{1}{3}\delta_{\bar{T}_v}(v) + \frac{1}{6}\delta_{T_u}(u) \geqslant \frac{1}{3} \times 2^{h-k}$.

(2) ⇒ (3): Without loss of generality, restrict attention to $v_h = u_h \in V(T_u)$. Let $g_{k-1} = m_{k-1}$ and $g_i = m_i + \lfloor \frac{1}{2} g_{i-1} \rfloor$, $i = k, k+1, \ldots, h$. Then

$$\delta_T(u_h) = m_h + \lfloor \tfrac{1}{2}(m_{h-1} + \lfloor \tfrac{1}{2}(m_{h-2} + \lfloor \tfrac{1}{2}(\ldots(m_k + \lfloor \tfrac{1}{2} m_{k-1} \rfloor)\ldots)\rfloor)\rfloor = g_h.$$

It suffices to prove that $g_h \geqslant 1$. First, we will prove that $g_j \geqslant 2^{h-j} - \sum_{i=j+1}^{h} 2^{i-j} m_i$ for $k-1 \leqslant j \leqslant h$ by induction on $j$. Note that $2^{h-j} - \sum_{i=j+1}^{h} 2^{i-j} m_i$ is even for $k-1 \leqslant j \leqslant h-1$. Since $\delta_{T_u} - \frac{1}{3}\delta_{T_u}(u) + \frac{1}{3}\delta_{\bar{T}_v}(v) + \frac{1}{6}\delta_{T_w}(w) \geqslant \frac{1}{3} \times 2^{h-k}$, we have $\sum_{i=k-1}^{h} (\frac{1}{2})^{h-i} m_i \geqslant 1$. Thus $\sum_{i=k-1}^{h} 2^{i-(k-1)} m_i \geqslant 2^{h-(k-1)}$ and $g_{k-1} = m_{k-1} \geqslant 2^{h-(k-1)} - \sum_{i=k}^{h} 2^{i-(k-1)} m_i$, so the assertion is true for $j = k-1$. Assume that $g_j \geqslant 2^{h-j} - \sum_{i=j+1}^{h} 2^{i-j} m_i$ for some $j$, $k-1 < j \leqslant h-1$, then $g_{j+1} = m_{j+1} + \lfloor \frac{1}{2} g_j \rfloor \geqslant m_{j+1} + 2^{h-j-1} - \sum_{i=j+1}^{h} 2^{i-j-1} m_i = 2^{h-(j+1)} - \sum_{i=(j+1)+1}^{h} 2^{i-(j+1)} m_i$, completing the induction. Setting $j = h$, we then have $g_h \geqslant 1$, so we are done.

(3) ⇒ (1): We shall use the 'bottom-up' idea to prove the implication. If $h = k+1$, by our hypothesis, either $\delta(u) \geqslant 2$, $\delta(w) \geqslant 2$, or $\delta_T(v) \geqslant 2$. This implies $\delta_T(v) \geqslant 1$ and we are done. Otherwise, let $h > k+1$. Assume that we have shown $\delta_T(v') \geqslant 1$ for each $v' \in V(T_u)$ in the levels greater than $j \geqslant k+1$ and let $v_j \in V(T_u)$ be an arbitrary vertex in the $j$th level which has two children $u_{j+1}$ and $w_{j+1}$. By assumption, we have $\delta_T(u_{j+1}) = \delta_{T_{u_{j+1}}}(u_{j+1}) + \lfloor \frac{1}{2}(\delta_{\bar{T}_{v_j}}(v_j) + \lfloor \frac{1}{2} \delta_{T_{w_{j+1}}}(w_{j+1}) \rfloor) \rfloor \geqslant 1$ and $\delta_T(w_{j+1}) = \delta_{T_{w_{j+1}}}(w_{j+1}) + \lfloor \frac{1}{2}(\delta_{\bar{T}_{v_j}}(v_j) + \lfloor \frac{1}{2} \delta_{T_{u_{j+1}}}(u_{j+1}) \rfloor) \rfloor \geqslant 1$. Note that $\delta_{T_{u_{j+1}}}(u_{j+1}) = \delta_{T_{w_{j+1}}}(w_{j+1}) = \sum_{i=j+1}^{h} y_i$

