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Fuzzy Sets and Systems 113 (2000) 185–203

FUZZY sets and systems

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Bidirectional approximate reasoning for rule-based systems using interval-valued fuzzy sets

Shyi-Ming Chen [a,*], Wen-Hoar Hsiao [b]

[a] *Department of Electronic Engineering, National Taiwan University of Science and Technology, Taipei, Taiwan, Republic of China*
[b] *Department of Computer and Information Science, National Chiao Tung University, Hsinchu, Taiwan, Republic of China*

Received January 1998; received in revised form July 1998

Abstract

In this paper, we extend the work of Chen et al. [Fuzzy Sets and Systems 91 (1997) 339–353] to present a new method to deal with bidirectional approximate reasoning for rule-based systems based on the direction of matching between interval-valued fuzzy sets. We also use some examples to illustrate the bidirectional approximate reasoning process. Because the proposed method can perform bidirectional approximate reasoning based on the direction of matching between interval-valued fuzzy sets, it is more reasonable and more powerful than the one presented in Chen et al., Fuzzy Sets and Systems 91 (1997) 339–353. The proposed method can provide a useful way to deal with bidirectional approximate reasoning for rule-based systems using interval-valued fuzzy sets.

Keywords: Bidirectional approximate reasoning; Direction-matching function; Interval-valued fuzzy sets; Rule-based system; Similarity function

1. Introduction

Since fuzzy set theory was proposed by Zadeh [19], some methods based on the fuzzy set theory for handling approximate (fuzzy) reasoning have been proposed, such as [1–7,9–14]. The following single-

* Corresponding author.
E-mail address: smchen@et.ntust.edu.tw (S.-M. Chen)

0165-0114/00/$ - see front matter
PII: S0165-0114(98)00351-0

input–single-output (SISO) approximate reasoning scheme has been discussed by many researchers

$$\begin{array}{l} R_1\text{: IF } X \text{ is } A_1 \text{ THEN } Y \text{ is } B_1 \\ R_2\text{: IF } X \text{ is } A_2 \text{ THEN } Y \text{ is } B_2 \\ \qquad\qquad \vdots \\ R_p\text{: IF } X \text{ is } A_p \text{ THEN } Y \text{ is } B_p \\ \text{Fact: } X \text{ is } A_0 \\ \hline \text{Consequence: } Y \text{ is } B_0 \end{array} \tag{1}$$

where R_i are fuzzy production rules [16], $1 \leqslant i \leqslant p$; X and Y are linguistic variables [20]; $A_0, A_1, A_2, \ldots, A_p$, B_1, $B_2, \ldots$ and B_p are linguistic terms represented by fuzzy sets [19]. In [1], Bien et al. presented an inference network for bidirectional approximate reasoning based on fuzzy sets, where the following SISO approximate reasoning scheme is also discussed in [1]

$$\begin{array}{l} R_1\text{: IF } X \text{ is } A_1 \text{ THEN } Y \text{ is } B_1 \\ R_2\text{: IF } X \text{ is } A_2 \text{ THEN } Y \text{ is } B_2 \\ \qquad\qquad \vdots \\ R_p\text{: IF } X \text{ is } A_p \text{ THEN } Y \text{ is } B_p \\ \text{Fact: } Y \text{ is } B_0 \\ \hline \text{Consequence: } X \text{ is } A_0 \end{array} \tag{2}$$

where R_i are fuzzy production rules, $1 \leqslant i \leqslant p$; X and Y are linguistic variables; $A_1, \ldots, A_p$, B_0, $B_1, B_2, \ldots$, and B_p are linguistic terms represented by fuzzy sets.

In [17], Turksen proposed the definitions of interval-valued fuzzy sets for the representation of combined concepts based on normal forms. In [13,14], Gorzalczany presented a method for interval-valued fuzzy reasoning based on the compatibility measure and described some properties about the interval-valued reasoning method, respectively. In [18], Yuan et al. use the normal form based interval-valued fuzzy set to deal with approximate reasoning. In [10], we have presented a method to deal with bidirectional approximate reasoning using interval-valued fuzzy sets, where the linguistic terms appearing in formulas (1) and (2) are represented by interval-valued fuzzy sets. However, the method presented in [10] has a drawback in dealing with bidirectional approximate reasoning of rule-based systems. For example, let us consider the following generalized modus ponens (GMP):

$$\begin{array}{l} \text{Rule: IF } X \text{ is } A \text{ THEN } Y \text{ is } B \\ \text{Fact: } X \text{ is } A^* \\ \hline \text{Consequence: } Y \text{ is } B^* \end{array}$$

where X and Y are linguistic variables; A^* and A are interval-valued fuzzy sets of the universe of discourse U, $U = \{u_1, u_2, \ldots, u_n\}$; B^* and B are interval-valued fuzzy sets of the universe of discourse V, $V = \{v_1, v_2, \ldots, v_m\}$; the interval-valued fuzzy sets A^*, A and B have the following forms

$$A^* = \{(u_1, [x_{11}, x_{12}]), (u_2, [x_{21}, x_{22}]), \ldots, (u_n, [x_{n1}, x_{n2}])\},$$
$$A = \{(u_1, [y_{11}, y_{12}]), (u_2, [y_{21}, y_{22}]), \ldots, (u_n, [y_{n1}, y_{n2}])\},$$
$$B = \{(v_1, [z_{11}, z_{12}]), (v_2, [z_{21}, z_{22}]), \ldots, (v_m, [z_{m1}, z_{m2}])\},$$

where $0 \leqslant x_{i1} \leqslant x_{i2} \leqslant 1$, $0 \leqslant y_{i1} \leqslant y_{i2} \leqslant 1$, $1 \leqslant i \leqslant n$, $0 \leqslant z_{j1} \leqslant z_{j2} \leqslant 1$, and $1 \leqslant j \leqslant m$. Then, based on the matching function M presented in [10], we can calculate the degree of matching between the interval-valued fuzzy sets A^* and A. Assume that $M(A, A^*) = k$, where $k \in [0, 1]$ and M is the matching function between A^* and A, then the deduced consequence of the rule is "Y is B^*", where the membership function of the interval-valued fuzzy set B^* is as follows:

$$B^* = \{(v_1, [w_{11}, w_{12}]), (v_2, [w_{21}, w_{22}]), \ldots, (v_m, [w_{m1}, w_{m2}])\}, \tag{3}$$

where $w_{i1} = k^* z_{i1}$, $w_{i2} = k^* z_{i2}$, and $1 \leqslant i \leqslant m$.

However, there is a drawback in the above reasoning scheme, i.e., when $A^* =$ very A or when $A^* =$ more or less A, the method presented in [10] cannot deal with the approximate reasoning properly due to the fact that the deduced interval-valued fuzzy set B^* presented in [10] is always a linear modification of the interval-valued fuzzy set B described above (i.e., when $A^* =$ very A, we can see that $B^* \neq$ very B; when $A^* =$ more or less A, we can see that $B^* \neq$ more or less B). Furthermore, let us consider the following reasoning scheme:

Rule: IF X is A THEN Y is B
Fact: Y is B^*

Consequence: X is A^*

We can see that the method presented in [10] also cannot handle the approximate reasoning properly, for example, when $B^* =$ very B, we can see that $A^* \neq$ very A; when $B^* =$ more or less B, we can see that $A^* \neq$ more or less A. Thus, it is necessary to develop a more powerful method to deal with bidirectional approximate reasoning using interval-valued fuzzy sets to overcome the drawbacks of the one presented in [10].

In this paper, we extend the work of [10] to develop a new method for bidirectional approximate reasoning based on interval-valued fuzzy sets to overcome the drawbacks of the one presented in [10]. Because the proposed method can perform bidirectional approximate reasoning based on the direction of matching between interval-valued fuzzy sets, it is more reasonable and powerful than the one presented in [10]. It can provide a useful way to deal with bidirectional approximate reasoning for rule-based systems using interval-valued fuzzy sets.

