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Short Communication

An interval method for computing the stability margin of real uncertainty problems

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SUMMARY

Frequently, in practical control system design, some designing parameters are uncertain. These uncertain parameters may vary with temperature, humidity or other environmental variable, and these variations will have an impact on the stability of the system. In this paper, we use a global optimal method–interval method, by which stability margins of these uncertain parameters can be computed. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: Stability margin; μ-analysis; real uncertainty; interval arithmetic; global optimization

1. INTRODUCTION

The original idea of interval analysis was to bound rounding errors. However, interval mathematics can be said to have begun with the appearance of Moore's book [1] *Interval Analysis* in 1966. Moore's work transformed this simple idea into a useful tool for error analysis. Since the appearance of Moore's book, several persons have used interval analysis to solve the global optimization problem and systems of non-linear equations [2, 3]. Thus, the interval analysis has tended gradually to become an important mathematical tool for solving the problems of global optimization and systems of nonlinear equations. And we can also use interval analysis in robust control-system design.

Consider a robustness stability problem associated with real uncertain parameters. A fundamental problem addressed in a large number of papers [4, 5] is: Determine the maximum uncertainty bound at which a system is stable. Our main technical objective in this paper is to

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show how interval analysis can be used to determine the stability margin of a robustness stability problem associated with real uncertain parameters. Although structured singular value μ-analysis can solve the type of problem, the stability margin yielded by μ-analysis is more conservative than that obtained by using the interval method; see Section 2 for a detailed description.

2. PROBLEM FORMULATION

Consider a robustness stability problem as shown in Figure 1, where $\mathbf{M}(s) \in RH_\infty^{n\times n}$, and the real uncertain part $\Delta \in \mathbb{R}^{n\times n}$. We know that this is a structured singular value problem. If we claim the closed-loop system is internally stable then the necessary and sufficient condition is

$$\det(\mathbf{I} - \mathbf{M}(jw)\Delta) \neq 0 \quad \text{for } w \in [0, \infty).$$

Hence, if the uncertain part, $\|\Delta\|_\infty \leqslant (\mu_\Delta(\mathbf{M}))^{-1}$, then the system is always stable. But the condition $\|\Delta\|_\infty \leqslant (\mu_\Delta(\mathbf{M}))^{-1}$ is stringent. Here, we will use the interval method to obtain a loosen condition for the uncertain part Δ under which the system is still stable.

For convenience, we suppose $\mathbf{M} \in \mathbb{C}^{n\times n}$ and Δ is a diagonal matrix with real uncertain elements (if the block structure of Δ is not diagonal, we can transform the non-diagonal-structure problem into diagonal-structure problem by using certain techniques from Cheng and DeMoor [6]). Let $\Delta = \{\mathrm{diag}(\delta_1 \mathbf{I}_{r1}, \ldots, \delta_n \mathbf{I}_{rn}) \mid \delta_i \in \mathbb{R}\}$, and $\mu_\Delta(\mathbf{M})$ is defined as

$$\mu_\Delta(\mathbf{M}) := \frac{1}{\min\{\bar{\sigma}(\Delta) \mid \Delta \in \Delta;\ \det(\mathbf{I} - \mathbf{M}\Delta) = 0\}}.$$

Let Δ_r is a subset of Δ defined as follows:

$$\Delta_r := \{\Delta : \Delta \in \Delta;\ |\delta_i| \leqslant r,\ i = 1, \ldots, n\} \quad \text{where } r \in \mathbb{R}^+$$

hence, we can say Δ_r is a "square" in which $|\delta_i| \leqslant r$, $i = 1, \ldots, n$, and the length of "square" is r.

At first we defined $r_{\sup} := \sup\{r: \det(\mathbf{I} - \mathbf{M}\Delta) \neq 0, \forall \Delta \in \Delta_r\}$. Then we will prove $\mu_\Delta(\mathbf{M}) = 1/r_{\sup}$ which mean the calculation of μ-norm is to determine the maximum 'square' within which $\det(\mathbf{I} - \mathbf{M}\Delta) \neq 0$.