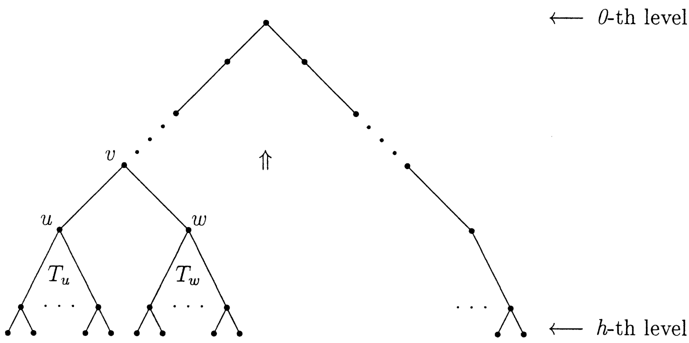


Fig. 3.

is even; this implies that either $\delta_{T_{u_{j+1}}}(u_{j+1})=\delta_{T_{w_{j+1}}}(w_{j+1})\geqslant 2$ or $\delta_{\bar{T}_{v_j}}(v_j)\geqslant 2$. Hence, $\delta_T(v_j)=\delta_{\bar{T}_{v_j}}(v_j)+\frac{1}{2}\delta_{T_{u_{j+1}}}(u_{j+1})+\frac{1}{2}\delta_{T_{w_{j+1}}}(w_{j+1})\geqslant 2$, and we obtain $\delta_T(v')\geqslant 2$ for each $v'\in V(T_u)$ in levels above $h$. Similarly, $\delta_T(v')\geqslant 2$ for each $v'\in V(T_w)$ in levels above $h$. Either of these two facts implies that $\delta_T(v)\geqslant 1$. This concludes the proof of this implication. □

**Lemma 3.2.** *Let $\delta$ be a symmetric d.c. of $T$ and $\delta=\langle x_0,x_1,\ldots,x_h\rangle_T$. Then $\delta$ is a pebbling of $T$ if and only if $\sum_{i=0}^{h}(2^i-\frac{1}{3})x_i\geqslant\frac{1}{3}\times 2^{h+1}$.*

**Proof.** Let $v$ be the root of $T$ which has two children $u$ and $w$. Then $\delta$ is symmetric on $T_u$ and $T_w$, respectively. By Lemma 3.1, we obtain that $\delta$ is a pebbling of $T$ if and only if

$$\delta_{T_u}-\frac{1}{3}\delta_{T_u}(u)+\frac{1}{3}\delta_{\bar{T}_v}(v)+\frac{1}{6}\delta_{T_w}(w)\geqslant\frac{1}{3}\times 2^h$$

and

$$\delta_{T_w}-\frac{1}{3}\delta_{T_w}(w)+\frac{1}{3}\delta_{\bar{T}_v}(v)+\frac{1}{6}\delta_{T_u}(u)\geqslant\frac{1}{3}\times 2^h.$$

Note that $\delta_{T_u}=\delta_{T_w}=\sum_{i=1}^{h}2^{i-1}x_i$, $\delta_{T_u}(u)=\delta_{T_w}(w)=\sum_{i=1}^{h}x_i$ and $\delta_{\bar{T}_v}(v)=x_0$. Hence, observing that the two inequalities are identical, we have the proof. □

**Lemma 3.3.** *For any pebbling $\delta$ of $T$, there exists a pebbling $\tilde{\delta}$ of $T$ such that $\tilde{\delta}$ is symmetric on $T$ and $\tilde{\delta}_T=\delta_T$.*

**Proof.** The proof follows by constructing $\tilde{\delta}$ recursively starting from the highest level. To do this, we let $\delta$ be a pebbling of $T$ which is symmetric on $T_u$ and $T_w$ where $u$ and $w$ are two children of $v$ (see Fig. 3). We will first construct another pebbling $\delta''$ of $T$ from $\delta$ which is symmetric on $T_u$ and $T_w$, by rearranging the pebbles on $T_w$. Then, by adjusting the number of pebbles on $v$, $w$, and $T_u$, we will have a pebbling $\tilde{\delta}$ symmetric on $T_v$ and with no more pebbles than $\delta$. The proof ends when $v$ reaches the root.