The rest of this paper is organized as follows. In Section 2, we briefly review some similarity measures between interval-valued fuzzy sets. Furthermore, we also present a method to measure the direction of matching between interval-valued fuzzy sets. In Section 3, we present a new method for bidirectional approximate reasoning based on the direction of matching between interval-valued fuzzy sets. In Section 4, we use some examples to illustrate the approximate reasoning process. The conclusions are discussed in Section 5.

2. Similarity measures between interval-valued fuzzy sets

In 1986, Turksen has proposed the definitions of interval-valued fuzzy sets [17]. In [13,14], Gorzalczany presented interval-valued fuzzy inference methods based on interval-valued fuzzy sets. If a fuzzy set is represented by an interval-valued membership function, then it is called an interval-valued fuzzy set. In [8] we have presented a method for handling multicriteria fuzzy decision-making problems using interval-valued fuzzy sets.

Definition 2.1. Let U be the universe of discourse, $U = \{u_1, u_2, \ldots, u_n\}$. An interval-valued fuzzy set A of the universe of discourse U can be represented by

$$A = \{(u_1, [a_{11}, a_{12}]), (u_2, [a_{21}, a_{22}]), \ldots, (u_n, [a_{n1}, a_{n2}])\},$$

where interval $[a_{i1}, a_{i2}]$ indicating the grade of membership of u_i in the interval-valued fuzzy set A is between a_{i1} and a_{i2}, where $0 \leqslant a_{i1} \leqslant a_{i2} \leqslant 1$ and $1 \leqslant i \leqslant n$.

Let A and B be two interval-valued fuzzy sets of the universe of discourse U, where

$$\begin{aligned} U &= \{u_1, u_2, \ldots, u_n\}, \\ A &= \{(u_1, [a_{11}, a_{12}]), (u_2, [a_{21}, a_{22}]), \ldots, (u_n, [a_{n1}, a_{n2}])\}, \\ &= \{(u_i, [a_{i1}, a_{i2}]) \mid 1 \leqslant i \leqslant n\}, \\ B &= \{(u_1, [b_{11}, b_{12}]), (u_2, [b_{21}, b_{22}]), \ldots, (u_n, [b_{n1}, b_{n2}])\}, \\ &= \{(u_i, [b_{i1}, b_{i2}]) \mid 1 \leqslant i \leqslant n\}. \end{aligned}$$

If $\forall i,\ a_{i1} = b_{i1}$ and $a_{i2} = b_{i2}$, where $1 \leqslant i \leqslant n$, then the interval-valued fuzzy sets A and B are called equal (i.e., $A = B$). The union operation between the interval-valued fuzzy sets A and B is defined as follows:

$$A \cup B = \{(u_i, [c_{i1}, c_{i2}]) \mid c_{i1} = \mathrm{Max}(a_{i1}, b_{i1}), c_{i2} = \mathrm{Max}(a_{i2}, b_{i2}) \text{ and } 1 \leqslant i \leqslant n\}.$$

Let f_A be the membership function of the interval-valued fuzzy set A, where $f_A(u_i) = [a_{i1}, a_{i2}],\ 0 \leqslant a_{i1} \leqslant a_{i2} \leqslant 1$, and $1 \leqslant i \leqslant n$. The support $\mathrm{Supp}(A)$ of the interval-valued fuzzy set A is a subset of the universe of discourse U defined as

$$\mathrm{Supp}(A) = \{u_i \mid f_A(u_i) = [a_{i1}, a_{i2}],\ a_{i2} > 0 \text{ and } 1 \leqslant i \leqslant n\}.$$

Let U be the universe of discourse, $U = \{u_1, u_2, \ldots, u_n\}$, and let A be an interval-valued fuzzy set of the universe of discourse U, where

$$A = \{(u_1, [a_{11}, a_{12}]), (u_2, [a_{21}, a_{22}]), \ldots, (u_n, [a_{n1}, a_{n2}])\}.$$

Then, the interval-valued fuzzy sets "very A" and "more or less A" are defined as follows:

$$\text{very } A = \{(u_1, [a_{11}^2,\ a_{12}^2]), (u_2, [a_{21}^2,\ a_{22}^2]), \ldots, (u_n, [a_{n1}^2,\ a_{n2}^2])\},$$

$$\text{more or less } A = \{(u_1, [a_{11}^{1/2},\ a_{12}^{1/2}]), (u_2, [a_{21}^{1/2},\ a_{22}^{1/2}]), \ldots, (u_n, [a_{n1}^{1/2},\ a_{n2}^{1/2}])\}.$$

In [15], Ke et al. have presented a similarity function S to measure the degree of similarity between two vectors. In [4], we have used the similarity function S to develop a method for handling fuzzy decision-making problems. In [10], we presented a matching function M to measure the degree of similarity between interval-valued fuzzy sets based on the similarity function S. In this paper, we present the definition of the direction of matching between interval-valued fuzzy sets. The definition of the similarity function S is reviewed as follows:

Definition 2.2. Let $\overline{a}$ and $\overline{b}$ be two vectors in R^n, where R is a set of real numbers between zero and one, i.e.,

$$\begin{aligned} \overline{a} &= \langle a_1, a_2, \ldots, a_n \rangle, \\ \overline{b} &= \langle b_1, b_2, \ldots, b_n \rangle, \end{aligned}$$

where $a_i \in [0, 1],\ b_i \in [0, 1]$, and $1 \leqslant i \leqslant n$. Then, the degree of similarity between the vectors $\overline{a}$ and $\overline{b}$ can be measured by the similarity function S,

$$S(\overline{a}, \overline{b}) = \frac{\overline{a} \cdot \overline{b}}{\mathrm{Max}(\overline{a} \cdot \overline{a}, \overline{b} \cdot \overline{b})}, \tag{4}$$

where $S(\overline{a},\ \overline{b}) \in [0, 1]$. The larger the value of $S(\overline{a},\ \overline{b})$, the more the similarity between the vectors $\overline{a}$ and $\overline{b}$.

In the following, we introduce the matching function M [10] to measure the degree of matching between interval-valued fuzzy sets based on the similarity function S. Let U be the universe of discourse, $U=\{u_1,u_2,\ldots,u_n\}$, and let A be an interval-valued fuzzy set of U,

$$\begin{aligned}A&=\{(u_1,[a_{11},a_{12}]),(u_2,[a_{21},a_{22}]),\ldots,(u_n,[a_{n1},a_{n2}])\}\\&=\{(u_i,[a_{i1},a_{i2}])\mid 1\leqslant i\leqslant n\},\end{aligned}$$

then the lower bound and the upper bound of the interval-valued fuzzy set A can be represented by the subscript vector $\overline{A}$ and the superscript vector $\overline{\overline{A}}$, respectively, where

$$\begin{aligned}\overline{A}&=\langle a_{11},a_{21},\ldots,a_{n1}\rangle,\\\overline{\overline{A}}&=\langle a_{12},a_{22},\ldots,a_{n2}\rangle.\end{aligned}$$

The degree of similarity between interval-valued fuzzy sets can be measured by the matching function M [10] reviewed as follows. Let U be the universe of discourse, $U=\{u_1,u_2,\ldots,u_n\}$, and let A and B be two interval-valued fuzzy sets of U, where

$$\begin{aligned}A&=\{(u_1,[a_{11},a_{12}]),(u_2,[a_{21},a_{22}]),\ldots,(u_n,[a_{n1},a_{n2}])\}\\&=\{(u_i,[a_{i1},a_{i2}])\mid 1\leqslant i\leqslant n\},\end{aligned}$$

$$\begin{aligned}B&=\{(u_1,[b_{11},b_{12}]),(u_2,[b_{21},b_{22}]),\ldots,(u_n,[b_{n1},b_{n2}])\}\\&=\{(u_i,[b_{i1},b_{i2}])\mid 1\leqslant i\leqslant n\}.\end{aligned}$$

The lower bound and the upper bound of the interval-valued fuzzy set A can be represented by the subscriptor vector $\overline{A}$ and the superscriptor vector $\overline{\overline{A}}$, respectively, the lower bound and the upper bound of the interval-valued fuzzy set B can be represented by the subscriptor vector $\overline{B}$ and the superscriptor vector $\overline{\overline{B}}$, respectively, where

$$\begin{aligned}\overline{A}&=\langle a_{11},a_{21},\ldots,a_{n1}\rangle,\\\overline{\overline{A}}&=\langle a_{12},a_{22},\ldots,a_{n2}\rangle,\\\overline{B}&=\langle b_{11},b_{21},\ldots,b_{n1}\rangle,\\\overline{\overline{B}}&=\langle b_{12},b_{22},\ldots,b_{n2}\rangle.\end{aligned}$$

then the degree of matching $M(A,B)$ between the interval-valued fuzzy sets A and B can be measured as follows:

$$M(A,B)=\frac{S(\overline{A},\overline{B})+S(\overline{\overline{A}},\overline{\overline{B}})}{2},\tag{5}$$

where $M(A,B)\in[0,1]$. The larger the value of $M(A,B)$, the higher the degree of matching between the interval-valued fuzzy sets A and B.