Let Δ^* is a solution of $\det(\mathbf{I} - \mathbf{M}\Delta) = 0$, where $\Delta^* \in \Delta$ and $\Delta^* = \mathrm{diag}(\delta_1^* \mathbf{I}_{r1}, \ldots, \delta_n^* \mathbf{I}_{rn})$. And let $\bar{\sigma}(\Delta^*) = \min\{\bar{\sigma}(\Delta) \mid \Delta \in \Delta;\ \det(\mathbf{I} - \mathbf{M}\Delta) = 0\}$. We know that $\bar{\sigma}(\Delta^*) = \max\{|\delta_i^*|: i = 1, \ldots, n\} = r^*$, it follows that $\mu_\Delta(\mathbf{M}) = 1/r^*$.

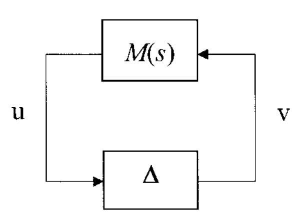

Figure 1. A closed-loop feedback system.

If $r_{\sup} > r^*$ then $|\delta_i^*| < r_{\sup}(i = 1, \ldots, n)$. Hence, $\Delta^* \in \boldsymbol{\Delta}_{r_{\sup}}$ and $\det(\mathbf{I} - \mathbf{M}\Delta^*) \neq 0$, we have the contradiction $\det(\mathbf{I} - \mathbf{M}\Delta^*) = 0$. If $r_{\sup} < r^*$ then there exist r' and Δ' where $r_{\sup} < r' < r^*$, and $\Delta' \in \boldsymbol{\Delta}_{r'}$ such that $\det(\mathbf{I} - \mathbf{M}\Delta') = 0$ and $\bar{\sigma}(\Delta') = r' < r^*$. It is contradicte with $\min\{\bar{\sigma}(\Delta) | \Delta \in \boldsymbol{\Delta};\ \det(\mathbf{I} - \mathbf{M}\Delta) = 0\} = \bar{\sigma}(\Delta^*) = r^*$. Therefore, $r_{\sup} = r^*$ and $\mu_{\Delta}(\mathbf{M}) = 1/r_{\sup}$.

From the above description, we know that the significance of μ-analysis is to determine the maximum 'square' of uncertain parameters within which $\det(\mathbf{I} - \mathbf{M}\Delta) \neq 0$ (i.e. the system is stable). Let $\det(\mathbf{I} - \mathbf{M}\Delta) = f_{\mathrm{R}}(\delta_1, \ldots, \delta_n) + f_{\mathrm{I}}(\delta_1, \ldots, \delta_n)\, j$, where f_{R}, f_{I} are the real and imaginary parts of $\det(\mathbf{I} - \mathbf{M}\Delta)$ dependent on $\delta_1, \ldots, \delta_n$. If $\det(\mathbf{I} - \mathbf{M}\Delta) \neq 0$, then $f_{\mathrm{R}}(\delta_1, \ldots, \delta_n) \neq 0$ or $f_{\mathrm{I}}(\delta_1, \ldots, \delta_n) \neq 0$.

We can illustrate the geometrical significance of $\mu_{\Delta}(\mathbf{M})$ by $n = 2$, the geometrical significance of $\mu_{\Delta}(\mathbf{M})$ is shown in Figure 2(a), which means $f_{\mathrm{R}} \neq 0$ or $f_{\mathrm{I}} \neq 0$ within the largest square on a plane consisting of δ_1, δ_2. Therefore, the structured singular value of $\mu_{\Delta} = 1/r$.

From Figure 2(a), we know that if the uncertain part $\|\Delta\|_{\infty} \leqslant (\mu_{\Delta}(\mathbf{M}))^{-1} = r$, i.e. $|\delta_1| \leqslant r$ and $|\delta_2| \leqslant r$, then the system is always stable. But this claim is stringent. In fact, i.e. we let δ_1 and δ_2 fall within the rectangle shown in Figure 2(b), then the system is still stable. Therefore, the robustness stability problem can be transformed into a problem of systems of equations:

$$\begin{aligned} f_{\mathrm{R}}(\delta_1, \ldots, \delta_n) &= 0 \\ f_{\mathrm{I}}(\delta_1, \ldots, \delta_n) &= 0 \end{aligned} \quad (\mathbf{M} \in \mathbb{C}^{n \times n}) \tag{1a}$$