We shall use the same notations as we have used in the proof of Lemma 3.1. Without loss of generality, let $\delta_{T_u} \geqslant \delta_{T_w}$. For convenience, we introduce three functions defined on the set of distributing configurations of $T$. For each d.c. of $T$, $\alpha$, let

$$\gamma_u(\alpha) = \alpha_{T_u} - \tfrac{1}{3}\alpha_{T_u}(u) + \tfrac{1}{3}\alpha_{\bar{T}_v}(v) + \tfrac{1}{6}\alpha_{T_w}(w),$$

$$\gamma_w(\alpha) = \alpha_{T_w} - \tfrac{1}{3}\alpha_{T_w}(w) + \tfrac{1}{3}\alpha_{\bar{T}_v}(v) + \tfrac{1}{6}\alpha_{T_u}(u)$$

and

$$\phi(\alpha) = \gamma_u(\alpha) - \gamma_w(\alpha).$$

Now, by observation, we have the following facts:

(1) $\phi(\alpha) = \alpha_{T_u} - \alpha_{T_w} - \frac{1}{2}(\alpha_{T_u}(u) - \alpha_{T_w}(w))$.

(2) By Lemma 3.1, since $\delta$ is a pebbling of $T$ which is symmetric on $T_u$ and $T_w$, we have $\gamma_u(\delta) \geqslant \frac{1}{3} \times 2^{h-k}$ and $\gamma_w(\delta) \geqslant \frac{1}{3} \times 2^{h-k}$.

(3) Let $\delta|_{T_w} = \langle y_{k+1}, y_{k+2}, \ldots, y_h \rangle_{T_w}$. If $\phi(\delta) < 0$, then there exists an index $j > k+1$ such that $y_j > 0$. (For otherwise, by the definition of $\phi$, $\phi(\delta) \geqslant 0$.) Now, define $\delta'$ by $\delta'|_{T_w} = \langle y_{k+1}, y_{k+2}, \ldots, y_{j-2}, y_{j-1}+4, y_j - 2, y_{j+1}, \ldots, y_h \rangle_{T_w}$, $\delta'|_{T_u} = \delta|_{T_u}$ and $\delta'|_{\bar{T}_v} = \delta|_{\bar{T}_v}$. Then $\delta'$ is symmetric on $T_u$ and $T_w$, $\delta'_T = \delta_T$, and $\delta'_{T_w}(w) = \delta_{T_w}(w) + 2$, which implies that $\phi(\delta') = \phi(\delta) + 1$, $\delta'_{T_v}(v) = \delta_{T_v}(v) + 1$ and $\gamma_u(\delta') = \gamma_u(\delta) + \frac{1}{3}$.

Using the above facts we can construct a d.c. $\delta''$ of $T$ by way of the following procedure:

**begin**
  Let $\delta^{(0)} := \delta$, $\phi := \phi(\delta^{(0)})$ and $i := 0$;
  **while** $\phi < -\frac{1}{2}$ **do**
    if $\delta^{(i)}|_{T_w} = \langle z_{k+1}^{(i)}, \ldots, z_h^{(i)} \rangle_{T_w}$, and
    $j > k+1$ is an index such that $z_j^{(i)} > 0$,
    let $\delta^{(i+1)}|_{\bar{T}_v} := \delta^i|_{\bar{T}_v}$, $\delta^{(i+1)}|_{T_u} := \delta^i|_{T_u}$, and
    let $\delta^{(i+1)}|_{T_w} = \langle z_{k+1}^{(i)}, z_{k+2}^{(i)}, \ldots, z_{j-2}^{(i)}, z_{j-1}^{(i)} + 4,\ z_j^{(i)} - 2, z_{j+1}^{(i)}, \ldots, z_h^{(i)} \rangle_{T_w}$;
    $\phi = \phi + 1$;
    $i = i + 1$;
**end**

Clearly, the d.c. $\delta'' = \delta^{(i)}$ obtained satisfies the following properties:

(a) $\delta''$ is symmetric on $T_u$ and $T_w$;
(b) $\delta''_T = \delta_T$;
(c) $\delta''_{T_u} = \delta_{T_u}$ and $\delta''_{T_w} = \delta_{T_w}$; this implies that $\delta''_{T_u} \geqslant \delta''_{T_w}$; and
(d) $\phi(\delta'') \geqslant -\frac{1}{2}$. Furthermore, by (1) and (3), if $\phi(\delta'') > 0$, then $\delta'' = \delta$.