In the following, we present the direction-matching function D between the interval-valued fuzzy sets A and B, where

$$D(A,B)=\sum_{i=1}^{n}[(a_{i1}-b_{i1})+(a_{i2}-b_{i2})].\tag{6}$$

If $D(A,B)\geqslant 0$, then the direction of matching from A to B is positive. Otherwise, the direction of matching from A to B is negative.

In the next section, we will use the direction-matching function D to develop a new bidirectional approximate reasoning method for rule-based systems using interval-valued fuzzy sets.

3. Bidirectional approximate reasoning using interval-valued fuzzy sets

Let us consider the following generalized modus ponens (GMP):

Rule: IF X is A THEN Y is B
Fact: X is A^*

Consequence: Y is B^*

where X and Y are linguistic variables, A^* and A are interval-valued fuzzy sets of the universe of discourse U, $U=\{u_1,u_2,\ldots,u_n\}$, and B^* and B are interval-valued fuzzy sets of the universe of discourse V, $V=\{v_1,v_2,\ldots,v_m\}$. Assume that the interval-valued fuzzy sets A^*, A, and B have the following forms

$$A^*=\{(u_1,[x_{11},x_{12}]),(u_2,[x_{21},x_{22}]),\ldots,(u_n,[x_{n1},x_{n2}])\},$$
$$A=\{(u_1,[y_{11},y_{12}]),(u_2,[y_{21},y_{22}]),\ldots,(u_n,[y_{n1},y_{n2}])\},$$
$$B=\{(v_1,[z_{11},z_{12}]),(v_2,[z_{21},z_{22}]),\ldots,(v_m,[z_{m1},z_{m2}])\},$$

where $0\leqslant x_{i1}\leqslant x_{i2}\leqslant 1$, $0\leqslant y_{i1}\leqslant y_{i2}\leqslant 1$, $1\leqslant i\leqslant n$, $0\leqslant z_{j1}\leqslant z_{j2}\leqslant 1$, and $1\leqslant j\leqslant m$. Let $\overline{A^*}$ and $\overline{A}$ be the subscript vectors of the interval-valued fuzzy sets A^* and A, respectively, and let $\overline{\overline{A^*}}$ and $\overline{\overline{A}}$ be the superscript vectors of the interval-valued fuzzy sets A^* and A, respectively, where

$$\overline{A^*}=\langle x_{11},x_{21},\ldots,x_{n1}\rangle,$$
$$\overline{A}=\langle y_{11},y_{21},\ldots,y_{n1}\rangle,$$
$$\overline{\overline{A^*}}=\langle x_{12},x_{22},\ldots,x_{n2}\rangle,$$
$$\overline{\overline{A}}=\langle y_{12},y_{22},\ldots,y_{n2}\rangle.$$

Then, based on formula (5), the degree of matching between the interval-valued fuzzy sets A^* and A can be measured. Furthermore, the direction of matching from A^* to A can be decided by formula (6). If $D(A^*,A)\geqslant 0$, then the direction of matching from A^* to A is positive. Otherwise, the direction of matching from A^* to A is negative. Assume that $M(A^*,A)=k$, where $k\in[0,1]$, then the deduced consequence of the rule is "Y is B^*", where the membership function of the interval-valued fuzzy set B^* is as follows:

$$B^*=\{(v_1,[w_{11},w_{12}]),(v_2,[w_{21},w_{22}]),\ldots,(v_m,[w_{m1},w_{m2}])\}, \tag{8}$$

where w_{j1} and $w_{j2}, 1\leqslant j\leqslant m$, can be evaluated as follows:

Case 1: IF $\mathrm{Supp}(A^*)=\mathrm{Supp}(A)$ THEN

IF $A^*=$ very A (i.e. $x_{i1}=y_{i1}^2$, $x_{i2}=y_{i2}^2$ and $1\leqslant i\leqslant n$)

THEN let

$$w_{j1}=z_{j1}^2, \tag{9}$$

$$w_{j2}=z_{j2}^2 \quad \text{where } 1\leqslant j\leqslant m; \tag{10}$$

IF $A^*=$ more or less A (i.e., $x_{i1}=y_{i1}^{1/2}$, $x_{i2}=y_{i2}^{1/2}$ and $1\leqslant i\leqslant n$)

THEN let

$$w_{j1} = z_{j1}^{1/2}, \tag{11}$$

$$w_{j2} = z_{j2}^{1/2}, \tag{12}$$

where $1 \leqslant j \leqslant m$;
IF the direction of matching from A^* to A is positive THEN

$$w_{j1} = z_{j1}^{k}, \tag{13}$$

$$w_{j2} = z_{j2}^{k} \quad \text{where } M(A^*, A) = k,\ k \in [0,1] \quad \text{and} \quad 1 \leqslant j \leqslant m \tag{14}$$

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$$w_{j1} = z_{j1}^{1/k}, \tag{15}$$

$$w_{j2} = z_{j2}^{1/k} \quad \text{where } M(A^*, A) = k,\ k \in [0,1] \quad \text{and} \quad 1 \leqslant j \leqslant m. \tag{16}$$

Case 2: IF $\text{Supp}(A^*) \neq \text{Supp}(A)$ THEN
IF $M(A^*, A) \geqslant 0.5$ THEN
IF the direction of matching from A^* to A is positive THEN

$$w_{j1} = z_{j1}^{k}, \tag{17}$$

$$w_{j2} = z_{j2}^{k} \quad \text{where } M(A^*, A) = k,\ k \in [0,1] \quad \text{and} \quad 1 \leqslant j \leqslant m \tag{18}$$

ELSE

$$w_{j1} = z_{j1}^{1/k}, \tag{19}$$

$$w_{j2} = z_{j2}^{1/k} \quad \text{where } M(A^*, A) = k,\ k \in [0,1] \quad \text{and} \quad 1 \leqslant j \leqslant m \tag{20}$$

ELSE

$$w_{j1} = z_{j1} * k, \tag{21}$$

$$w_{j2} = z_{j2} * k \quad \text{where } M(A^*, A) = k,\ k \in [0,1] \quad \text{and} \quad 1 \leqslant j \leqslant m. \tag{22}$$

It is obvious that if A^* and A are identical interval-valued fuzzy sets (i.e., $A^* = A$), then $M(A^*, A) = 1$ and B^* is equal to B.