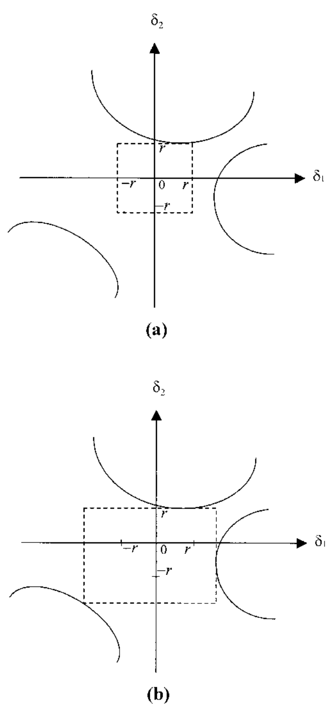

Figure 2. The plot of $f_{\mathrm{R}} = f_{\mathrm{I}} = 0$.

or

$$\begin{aligned} f_{\mathrm{R}}(\delta_1, \ldots, \delta_n, \ w) = 0 \\ f_{\mathrm{I}}(\delta_1, \ldots, \delta_n, \ w) = 0 \end{aligned} \quad (\mathbf{M}(s) \in RH_\infty^{n\times n}) \tag{1b}$$

where $\delta_i \in [\underline{\delta_i}, \overline{\delta_i}]$; $i = 1, \ldots, n$; $w \in [0, \infty)$.

Our aim is to determine the maximum uncertainty bound δ_i at which the systems of equations (1) have no solutions. In the next section, we introduce a method for using interval analysis to solve this problem.

3. INTERVAL ALGORITHM

Consider a function $f(x_1, \ldots, x_n)$, $x_i \in X_i$ $(i = 1, \ldots, n)$, we expand f with Taylor's theorem [2].

$$f(y_1, \ldots, y_n) = f(x_1, \ldots, x_n) + \sum_{i=1}^{n} g_i(\zeta_1, \ldots, \zeta_i, x_{i+1}, \ldots, x_n)\,(y_i - x_i)$$

where $g_i = (\partial f/\partial x_i)$ $(i = 1, \ldots, n)$. If $x_i \in X_i$ and $y_i \in X_i$, then this holds for some number $\zeta_i \in X_i$. In our applications, we sometimes want a linear bound on $f(y_1, \ldots, y_n)$ for all $y_i \in X_i$ $(i = 1, \ldots, n)$. Thus, we replace ζ_i with the bounding interval X_i $(i = 1, \ldots, n)$ and obtain

$$f(y_1, \ldots, y_n) \in f(x_1, \ldots, x_n) + \sum_{i=1}^{n} g_i\,(X_1, \ldots, X_i, x_{i+1}, \ldots, x_n)\,(y_i - x_i) \tag{2}$$

If $\mathbf{y}$ is a zero of f, then $f(\mathbf{y}) = 0$ and we replace Equation (2) with

$$f(\mathbf{x}) + \mathbf{g}(\mathbf{x}, \mathbf{X})\,(\mathbf{y} - \mathbf{x}) = 0. \tag{3}$$

We define the solution set of Equation (3) to be $\mathbf{S} = \{\mathbf{y}: f(\mathbf{x}) + \mathbf{g}(\mathbf{x}, \zeta)\,(\mathbf{y} - \mathbf{x}) = 0\}$ for all $\zeta \in \mathbf{X}$. This set contains any point $\mathbf{y} \in \mathbf{X}$ for which $f(\mathbf{y}) = 0$. From Equation (3), we let

$$Y_i = x_i - \frac{f(x_1, \ldots, x_n) + \sum_{j=1}^{i-1} g_j\cdot(y_j - x_j) + \sum_{j=i+1}^{n} g_j\cdot(y_j - x_j)}{g_i(X_1, \ldots, X_i, \ x_{i+1}, \ldots, x_n)} \quad (i = 1, \ldots, n) \tag{4}$$

and the set $\mathbf{Y} = \{(Y_1, \ldots, Y_n)\}$. Thus, the set $\mathbf{S} \subset \mathbf{Y}$, where the right-hand member of (4) is obtained by simple evaluation using interval arithmetic [3].

For future reference, it is desirable to have a distinctive notation for the solution of Equation (3). In place of Y_i and $\mathbf{Y}$, we shall use the notation $N_i(\mathbf{x}, \mathbf{X})$ and $\mathbf{N}(\mathbf{x}, \mathbf{X})$, which emphasizes the dependence on $\mathbf{X}$ and $\mathbf{x}$.