Most importantly, we have to prove the following property:

(e) $\delta''$ is a pebbling of $T$.

If $\delta''=\delta$, this was assumed. Otherwise, by (3), we have $\delta''_{T_v}(v)\geqslant\delta_{T_v}(v)$ which implies that $\delta''_T(v')\geqslant\delta_T(v')\geqslant 1$ for each $v'\in V(\bar{T}_v)$. Furthermore, $\gamma_u(\delta'')\geqslant\gamma_u(\delta)\geqslant\frac{1}{3}\times 2^{h-k}$. By (d), we have $\phi(\delta'')\leqslant 0$; this implies that $\gamma_w(\delta'')\geqslant\gamma_u(\delta'')\geqslant\frac{1}{3}\times 2^{h-k}$. If $\delta''\neq\delta$ we must have $h>k+1$, so, by Lemma 3.1, we have the claim.

Now we are ready to define $\tilde{\delta}$. Let $\delta''|_{T_w}=\langle z_{k+1},z_{k+2},\dots,z_h\rangle_{T_w}$ and let $\lambda=0$ if $z_{k+1}$ is even and 1 if $z_{k+1}$ is odd. We define $\tilde{\delta}$ by $\tilde{\delta}(v')=\delta''(v')$ for each vertex $v'\in V(\bar{T}_v)\setminus\{v\}$ and $\tilde{\delta}|_{T_v}=\langle x_k,x_{k+1},\dots,x_h\rangle_{T_v}$ where $x_k=\delta''(v)+\delta''_{T_u}-\delta''_{T_w}+2\lambda$, $x_{k+1}=z_{k+1}-\lambda$ and $x_j=z_j$ for each $j>k+1$. Clearly, $\tilde{\delta}_{T_v}=\delta''_{T_v}$, $\tilde{\delta}_{T_u}(u)=\tilde{\delta}_{T_w}(w)=\delta''_{T_w}(w)-\lambda$ is even, and $x_j$ is even for each $j>k$. This implies that $\gamma_u(\tilde{\delta})=\gamma_w(\tilde{\delta})=\tilde{\delta}_{T_w}-\frac{1}{6}\tilde{\delta}_{T_w}(w)+\frac{1}{3}\tilde{\delta}_{\bar{T}_v}(v)=\frac{1}{3}l$ for some integer $l$, $\lfloor\frac{1}{2}\tilde{\delta}_{T_w}(w)\rfloor=\frac{1}{2}\tilde{\delta}_{T_w}(w)=\frac{1}{2}(\delta''_{T_w}(w)-\lambda)$, $\tilde{\delta}_T=\delta''_T$, and $\tilde{\delta}$ is symmetric on $T_v$.

Again, we have to prove that $\tilde{\delta}$ is a pebbling of $T$. Since $\delta''$ is a pebbling of $T$, and $\tilde{\delta}(u)=\tilde{\delta}(w)$ is even and $\tilde{\delta}(v')=\delta''(v')$ for each vertex $v'\in V(\bar{T}_v)\setminus\{v\}$, it suffices to prove that $\tilde{\delta}_{T_v}(v)\geqslant\delta''_{T_v}(v)$ and $\gamma_u(\tilde{\delta})=\gamma_w(\tilde{\delta})\geqslant\frac{1}{3}\times 2^{h-k}$. Since $\phi(\delta'')\geqslant -\frac{1}{2}$, we have

$$\begin{aligned}\delta''_{T_u}-\delta''_{T_w} &= \phi(\delta'')+\tfrac{1}{2}(\delta''_{T_u}(u)-\delta''_{T_w}(w))\\ &\geqslant \lceil\tfrac{1}{2}(\delta''_{T_u}(u)-\delta''_{T_w}(w)-1)\rceil=\lfloor\tfrac{1}{2}(\delta''_{T_u}(u)-\delta''_{T_w}(w))\rfloor.\end{aligned}$$