Furthermore, consider the following single-input–single-output (SISO) approximate reasoning scheme:

R_1: IF X is A_1 THEN Y is B_1
R_2: IF X is A_2 THEN Y is B_2
$\vdots$
R_p: IF X is A_p THEN Y is B_p
Fact: X is A_0

Consequence: Y is B_0

where $A_0, A_1, A_2, \ldots, A_p$ are interval-valued fuzzy sets of the universe of discourse U, $U = \{u_1, u_2, \ldots, u_n\}$, and $B_1, B_2, \ldots$ and B_p are interval-valued fuzzy sets of the universe of discourse V, $V = \{v_1, v_2, \ldots, v_m\}$. Assume that

$$A_i = \{(u_1, [x_{i1}, x_{i1}^*]), (u_2, [x_{i2}, x_{i2}^*]), \ldots, (u_n, [x_{in}, x_{in}^*])\},$$
$$B_j = \{(v_1, [y_{j1}, y_{j1}^*]), (v_2, [y_{j2}, y_{j2}^*]), \ldots, (v_m, [y_{jm}, y_{jm}^*])\},$$

where $0 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant p$. Based on the previous discussions, the interval-valued fuzzy sets A_i can be represented by the subscript vectors $\overline{A_i}$ and the superscript vectors $\overline{\overline{A_i}}$, $0 \leqslant i \leqslant p$, where

$$\overline{A_0} = \langle x_{01}, x_{02}, \ldots, x_{0n} \rangle,$$
$$\overline{A_1} = \langle x_{11}, x_{12}, \ldots, x_{1n} \rangle,$$
$$\overline{A_2} = \langle x_{21}, x_{22}, \ldots, x_{2n} \rangle,$$
$$\vdots$$
$$\overline{A_p} = \langle x_{p1}, x_{p2}, \ldots, x_{pn} \rangle,$$
$$\overline{\overline{A_0}} = \langle x_{01}^*, x_{02}^*, \ldots, x_{0n}^* \rangle,$$
$$\overline{\overline{A_1}} = \langle x_{11}^*, x_{12}^*, \ldots, x_{1n}^* \rangle,$$
$$\overline{\overline{A_2}} = \langle x_{21}^*, x_{22}^*, \ldots, x_{2n}^* \rangle,$$
$$\vdots$$
$$\overline{\overline{A_p}} = \langle x_{p1}^*, x_{p2}^*, \ldots, x_{pn}^* \rangle.$$

Assume that $M(A_0, A_i) = k_i$, where $k_i \in [0, 1]$, and the direction of matching from A_0 to A_i can be decided by formula (6), then the deduced consequence of rule R_i is "Y is B_i^*", and the membership function of the interval-valued fuzzy set B_i^*, $1 \leqslant i \leqslant p$, is as follows:

$$B_i^* = \{(v_1, [w_{i1}, w_{i1}^*]), (v_2, [w_{i2}, w_{i2}^*]), \ldots, (v_m, [w_{im}, w_{im}^*])\}, \tag{23}$$

where w_{ij} and w_{ij}^*, $1 \leqslant j \leqslant m$, can be evaluated by the following two cases

Case 1: IF $\mathrm{Supp}(A_0) = \mathrm{Supp}(A_i)$ THEN

IF $A_0 =$ very A_i (i.e., $x_{0k} = x_{ik}^2$, $x_{0k}^* = x_{ik}^{*2}$ and $1 \leqslant k \leqslant n$)
THEN let

$$w_{ij} = y_{ij}^2, \tag{24}$$

$$w_{ij}^* = y_{ij}^{*2} \quad \text{where } 1 \leqslant j \leqslant m; \tag{25}$$

IF $A_0 =$ more or less A_i (i.e., $x_{0k} = x_{ik}^{1/2}$, $x_{0k}^* = x_{ik}^{*1/2}$ and $1 \leqslant k \leqslant n$)
THEN let

$$w_{ij} = y_{ij}^{1/2}, \tag{26}$$

$$w_{ij}^* = y_{ij}^{*1/2} \quad \text{where } 1 \leqslant j \leqslant m; \tag{27}$$

IF the direction of matching from A_0 to A_i is positive THEN

$$w_{ij} = y_{ij}^{k_i}, \tag{28}$$

$$w_{ij}^* = y_{ij}^{*k_i} \quad \text{where } M(A_0, A_i) = k_i,\ k_i \in [0,1] \quad \text{and} \quad 1 \leqslant j \leqslant m \tag{29}$$

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$$w_{ij} = y_{ij}^{1/k_i}, \tag{30}$$

$$w_{ij}^* = y_{ij}^{*1/k_i} \quad \text{where } M(A_0, A_i) = k_i,\ k_i \in [0,1] \quad \text{and} \quad 1 \leqslant j \leqslant m. \tag{31}$$

Case 2: IF $\text{Supp}(A_0) \neq \text{Supp}(A_i)$ THEN
IF $M(A_0, A_i) \geqslant 0.5$ THEN
IF the direction of matching from A^* to A is positive THEN

$$w_{ij} = y_{ij}^{k_i}, \tag{32}$$

$$w_{ij}^* = y_{ij}^{*k_i} \quad \text{where } M(A_0, A_i) = k_i,\ k_i \in [0,1] \quad \text{and} \quad 1 \leqslant j \leqslant m \tag{33}$$

ELSE

$$w_{ij} = y_{ij}^{1/k_i}, \tag{34}$$

$$w_{ij}^* = y_{ij}^{*1/k_i} \quad \text{where } M(A_0, A_i) = k_i,\ k_i \in [0,1] \quad \text{and} \quad 1 \leqslant j \leqslant m \tag{35}$$

ELSE

$$w_{ij} = y_{ij} * k_i, \tag{36}$$

$$w_{ij}^* = y_{ij}^* * k_i \quad \text{where } M(A_0, A_i) = k_i,\ k_i \in [0,1] \quad \text{and} \quad 1 \leqslant j \leqslant m. \tag{37}$$

Thus, the deduced consequence of the SISO interval-valued approximate reasoning scheme is "Y is B_0", where

$$B_0 = B_1^* \cup B_2^* \cup \cdots \cup B_p^*, \tag{38}$$

and "$\cup$" is the union operator of the interval-valued fuzzy sets.

Conversely, consider the following SISO interval-valued approximate reasoning scheme:

R_1: IF X is A_1 THEN Y is B_1
R_2: IF X is A_2 THEN Y is B_2
$\vdots$
R_p: IF X is A_p THEN Y is B_p
Fact: Y is B_0

Consequence: X is A_0

where

$$A_i = \{(u_1, [x_{i1}, x_{i1}^*]), (u_2, [x_{i2}, x_{i2}^*]), \ldots, (u_n, [x_{in}, x_{in}^*])\},$$
$$B_j = \{(v_1, [y_{j1}, y_{j1}^*]), (v_2, [y_{j2}, y_{j2}^*]), \ldots, (v_m, [y_{jm}, y_{jm}^*])\},$$

where $1 \leqslant i \leqslant p$ and $0 \leqslant j \leqslant p$. Based on the previous discussions, the interval-valued fuzzy sets B_j can be represented by the subscript vectors $\overline{B_j}$ and the superscript vectors $\overline{\overline{B_j}}$, $0 \leqslant j \leqslant p$, where

$$\begin{aligned}
&\overline{B_0} = \langle y_{01}, y_{02}, \ldots, y_{0m} \rangle,\\
&\overline{B_1} = \langle y_{11}, y_{12}, \ldots, y_{1m} \rangle,\\
&\overline{B_2} = \langle y_{21}, y_{22}, \ldots, y_{2m} \rangle,\\
&\quad\vdots\\
&\overline{B_p} = \langle y_{p1}, \ y_{p2}, \ldots, \ y_{pm} \rangle,\\
&\overline{\overline{B_0}} = \langle y^*_{01}, y^*_{02}, \ldots, y^*_{0m} \rangle,\\
&\overline{\overline{B_1}} = \langle y^*_{11}, y^*_{12}, \ldots, y^*_{1m} \rangle,\\
&\overline{\overline{B_2}} = \langle y^*_{21}, y^*_{22}, \ldots, y^*_{2m} \rangle,\\
&\quad\vdots\\
&\overline{\overline{B_p}} = \langle y^*_{p1}, y^*_{p2}, \ldots, y^*_{pm} \rangle.
\end{aligned}$$

Assume that $M(B_0, B_i) = k_i$, where $k_i \in [0,1]$, and the direction of matching from B_0 to B_i can be decided by formula (6), then the deduced consequence of rule R_i is "X is A_i^*", and the membership function of the interval-valued fuzzy set A_i^*, $1 \leqslant i \leqslant p$, is as follows:

$$A_i^* = \{(u_1, [r_{i1}, r^*_{i1}]), (u_2, [r_{i2}, r^*_{i2}]), \ldots, (u_n, [r_{in}, r^*_{in}])\}, \tag{39}$$

where r_{iw} and r^*_{iw}, $1 \leqslant w \leqslant n$, can be evaluated by the following two cases:

Case 1: IF $\mathrm{Supp}(B_0) = \mathrm{Supp}(B_i)$ THEN

IF $B_0 =$ very B_i (i.e., $y_{os} = y_{is}^2$, $y^*_{os} = y^{*2}_{is}$ and $1 \leqslant s \leqslant m$)
THEN let

$$r_{iw} = x_{iw}^2, \tag{40}$$

$$r^*_{iw} = x^{*2}_{iw} \quad \text{where } 1 \leqslant w \leqslant n; \tag{41}$$

IF $B_0 =$ more or less B_i (i.e., $y_{os} = y_{is}^{1/2}$, $y^*_{os} = y^{*1/2}_{is}$ and $1 \leqslant s \leqslant m$)
THEN let

$$r_{iw} = x_{iw}^{1/2}, \tag{42}$$

$$r^*_{iw} = x^{*1/2}_{iw} \quad \text{where } 1 \leqslant w \leqslant n; \tag{43}$$

IF the direction of matching from B_0 to B_i is positive THEN

$$r_{iw} = x_{iw}^{k_i}, \tag{44}$$

$$r^*_{iw} = x^{*k_i}_{iw} \quad \text{where } M(B_0, B_i) = k_i,\ k_i \in [0,1] \quad \text{and} \quad 1 \leqslant w \leqslant n \tag{45}$$

ELSE

$$r_{iw} = x_{iw}^{1/k_i}, \tag{46}$$

$$r^*_{iw} = x^{*1/k_i}_{iw} \quad \text{where } M(B_0, B_i) = k_i,\ k_i \in [0,1] \quad \text{and} \quad 1 \leqslant w \leqslant n. \tag{47}$$

Case 2: IF $\mathrm{Supp}(B_0) \neq \mathrm{Supp}(B_j)$ THEN

IF $M(B_0, B_j) \geqslant 0.5$ THEN
IF the direction of matching from B_0 to B_j is positive THEN

$$r_{iw} = x_{iw}^{k_i}, \tag{48}$$

$$r_{iw}^* = x_{iw}^{*k_i} \quad \text{where } M(B_0, B_i) = k_i,\ k_i \in [0,1] \quad \text{and} \quad 1 \leqslant w \leqslant n \tag{49}$$

ELSE

$$r_{iw} = x_{iw}^{1/k_i}, \tag{50}$$

$$r_{iw}^* = x_{iw}^{*1/k_i} \quad \text{where } M(B_0, B_i) = k_i,\ k_i \in [0,1] \quad \text{and} \quad 1 \leqslant w \leqslant n \tag{51}$$

ELSE

$$r_{iw} = x_{iw} * k_i, \tag{52}$$

$$r_{iw}^* = x_{iw}^* * k_i \quad \text{where } M(B_0, B_i) = k_i,\ k_i \in [0,1] \quad \text{and} \quad 1 \leqslant w \leqslant n. \tag{53}$$

Thus, the deduced consequence of the SISO interval-valued approximate reasoning scheme is "X is A_0", where

$$A_0 = A_1^* \cup A_2^* \cup \cdots \cup A_p^*, \tag{54}$$

and "$\cup$" is the union operator of the interval-valued fuzzy sets.

4. Examples

In this section, we use some examples to illustrate the bidirectional approximate reasoning process based on the direction of matching between interval-valued fuzzy sets.

Example 4.1. Consider the following single-input–single-output interval-valued approximate reasoning scheme

R_1: IF X is A_1 THEN Y is B_1
R_2: IF X is A_2 THEN Y is B_2
R_3: IF X is A_3 THEN Y is B_3
R_4: IF X is A_4 THEN Y is B_4
R_5: IF X is A_5 THEN Y is B_5
Fact: X is A_0

Consequence: Y is B_0

where A_0, A_1, $A_2, \ldots$, and A_5 are interval-valued fuzzy sets of the universe of discourse U, $U = \{u_1, u_2, \ldots, u_{14}\}$, and B_0, B_1, $B_2, \ldots$, and B_5 are interval-valued fuzzy sets of the universe of discourse V, $V = \{v_1, v_2, \ldots, v_{14}\}$. These interval-valued fuzzy sets are shown as follows:

$$\begin{aligned} A_0 = \{&(u_1,[0,0]),(u_2,[0,0]),(u_3,[0.90,0.95]),(u_4,[1,1]),(u_5,[0.90,0.95]),(u_6,[0,0.8]),\\ &(u_7,[0,0]),(u_8,[0,0]),(u_9,[0,0]),(u_{10},[0,0]),(u_{11},[0,0]),(u_{12},[0,0]),\\ &(u_{13},[0,0]),(u_{14},[0,0])\}, \end{aligned}$$

$$\begin{aligned} A_1 = \{&(u_1,[1,1]),(u_2,[1,1]),(u_3,[0.82,0.95]),(u_4,[0,0.7]),(u_5,[0,0]),(u_6,[0,0]),\\ &(u_7,[0,0]),(u_8,[0,0]),(u_9,[0,0]),(u_{10},[0,0]),(u_{11},[0,0]),(u_{12},[0,0]),\\ &(u_{13},[0,0]),(u_{14},[0,0])\}, \end{aligned}$$

$$A_2 = \{(u_1,[0,0]),(u_2,[0,0]),(u_3,[0,0.5]),(u_4,[0.75,0.8]),(u_5,[0.94,0.95]),(u_6,[1,1]),$$
$$(u_7,[0.94,0.95]),(u_8,[0.75,0.83]),(u_9,[0,0.5]),(u_{10},[0,0]),(u_{11},[0,0]),$$
$$(u_{12},[0,0]),(u_{13},[0,0]),(u_{14},[0,0])\},$$
$$A_3 = \{(u_1,[0,0]),(u_2,[0,0]),(u_3,[0,0]),(u_4,[0,0]),(u_5,[0,0]),(u_6,[0,0]),(u_7,[0,0.6]),$$
$$(u_8,[0.87,0.92]),(u_9,[1,1]),(u_{10},[0.87,0.92]),(u_{11},[0,0.6]),$$
$$(u_{12},[0,0]),(u_{13},[0,0]),(u_{14},[0,0])\},$$
$$A_4 = \{(u_1,[0,0]),(u_2,[0,0]),(u_3,[0,0]),(u_4,[0,0]),(u_5,[0,0]),(u_6,[0,0]),(u_7,[0,0]),$$
$$(u_8,[0,0]),(u_9,[0,0.6]),(u_{10},[0.87,0.92]),(u_{11},[1,1]),$$
$$(u_{12},[0.87,0.92]),(u_{13},[0,0.6]),(u_{14},[0,0])\},$$
$$A_5 = \{(u_1,[0,0]),(u_2,[0,0]),(u_3,[0,0]),(u_4,[0,0]),(u_5,[0,0]),(u_6,[0,0]),(u_7,[0,0]),$$
$$(u_8,[0,0]),(u_9,[0,0]),(u_{10},[0,0]),(u_{11},[0,0]),(u_{12},[0,0.6]),$$
$$(u_{13},[0.87,0.92]),(u_{14},[1,1])\},$$
$$B_1 = \{(v_1,[1,1]),(v_2,[0.94,0.96]),(v_3,[0,0.65]),(v_4,[0,0]),(v_5,[0,0]),$$
$$(v_6,[0,0]),(v_7,[0,0]),(v_8,[0,0]),(v_9,[0,0]),(v_{10},[0,0]),(v_{11},[0,0]),(v_{12},[0,0]),$$
$$(v_{13},[0,0]),(v_{14},[0,0])\},$$
$$B_2 = \{(v_1,[0,0]),(v_2,[0,0.6]),(v_3,[0.87,0.92]),(v_4,[1,1]),(v_5,[0.87,0.92]),$$
$$(v_6,[0,0.6]),(v_7,[0,0]),(v_8,[0,0]),(v_9,[0,0]),(v_{10},[0,0]),(v_{11},[0,0]),$$
$$(v_{12},[0,0]),(v_{13},[0,0]),(v_{14},[0,0])\},$$
$$B_3 = \{(v_1,[0,0]),(v_2,[0,0]),(v_3,[0,0]),(v_4,[0,0.5]),(v_5,[0.74,0.82]),$$
$$(v_6,[0.94,0.95]),(v_7,[1,1]),(v_8,[0.94,0.95]),(v_9,[0.74,0.82]),(v_{10},[0,0.5]),$$
$$(v_{11},[0,0]),(v_{12},[0,0]),(v_{13},[0,0]),(v_{14},[0,0])\},$$
$$B_4 = \{(v_1,[0,0]),(v_2,[0,0]),(v_3,[0,0]),(v_4,[0,0]),(v_5,[0,0]),(v_6,[0,0]),$$
$$(v_7,[0,0.5]),(v_8,[0.74,0.82]),(v_9,[0.94,0.95]),(v_{10},[1,1]),(v_{11},[0.94,0.95]),$$
$$(v_{12},[0.74,0.82]),(v_{13},[0,0.5]),(v_{14},[0,0])\},$$
$$B_5 = \{(v_1,[0,0]),(v_2,[0,0]),(v_3,[0,0]),(v_4,[0,0]),(v_5,[0,0]),(v_6,[0,0]),$$
$$(v_7,[0,0]),(v_8,[0,0]),(v_9,[0,0]),(v_{10},[0,0]),(v_{11},[0,0.6]),(v_{12},[0.87,0.92]),$$
$$(v_{13},[1,1]),(v_{14},[1,1])\}.$$