From Equation (3), we define an iterative algorithm of the form

$$f(\mathbf{x}^{(k)}) + \mathbf{g}(\mathbf{x}^{(k)}, \mathbf{X}^{(k)})\,[\mathbf{N}(\mathbf{x}^{(k)}, \mathbf{X}^{(k)}) - \mathbf{x}^{(k)}] = 0 \tag{5a}$$

$$\mathbf{X}^{(k+1)} = \mathbf{X}^{(k)} \cap \mathbf{N}(\mathbf{x}^{(k)}, \mathbf{X}^{(k)}) \tag{5b}$$

for $k = 0, 1, 2, \ldots,$ where $\mathbf{x}^{(k)}$ is the centre of $\mathbf{X}^{(k)}$.

The components of $\mathbf{N}(\mathbf{x}^{(k)}, \mathbf{X}^{(k)})$ will be computed sequentially. The intersection in (5b) should be performed as soon as a new component is obtained so that components computed later will be narrower intervals. And proof of convergence for Equation (5) can be found in Moore [1], Krawczky [7] and Alefeld [8].

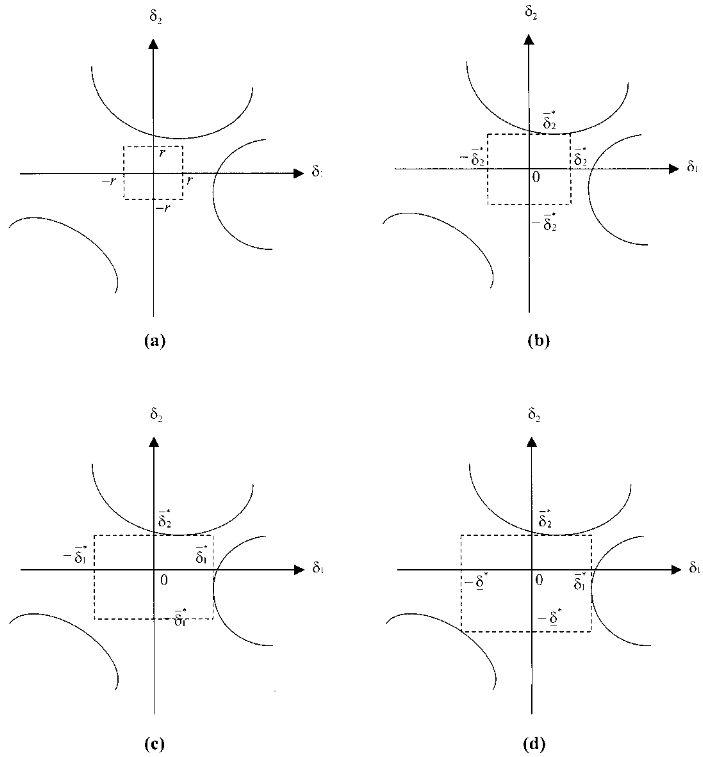

Figure 3. The steps of interval method.

Now, we consider the robustness stability problem in Section 2. If $\underline{\delta}_i$, $\bar{\delta}_i$, $i = 1, \ldots, n$, is given in Equation (1), we can use the iterative algorithm given in Equations (5) to solve Equation (1). If there are solutions in box $\delta_i \in [\underline{\delta}_i, \bar{\delta}_i]$ $(i = 1, \ldots, n)$, then we will decrease the size of box, otherwise, we will increase the size of the box. We can tune the size of box successively until we get a maximum box from numerical computation such that no solutions exist in the box for Equation (1).