This implies that

$$\begin{aligned}\tilde{\delta}_{T_v}(v) &= \delta''(v)+\delta''_{T_u}-\delta''_{T_w}+2\lambda+2\lfloor\tfrac{1}{2}\tilde{\delta}_{T_w}(w)\rfloor\\ &\geqslant \delta''(v)+\lfloor\tfrac{1}{2}(\delta''_{T_u}(u)-\delta''_{T_w}(w))\rfloor+2\lambda+2\times\tfrac{1}{2}(\delta''_{T_w}(w)-\lambda)\\ &= \delta''(v)+\lfloor\tfrac{1}{2}\delta''_{T_u}(u)+\tfrac{1}{2}\delta''_{T_w}(w)\rfloor+\lambda\\ &\geqslant \delta''(v)+\lfloor\tfrac{1}{2}\delta''_{T_u}(u)\rfloor+\lfloor\tfrac{1}{2}\delta''_{T_w}(w)\rfloor=\delta''_{T_v}(v).\end{aligned}$$

Note that $\tilde{\delta}_{T_w}=\delta''_{T_w}-\lambda$, $\tilde{\delta}_{T_w}(w)=\delta''_{T_w}(w)-\lambda$ and $\tilde{\delta}_{\bar{T}_v}(v)=\delta''_{\bar{T}_v}(v)+\delta''_{T_u}-\delta''_{T_w}+2\lambda=\delta''_{\bar{T}_v}(v)+\phi(\delta'')+\frac{1}{2}\delta''_{T_u}(u)-\frac{1}{2}\delta''_{T_w}(w)+2\lambda$. Therefore, we have

$$\begin{aligned}\gamma_u(\tilde{\delta}) &= \gamma_w(\tilde{\delta})=\tilde{\delta}_{T_w}-\tfrac{1}{6}\tilde{\delta}_{T_w}(w)+\tfrac{1}{3}\tilde{\delta}_{\bar{T}_v}(v)\\ &= (\delta''_{T_w}-\lambda)-\tfrac{1}{6}(\delta''_{T_w}(w)-\lambda)+\tfrac{1}{3}(\delta''_{\bar{T}_v}(v)+\phi(\delta'')\\ &\quad +\tfrac{1}{2}\delta''_{T_u}(u)-\tfrac{1}{2}\delta''_{T_w}(w)+2\lambda)\\ &= \delta''_{T_w}-\tfrac{1}{3}\delta''_{T_w}(w)+\tfrac{1}{3}\delta''_{\bar{T}_v}(v)+\tfrac{1}{6}\delta''_{T_u}(u)+\tfrac{1}{3}\phi(\delta'')-\tfrac{1}{6}\lambda\\ &\geqslant \gamma_w(\delta'')+\tfrac{1}{3}\phi(\delta'')-\tfrac{1}{6}.\end{aligned}$$

Now, if $\phi(\delta'')>0$, then $\gamma_w(\delta'')+\frac{1}{3}\phi(\delta'')\geqslant\frac{1}{3}\times 2^{h-k}$. Otherwise we have $\gamma_w(\delta'')+\frac{1}{3}\phi(\delta'')\geqslant\gamma_w(\delta'')+\phi(\delta'')=\gamma_u(\delta'')\geqslant\frac{1}{3}\times 2^{h-k}$. This implies that $\gamma_u(\tilde{\delta})=\gamma_w(\tilde{\delta})\geqslant\frac{1}{3}\times 2^{h-k}-\frac{1}{6}$. But $\gamma_u(\tilde{\delta})=\gamma_w(\tilde{\delta})=\frac{1}{3}l$, $l$ an integer; therefore $\gamma_u(\tilde{\delta})=\gamma_w(\tilde{\delta})\geqslant\frac{1}{3}\times 2^{h-k}$. Hence by Lemma 3.1, we conclude that $\tilde{\delta}$ is a pebbling of $T$. □

**Theorem 3.4.** $f'(T) = \min\{\sum_{i=0}^{h} 2^i x_i \mid \sum_{i=0}^{h}(2^i - \frac{1}{3})x_i \geqslant \frac{1}{3} \times 2^{h+1},\ x_0 \in \{0,1,2,3\}$, *and* $x_i \in \{0,2\},\ i = 1,2,\ldots,h\}$.