The membership function curves of these interval-valued fuzzy sets are shown in Fig. 1. The interval-valued fuzzy sets A_i can be represented by the subscript vectors $\overline{A_i}$ and the superscript vectors $\overline{\overline{A_i}}$, $0 \leqslant i \leqslant 5$, where

$$\overline{A_0} = \langle 0,0,0.90,1,0.90,0,0,0,0,0,0,0,0,0\rangle,$$
$$\overline{A_1} = \langle 1,1,0.82,0,0,0,0,0,0,0,0,0,0,0\rangle,$$
$$\overline{A_2} = \langle 0,0,0,0.75,0.94,1,0.94,0.75,0,0,0,0,0,0\rangle,$$
$$\overline{A_3} = \langle 0,0,0,0,0,0,0,0.87,1,0.87,0,0,0,0\rangle,$$

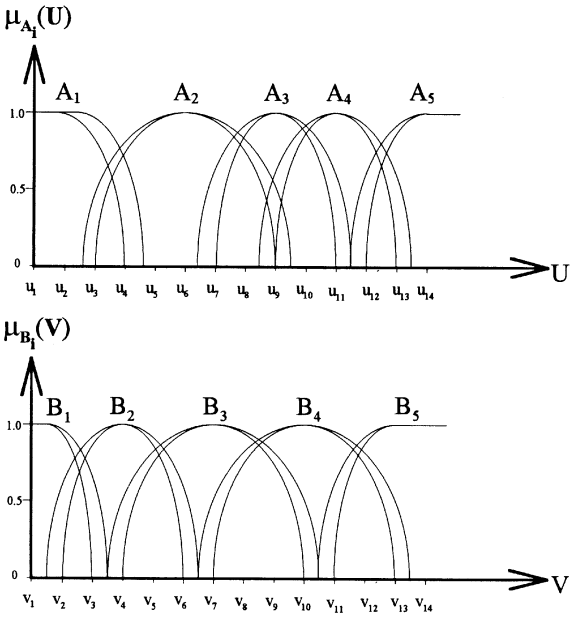

Fig. 1. The membership functions of A_i and B_i, $i = 1, 2, \ldots, 5$.

$$\overline{A_4} = \langle 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.87, 1, 0.87, 0, 0\rangle,$$
$$\overline{A_5} = \langle 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.87, 1\rangle,$$
$$\overline{\overline{A_0}} = \langle 0, 0, 0.95, 1, 0.95, 0.8, 0, 0, 0, 0, 0, 0, 0, 0\rangle,$$
$$\overline{\overline{A_1}} = \langle 1, 1, 0.95, 0.7, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\rangle,$$
$$\overline{\overline{A_2}} = \langle 0, 0, 0.5, 0.8, 0.95, 1, 0.95, 0.83, 0.5, 0, 0, 0, 0, 0\rangle,$$
$$\overline{\overline{A_3}} = \langle 0, 0, 0, 0, 0, 0, 0.6, 0.92, 1, 0.92, 0.6, 0, 0, 0\rangle,$$
$$\overline{\overline{A_4}} = \langle 0, 0, 0, 0, 0, 0, 0, 0, 0.6, 0.92, 1, 0.92, 0.6, 0\rangle,$$
$$\overline{\overline{A_5}} = \langle 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.6, 0.92, 1\rangle,$$

and the interval-valued fuzzy sets B_1, B_2, B_3, B_4, B_5 can also be represented by the subscript vectors $\overline{B_1}$, $\overline{B_2}$, $\overline{B_3}$, $\overline{B_4}$, $\overline{B_5}$ and the superscript vectors $\overline{\overline{B_1}}$, $\overline{\overline{B_2}}$, $\overline{\overline{B_3}}$, $\overline{\overline{B_4}}$, $\overline{\overline{B_5}}$, respectively, where

$$\overline{B_1} = \langle 1, 0.94, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\rangle,$$
$$\overline{B_2} = \langle 0, 0, 0.87, 1, 0.87, 0, 0, 0, 0, 0, 0, 0, 0, 0\rangle,$$
$$\overline{B_3} = \langle 0, 0, 0, 0, 0.74, 0.94, 1, 0.94, 0.74, 0, 0, 0, 0, 0\rangle,$$
$$\overline{B_4} = \langle 0, 0, 0, 0, 0, 0, 0, 0.74, 0.94, 1, 0.94, 0.74, 0, 0\rangle,$$
$$\overline{B_5} = \langle 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.87, 1, 1\rangle,$$

$$\overline{\overline{B_1}} = \langle 1, 0.96, 0.65, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\rangle,$$

$$\overline{\overline{B_2}} = \langle 0, 0.6, 0.92, 1, 0.92, 0.6, 0, 0, 0, 0, 0, 0, 0, 0\rangle,$$

$$\overline{\overline{B_3}} = \langle 0, 0, 0, 0.5, 0.82, 0.95, 1, 0.95, 0.82, 0.5, 0, 0, 0, 0\rangle,$$

$$\overline{\overline{B_4}} = \langle 0, 0, 0, 0, 0, 0, 0.5, 0.82, 0.95, 1, 0.95, 0.82, 0.5, 0\rangle,$$

$$\overline{\overline{B_5}} = \langle 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.6, 0.92, 1, 1\rangle,$$

then

(i) Because $k_1 = M(A_0, A_1) = 0.47$ and because $\text{Supp}(A_0) \neq \text{Supp}(A_1)$, by formulas (36)–(37), we can get

$$\begin{aligned} B_1^* = \{&(v_1, [0.47, 0.47]), (v_2, [0.44, 0.45]), (v_3, [0, 0.3]), (v_4, [0, 0]), (v_5, [0, 0]),\\ &(v_6, [0, 0]), (v_7, [0, 0]), (v_8, [0, 0]), (v_9, [0, 0]), (v_{10}, [0, 0]), (v_{11}, [0, 0]),\\ &(v_{12}, [0, 0]), (v_{13}, [0, 0]), (v_{14}, [0, 0])\}. \end{aligned}$$

(ii) Because $k_2 = M(A_0, A_2) = 0.53$, and by formula (6) we can see that the direction of matching from A_0 to A_2 is negative, and because $\text{Supp}(A_0) \neq \text{Supp}(A_2)$, by formulas (34)–(35), we can get

$$\begin{aligned} B_2^* = \{&(v_1, [0, 0]), (v_2, [0, 0.38]), (v_3, [0.77, 0.85]), (v_4, [1, 1]), (v_5, [0.77, 0.85]),\\ &(v_6, [0, 0.38]), (v_7, [0, 0]), (v_8, [0, 0]), (v_9, [0, 0]), (v_{10}, [0, 0]), (v_{11}, [0, 0]), (v_{12}, [0, 0]),\\ &(v_{13}, [0, 0]), (v_{14}, [0, 0])\}. \end{aligned}$$