**Proof.** By Lemma 3.3, since each pebbling of $T$, $\delta$, has a symmetric pebbling $\tilde{\delta}$ such that $\delta_T = \tilde{\delta}_T$, $f'(T)$ can be obtained by minimizing $\sum_{i=0}^{h} 2^i x_i$ where $\langle x_0, x_1, \ldots, x_h\rangle$ is a symmetric pebbling of $T$. Now it suffices to claim that there exists an optimal symmetric pebbling of $T$, $\langle x_0, x_1, \ldots, x_h\rangle$, such that $x_i < 4$ for each $i \in \{0, 1, \ldots, h\}$. Suppose not. Then in each optimal symmetric pebbling of $T$, $\langle y_0, y_1, \ldots, y_h\rangle$, there exists a smallest index $0 \leqslant j \leqslant h$ such that $y_j \geqslant 4$. Since $f'(T) \leqslant 2^h$, we must have $j < h$. By Lemma 3.2, it is easy to check that $\delta^{(j)} = \langle y_0, \ldots, y_{j-1}, y_j - 4, y_{j+1} + 2, y_{j+2}, \ldots, y_h\rangle$ is also an optimal symmetric pebbling of $T$. By applying this operation repeatedly we obtain an optimal symmetric pebbling of $T$, $\langle x_0^*, x_1^*, \ldots, x_h^*\rangle$, such that $x_j^* < 4$ for $j = 0, 1, \ldots, h$. This is a contradiction. Thus we have the proof. $\square$

Clearly, by Theorem 3.4, we can transform the optimal pebbling problem of complete binary tree to the following instance of ILP:

$$\min \sum_{i=0}^{h} 2^i x_i$$

$$\sum_{i=0}^{h} (2^i - \tfrac{1}{3}) x_i \geqslant \tfrac{1}{3} \times 2^{h+1},$$

$$x_0 \in \{0,1,2,3\} \quad \text{and} \quad x_i \in \{0,2\},\ i = 1,2,\ldots,h. \qquad (**)$$

Although ILP is NP-complete, we have an efficient algorithm to solve $(**)$, and thus we can quickly find the optimal pebbling number of the complete binary tree. In what follows we present the algorithm with full details.

Algorithm OPCBT(h)
*Input*: $h$ (a positive integer).
*Output*: $(x_0, x_1, \ldots, x_h)$ (an optimal solution of $(**)$).
**begin**
{Step 1: initialization of $c = \sum_{i=0}^{h} (2^i - \frac{1}{3})x_i$ and $x_i$, $i = 0, 1, \ldots, h$.}
$x_0 := 3$;
$c := (1 - \frac{1}{3})x_0$;
**for** $i = 1$ **to** $h$ **do**
$x_i := 2$;
$c := c + (2^i - \frac{1}{3})x_i$;
{Step 2: testing $x_i = 0$ or $2$, $i = 1, 2, \ldots, h$.}
**for** $i = h$ **to** $1$ **step** $-1$ **do**
**if** $c - (2^i - \frac{1}{3})x_i \geqslant \frac{1}{3} \times 2^{h+1}$ **then**
$c := c - (2^i - \frac{1}{3})x_i$;
$x_i := 0$;

{Step 3: testing $x_0 = 0, 1, 2$ or 3.}
**while** $c - (1 - \frac{1}{3}) \geqslant \frac{1}{3} \times 2^{h+1}$ **and** $x_0 > 0$ **do**
$c := c - (1 - \frac{1}{3})$;
$x_0 := x_0 - 1$;
**end**

**Theorem 3.5.** *Algorithm OPCBT produces an optimal solution of* $(**)$, *and its time complexity is* $O(h)$.

**Proof.** Let $(n_0, n_1, \ldots, n_h)$ be the output of Algorithm OPCBT. Clearly, $(3, 2, \ldots, 2)$ is a feasible solution of $(**)$ in Step 1; hence an optimal solution of $(**)$ exists. Now, we shall claim that $(n_0, n_1, \ldots, n_h)$ is the unique optimal solution of $(**)$. Observe that for each integer $k \geqslant 1$, $2 \cdot 2^k > 3 + \sum_{i=1}^{k-1} 2 \times 2^i$. Therefore, if $(3, 2, \ldots, 2, 0, n_{k+1}, \ldots, n_h)$ is a feasible solution of $(**)$, then any feasible solution $(x_0, x_1, \ldots, x_{k-1}, 2, n_{k+1}, \ldots, n_h)$ is not optimal. On the other hand, if $(3, 2, \ldots, 2, 0, n_{k+1}, \ldots, n_h)$ is not a feasible solution of $(**)$, then for each feasible solution $(x_0, x_1, \ldots, x_{k-1}, x_k, n_{k+1}, \ldots, n_h)$, $x_k = 2$. Hence, if $(x_0^*, x_1^*, \ldots, x_h^*)$ is an optimal solution, by Step 2, starting from $h$, we see that $x_h^* = n_h$, and then $x_{h-1}^* = n_{h-1}, \ldots, x_1^* = n_1$. Finally, by Step 3, we have $x_0^* = n_0$. This implies that the output is the unique optimal solution. Since the time complexity of Algorithm OPCBT is easy to see, we conclude the proof. □