(iii) Because $k_3 = M(A_0, A_3) = 0$ and $\text{Supp}(A_0) \neq \text{Supp}(A_3)$, by formulas (36)–(37), we can get

$$B_3^* = \{(v_i, [0, 0]) \mid 1 \leqslant i \leqslant 14\}.$$

(iv) Because $k_4 = M(A_0,\ A_4) = 0$ and $\text{Supp}(A_0) \neq \text{Supp}(A_4)$, by formulas (36)–(37), we can get

$$B_4^* = \{(v_i, [0, 0]) \mid 1 \leqslant i \leqslant 14\}.$$

(v) Because $k_5 = M(A_0, A_5) = 0$ and $\text{Supp}(A_0) \neq \text{Supp}(A_5)$, by formulas (36)–(37), we can get

$$B_5^* = \{(v_i, [0, 0]) \mid 1 \leqslant i \leqslant 14\}.$$

Finally, we can get the deduced consequence "Y is B_0" of the SISO interval-valued approximate reasoning scheme, where

$$\begin{aligned} B_0 &= B_1^* \cup B_2^* \cup B_3^* \cup B_4^* \cup B_5^*\\ &= \{(v_1, [0.47, 0.47]), (v_2, [0.44, 0.45]), (v_3, [0.77, 0.85]),\\ &\quad (v_4, [1, 1]), (v_5, [0.77, 0.85]), (v_6, [0, 0.38]), (v_7, [0, 0]), (v_8, [0, 0]), (v_9, [0, 0]), (v_{10}, [0, 0]), (v_{11}, [0, 0]),\\ &\quad (v_{12}, [0, 0]), (v_{13}, [0, 0]), (v_{14}, [0, 0])\}. \end{aligned}$$

The reasoning result is shown in Fig. 2.

Example 4.2. Consider the single-input–single-output approximate reasoning scheme as shown in Example 4.1. Assume that given "$A_0 =$ more or less A_3", where

$$\begin{aligned} A_0 = \{&(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0, 0]), (u_4, [0, 0]), (u_5, [0, 0]), (u_6, [0, 0]), (u_7, [0, 0.775]),\\ &(u_8, [0.933, 0.959]), (u_9, [1, 1]), (u_{10}, [0.933, 0.959]), (u_{11}, [0, 0.775]),\\ &(u_{12}, [0, 0]), (u_{13}, [0, 0]), (u_{14}, [0, 0])\}, \end{aligned}$$

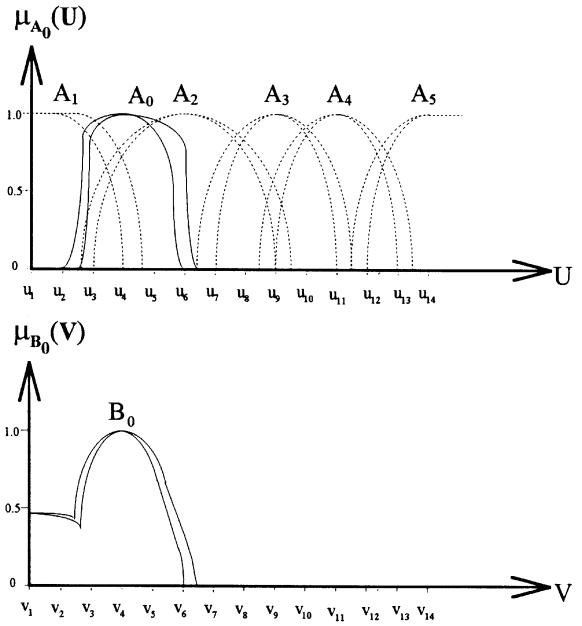

Fig. 2. The reasoning result of Example 4.1.

then

(i) Because $k_1 = M(A_0, A_1) = 0$ and $\text{Supp}(A_0) \neq \text{Supp}(A_1)$, from formulas (36)–(37), we can get

$$B_1^* = \{(v_i, [0,0]) \mid 1 \leqslant i \leqslant 14\}.$$

(ii) Because $k_2 = M(A_0, A_2) = 0.3$ and because $\text{Supp}(A_0) \neq \text{Supp}(A_2)$, from formulas (36)–(37), we can get

$$\begin{aligned} B_2^* = \{&(v_1, [0,0]), (v_2, [0, 0.18]), (v_3, [0.26, 0.28]), (v_4, [0.3, 0.3]), (v_5, [0.26, 0.28]), \\ &(v_6, [0, 0.18]), (v_7, [0,0]), (v_8, [0,0]), (v_9, [0,0]), (v_{10}, [0,0]), \\ &(v_{11}, [0,0]), (v_{12}, [0,0]), (v_{13}, [0,0]), (v_{14}, [0,0])\}. \end{aligned}$$

(iii) Because $\text{Supp}(A_0) = \text{Supp}(A_3)$ and because $A_0 =$ more or less A_3, from formulas (26)–(27), we can get

$$\begin{aligned} B_3^* = \{&(v_1, [0,0]), (v_2, [0,0]), (v_3, [0,0]), (v_4, [0, 0.707]), (v_5, [0.860, 0.906]), \\ &(v_6, [0.970, 0.975]), (v_7, [1,1]), (v_8, [0.970, 0.975]), (v_9, [0.860, 0.906]), \\ &(v_{10}, [0, 0.707]), (v_{11}, [0,0]), (v_{12}, [0,0]), (v_{13}, [0,0]), (v_{14}, [0,0])\}. \end{aligned}$$

(iv) Because $k_4 = M(A_0, A_4) = 0.43$ and because $\text{Supp}(A_0) \neq \text{Supp}(A_4)$, from formulas (36)–(37), we can get

$$\begin{aligned} B_4^* = \{&(v_1, [0,0]), (v_2, [0,0]), (v_3, [0,0]), (v_4, [0,0]), (v_5, [0,0]), (v_6, [0,0]), (v_7, [0, 0.22]), \\ &(v_8, [0.32, 0.35]), (v_9, [0.40, 0.41]), (v_{10}, [0.43, 0.43]), (v_{11}, [0.40, 0.41]), \\ &(v_{12}, [0.32, 0.35]), (v_{13}, [0, 0.22]), (v_{14}, [0,0])\}. \end{aligned}$$

(v) Because $k_5 = M(A_0, A_5) = 0$ and $\text{Supp}(A_0) \neq \text{Supp}(A_5)$, from formulas (36)–(37), we can get

$$B_5^* = \{(v_i, [0,0]) \mid 1 \leqslant i \leqslant 14\}.$$

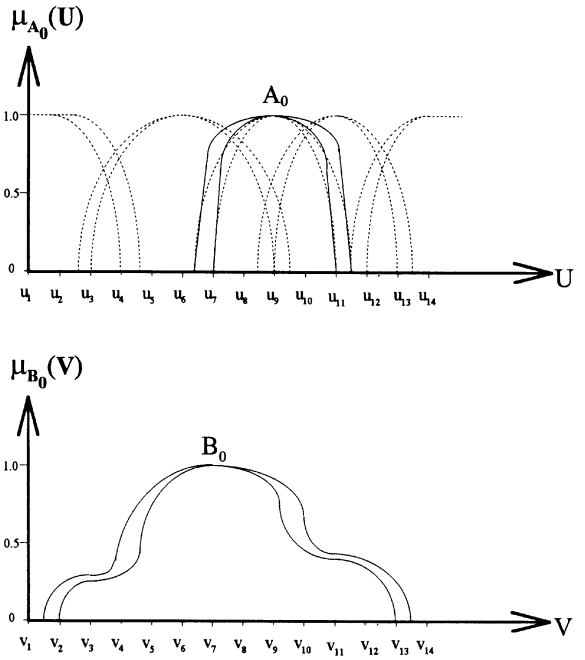

Fig. 3. The reasoning result of Example 4.2.