**Concluding remark.** Let $\delta = \langle x_0, x_1, \ldots, x_h \rangle$ be the optimal pebbling of $T$ obtained from Algorithm OPCBT. Then by Theorem 3.4, for any $k < h$, we have

$$\delta_T = \sum_{i=0}^{h} 2^i x_i \geqslant \tfrac{1}{3} \times 2^{h+1} + \sum_{i=0}^{h} \tfrac{1}{3} x_i$$

$$= \sum_{i=1}^{k} (\tfrac{1}{4})^i \times 2^{h+1} + \tfrac{1}{3} \times (\tfrac{1}{4})^k \times 2^{h+1} + \sum_{i=0}^{h} \tfrac{1}{3} x_i$$

$$= 2(\tfrac{1}{4} \times 2^h + \tfrac{1}{4^2} \times 2^h + \cdots + \tfrac{1}{4^k} \times 2^h) + 2 \times \tfrac{1}{3} \times \tfrac{1}{4^k} \times 2^h + \sum_{i=0}^{h} \tfrac{1}{3} x_i$$

$$= 2(2^{h-2} + 2^{h-4} + \cdots + 2^{h-2k}) + 2 \times \tfrac{1}{3} \times 2^{h-2k} + \sum_{i=0}^{h} \tfrac{1}{3} x_i. \qquad (***)$$

Eq. $(***)$ then suggests that, in order to obtain an optimal pebbling of $T$, 2 pebbles should be placed on each vertex of the $(h-2)$th, $(h-4)$th, $\ldots$, and $(h-2k)$th levels, using the lower levels to ensure that $\sum_{i=0}^{h-2k-1} 2^i x_i \geqslant 2 \times \frac{1}{3} \times 2^{h-2k} + \sum_{i=0}^{h} \frac{1}{3} x_i$. Therefore, we conclude that when $h$ is sufficiently large there exists an $h'$ such that $\langle x_{h'}, x_{h'+1}, \ldots, x_h \rangle = \langle 0, 2, \ldots, 0, 2, 0, 2, 0, 0 \rangle$, and that $f'(T)$ can be approximated by $\frac{1}{3}(2^{h+1} + h)$ with small errors. To be more precise,

$$f'(T_h^2) = (2^{h+1} + h)/3 + O(\log h)$$

and for $i > \mathrm{O}(\log h)$, the value of $x_i$ in an optimal pebbling of $T_h^2$ is

2, if $h-i$ is even and positive, and

0, otherwise.

To prove this, let each $n \in \{0,1,\ldots,2^{h+2}-1\}$ have binary expansion $n=\sum_{i=0}^{h+1} y_i(n)2^i$. Write

$$\psi(n) = y_0(n) + 2\sum_{i=1}^{h+1} y_i(n) \quad \text{and} \quad \chi(n) = n - \psi(n)/3.$$

Let $\delta = \langle x_0, x_1, \ldots, x_h\rangle$ be a symmetrical d.c. of $T_h^2$ with $x_0 \in \{0,1,2,3\}$, $x_i \in \{0,2\}$ for $i=1,2,\ldots,h$. Write $n=\delta_T$. Then

$$x_0 = y_0(n) + 2y_1(n),\ x_1 = 2y_2(n), \ldots, x_h = 2y_{h+1}(n), \tag{$****$}$$

so by Lemma 3.2, $\delta$ will be a pebbling of $T$ if and only if

$$\chi(n) = n - \psi(n)/3 \geqslant \tfrac{1}{3} \times 2^{h+1}. \tag{$*****$}$$

$f'(T_h^2)$ will therefore be the minimal $n$ for which $(*****)$ holds. Since $\psi(n+1) \leqslant \psi(n)+1$ for all $n$, $\chi$ is increasing with $n$. Let $n_0 = 2\lfloor 2^h/3 \rfloor$; then $y_i(n_0)$ will be 1 if $h-i$ is odd and $0<i<h$, and 0 otherwise; therefore, $\psi(n_0) = 2\lfloor h/2\rfloor$, and $\chi(n_0) = n_0 - \frac{2}{3}\lfloor h/2 \rfloor$. Set $k = \lfloor h/2 \rfloor - \lfloor \frac{1}{2}\log_2 h \rfloor - 1$, and let $h \geqslant 2$, so that $k \geqslant 0$. Evidently, $h > 2k$, so $y_{h-2k}(n_0) = 0$. Therefore, for all $i \geqslant 0$ no bigger than $2^{h-2k}$, no carry out of $y_{h-2k}(n_0)$ will occur when adding $i$ to $n_0$, and so

$$y_{h-2k+1}(n_0+i) = y_{h-2k+1}(n_0), \ldots, y_{h+1}(n_0+i) = y_{h+1}(n_0). \tag{$******$}$$

Therefore, for $0 \leqslant i \leqslant 2^{h-2k}$,

$$|\psi(n_0+i) - 2\lfloor \tfrac{h}{2} \rfloor| \leqslant 2(h-2k)+1 \leqslant 7 + 4\lfloor \tfrac{1}{2}\log_2 h \rfloor \leqslant 7 + 2\log_2 h$$

and

$$|\chi(n_0+i) - (n_0 + i - \tfrac{2}{3}\lfloor \tfrac{h}{2} \rfloor)| \leqslant \tfrac{7}{3} + \tfrac{2}{3}\log_2 h. \tag{$*******$}$$

Let $i_0 = h/3 - \log_2 h$ and $i_1 = h/3 + \log_2 h$. We have $2^{h-2k} \geqslant 2^{2(\lfloor \frac{1}{2}\log_2 h \rfloor + 1)} \geqslant h$, so for $h$ large enough, $0 \leqslant i_0 \leqslant i_1 \leqslant 2^{h-2k}$, and we may apply $(*******)$ to get

$$\chi(n_0+i_0) \leqslant n_0 + i_0 - \tfrac{2}{3}\lfloor \tfrac{h}{2} \rfloor + \tfrac{7}{3} + \tfrac{2}{3}\log_2 h < \tfrac{1}{3} \times 2^{h+1}, \text{ for } h \text{ sufficiently large,}$$

so $(*****)$ does not hold for $n = n_0 + i_0$, and

$$\chi(n_0+i_1) \geqslant n_0 + i_1 - \tfrac{2}{3}\lfloor \tfrac{h}{2} \rfloor - \tfrac{7}{3} - \tfrac{2}{3}\log_2 h \geqslant \tfrac{1}{3} \times 2^{h+1}, \text{ for } h \text{ sufficiently large,}$$

so $(*****)$ does hold for $n = n_0 + i_1$. Therefore, if $h$ is large enough, $n_0 + i_0 < f'(T_h^2) \leqslant n_0 + i_1$ and $f'(T_h^2) = 2^{h+1}/3 + h/3 + \mathrm{O}(\log h)$, as desired. The remark about the value of $x_i$ in an optimal pebbling now follows immediatedly from $(****)$ and $(******)$.

## Acknowledgements

The authors would like to express their appreciation to the referees for their efforts in improving the exposition of this paper. A special thanks goes to an anonymous referee who indeed rewrote the concluding remark for this paper.

## References

[1] F.R.K. Chung, Pebbling in hypercubes, SIAM J. Discrete Math. 2 (4) (1989) 467–472.
[2] T.A. Clarke, R.A. Hochberg, G.H. Hurlbert, Pebbling in diameter two graphs and products of paths, J. Graph Theory 25 (1997) 119–128.
[3] H.L. Fu, C.L. Shiue, Optimal pebbling graph, submitted.
[4] P. Lemke, D. Kleitman, An addition theorem on the integers modulo *n*, J. Number Theory 31 (3) (1989) 335–345.
[5] D. Moews, Pebbling graphs, J. Combin. Theory Ser. B 55 (1992) 244–252.
[6] D. Moews, Optimally pebbling hypercubes and powers, Discrete Math. 190 (1998) 271–276.
[7] L. Pachter, H. Snevily, B. Voxman, On pebbling graphs, Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory Comput. Congr. Numer. 107 (1995) 65–80.