Finally, based on formula (38), we can get the deduced consequence "Y is B_0" of the SISO interval-valued approximate reasoning scheme, where

$$\begin{aligned} B_0 &= B_1^* \cup B_2^* \cup B_3^* \cup B_4^* \cup B_5^* \\ &= \{(v_1,[0,0]),(v_2,[0,0.18]),(v_3,[0.26,0.28]),(v_4,[0.3,0.707]),(v_5,[0.860,0.906]), \\ &\quad (v_6,[0.970,0.975]),(v_7,[1,1]),(v_8,[0.970,0.975]),(v_9,[0.860,0.906]), \\ &\quad (v_{10},[0.43,0.707]),(v_{11},[0.40,0.41]),(v_{12},[0.32,0.35]),(v_{13},[0,0.22]),(v_{14},[0,0])\}. \end{aligned}$$

The reasoning result is shown in Fig. 3.

Example 4.3. Consider the following single-input–single-output (SISO) approximate reasoning scheme:

R_1: IF X is A_1 THEN Y is B_1
R_2: IF X is A_2 THEN Y is B_2
R_3: IF X is A_3 THEN Y is B_3
R_4: IF X is A_4 THEN Y is B_4
R_5: IF X is A_5 THEN Y is B_5
Fact: Y is B_0

Consequence: X is A_0

where X and Y are linguistic variables, A_0, A_1, A_2, A_3, A_4, and A_5 are interval-valued fuzzy sets of the universe of discourse U, $U=\{u_1,u_2,\ldots,u_{14}\}$, B_1, B_2, B_3, B_4, and B_5 are interval-valued fuzzy sets of the

universe of discourse V, $V=\{v_1,v_2,\ldots,v_{14}\}$. The membership functions of these interval-valued fuzzy sets are the same as those shown in Example 4.1. Assume that given "$B_0=$ very B_3", where

$$B_0=\{(v_1,[0,0]),(v_2,[0,0]),(v_3,[0,0]),(v_4,[0,0.25]),(v_5,[0.548,0.672]),(v_6,[0.884,0.903]),$$
$$(v_7,[1,1]),(v_8,[0.884,0.903]),(v_9,[0.548,0.672]),(v_{10},[0,0.25]),$$
$$(v_{11},[0,0]),(v_{12},[0,0]),(v_{13},[0,0]),(v_{14},[0,0])\},$$

then

(i) Because $k_1=M(B_0,B_1)=0$ and $\mathrm{Supp}(B_0)\neq\mathrm{Supp}(B_1)$, from formulas (52)–(53), we can get

$$A_1^*=\{(u_i,[0,0])\mid 1\leqslant i\leqslant 14\}.$$

(ii) Because $k_2=M(B_0,B_2)=0.27$ and because $\mathrm{Supp}(B_0)\neq\mathrm{Supp}(B_2)$, from formulas (52)–(53), we can get

$$A_2^*=\{(u_1,[0,0]),(u_2,[0,0]),(u_3,[0,0.14]),(u_4,[0.20,0.22]),(u_5,[0.25,0.26]),$$
$$(u_6,[0.27,0.27]),(u_7,[0.25,0.26]),(u_8,[0.20,0.22]),(u_9,[0,0.14]),$$
$$(u_{10},[0,0]),(u_{11},[0,0]),(u_{12},[0,0]),(u_{13},[0,0]),(u_{14},[0,0])\}.$$

(iii) Because $B_0=$ very B_3, from formulas (40)–(41), we can get

$$A_3^*=\{(u_1,[0,0]),(u_2,[0,0]),(u_3,[0,0]),(u_4,[0,0]),(u_5,[0,0]),(u_6,[0,0]),$$
$$(u_7,[0,0.36]),(u_8,[0.76,0.85]),(u_9,[1,1]),(u_{10},[0.76,0.85]),(u_{11},[0,0.36]),$$
$$(u_{12},[0,0]),(u_{13},[0,0]),(u_{14},[0,0])\}.$$

(iv) Because $k_4=M(B_0,B_4)=0.38$ and because $\mathrm{Supp}(B_0)\neq\mathrm{Supp}(B_4)$, from formulas (52)–(53), we can get

$$A_4^*=\{(u_1,[0,0]),(u_2,[0,0]),(u_3,[0,0]),(u_4,[0,0]),(u_5,[0,0]),(u_6,[0,0]),$$
$$(u_7,[0,0]),(u_8,[0,0]),(u_9,[0,0.23]),(u_{10},[0.33,0.35]),(u_{11},[0.38,0.38]),$$
$$(u_{12},[0.33,0.35]),(u_{13},[0,0.23]),(u_{14},[0,0])\}.$$

(v) Because $k_5=M(B_0,B_5)=0$ and $\mathrm{Supp}(B_0)\neq\mathrm{Supp}(B_5)$, from formulas (52)–(53), we can get

$$A_5^*=\{(u_i,[0,0])\mid 1\leqslant i\leqslant 14\}.$$

Finally, based on formula (54), we can get the deduced consequence "X is A_0" of the SISO interval-valued approximate reasoning scheme, where

$$A_0=A_1^*\cup A_2^*\cup A_3^*\cup A_4^*\cup A_5^*$$
$$=\{(u_1,[0,0]),(u_2,[0,0]),(u_3,[0,0.14]),(u_4,[0.20,0.22]),(u_5,[0.25,0.26]),$$
$$(u_6,[0.27,0.27]),(u_7,[0.25,0.36]),(u_8,[0.76,0.85]),(u_9,[1,1]),(u_{10},[0.76,0.85]),$$
$$(u_{11},[0.38,0.38]),(u_{12},[0.33,0.35]),(u_{13},[0,0.23]),(u_{14},[0,0])\}.$$

The reasoning result is shown in Fig. 4.

5. Conclusions

In this paper, we have extended the work of [10] to present a new bidirectional approximate reasoning method for rule-based systems based on the direction of matching between interval-valued fuzzy sets, where

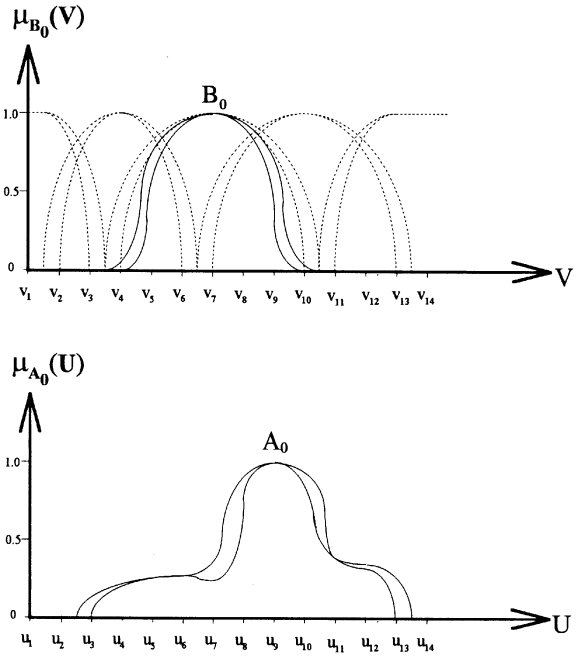

Fig. 4. The reasoning result of Example 4.3.

the concept of "direction of matching" is presented in order to intelligently modify the consequences of the deduced rules. We also have used some examples to illustrate the bidirectional approximate reasoning process of the rule-based systems based on the direction of matching between interval-valued fuzzy sets. From the examples shown in Section 4, we can see the proposed interval-valued bidirectional approximate reasoning method is more reasonable and more powerful than the one presented in [10]. The proposed method can overcome the drawbacks of the one we presented in [10]. It can provide a useful way to deal with bidirectional approximate reasoning for rule-based systems using interval-valued fuzzy sets.

Acknowledgements

This work was supported in part by the National Science Council, Republic of China, under Grant NSC 87-2213-E-009-011.

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