To illustrate the method described above, suppose $n = 2$, due to this being a two-dimensional case, hence there are four variables $\underline{\delta}_1$, $\bar{\delta}_1$, $\underline{\delta}_2$ and $\bar{\delta}_2$ to be determined. First, let $\underline{\delta}_1 = \underline{\delta}_2 = -r$, $\bar{\delta}_1 = \bar{\delta}_2 = r$, as shown in Figure 3(a). If there are no solutions in box $\delta_1 \in [\underline{\delta}_1, \bar{\delta}_1]$ and $\delta_2 \in [\underline{\delta}_2, \bar{\delta}_2]$, then we increase the magnitude of r. Otherwise, we decrease the magnitude of r. Until $r = \bar{\delta}_2^*$, as shown in Figure 3(b). Hold $\bar{\delta}_2 = \bar{\delta}_2^*$ and let $\underline{\delta}_1 = \underline{\delta}_2 = -r$, $\bar{\delta}_1 = r$, then tune the magnitude of r until $r = \bar{\delta}_1^*$, as shown in Figure 3(c). Then hold $\bar{\delta}_1 = \bar{\delta}_1^*$ and tune the magnitude of $|\underline{\delta}_1|$ and $|\underline{\delta}_2|$ again, until $|\underline{\delta}_1| = |\underline{\delta}_2| = |\underline{\delta}^*|$, as shown in Figure 3(d). Thus, we can decide the magnitude of $\underline{\delta}_1$, $\bar{\delta}_1$, $\underline{\delta}_2$ and $\bar{\delta}_2$ and determine the maximum uncertain bounds for δ_1 and δ_2. Simultaneously, to avoid the maximum uncertain bound values tend to infinite, we will limit the maximum uncertain bound values less than α which is a very large number.

Finally, we have to note that $\mathbf{M}(s) \in RH_{\infty}^{n \times n}$ in the real uncertainty problem. Let $\mathbf{M}(\mathrm{j}w) = [\mathbf{R}_{i,j}(w) + \mathbf{I}_{i,j}(w)\mathrm{j}]$; $i, j = 1, \ldots, n$; $w \in W = [0, \infty)$, where $\mathbf{R}_{i,j}(w)$ and $\mathbf{I}_{i,j}(w)$ are the real and imaginary parts of $\mathbf{M}(jw)$, and $\mathbf{M}(jW) = [\mathbf{R}_{i,j}(W) + \mathbf{I}_{i,j}(W)j]$ $(i, j = 1, \ldots, n)$ is an interval matrix.

Although, we cannot use interval arithmetic to compute $\mathbf{R}_{i,j}(W)$ and $\mathbf{I}_{i,j}(W)$ $(i, j = 1, \ldots, n)$ since $W = [0, \infty]$ is an unbounded interval. We can restrict W within $[0, \bar{w}]$, where $\bar{w}$ is a very large number or use the eigenvalue technique [9] to obtain the interval of $\mathbf{R}_{i,j}(W)$ and $\mathbf{I}_{i,j}(W)$ for $W = [0, \infty]$.

We now describe the steps in the interval algorithm. The subroutine for the iterative algorithm (5) solving Equation (1) is omitted [9, 10].

Step 1: Input r, Δr, ξ and α, let $\underline{r}_i = \bar{r}_i = r$, $\Delta\underline{r}_i = \Delta\bar{r}_i = \Delta r$; $i = 1, \ldots, n$.
Step 2: Let $I = J = 0$.
Step 3: Let $\underline{\delta}_i = -\underline{r}_i$, $\bar{\delta}_i = \bar{r}_i$; $i = 1, \ldots, n$.
Step 4: Using the iterative algorithm (5a) and (5b), if there are solutions in the box $\delta_i \in [\underline{\delta}_i, \bar{\delta}_i]$ $(i = 1, \ldots, n)$, $w \in [0, \infty]$, then decrease the size of box: let $I = 1$, $\Delta\underline{r}_i = (0.5)^J \Delta\underline{r}_i$ and $\Delta\bar{r}_i = (0.5)^J \Delta\bar{r}_i$, $\underline{r}_i = \underline{r}_i - \Delta\underline{r}_i$ and $\bar{r}_i = \bar{r}_i - \Delta\bar{r}_i$ $(i = 1, \ldots, n)$.
Step 5: Otherwise, increase the size of box, let $J = 1$, $\Delta\underline{r}_i = (0.5)^I \Delta\underline{r}_i$ and $\Delta\bar{r}_i = (0.5)^I \Delta\bar{r}_i$, $\underline{r}_i = \underline{r}_i + \Delta\underline{r}_i$ and $\bar{r}_i = \bar{r}_i + \Delta\bar{r}_i$ $(i = 1, \ldots, n)$.
Step 6: If $\max(\Delta\underline{r}_1, \ldots, \Delta\underline{r}_n, \Delta\bar{r}_1, \ldots, \Delta\bar{r}_n) \leqslant \xi$, then determine which $\underline{r}_j$ or $\bar{r}_j$ should be held; let the $\underline{r}_j$ or $\bar{r}_j$ be held at $\underline{r}_j - \xi$ or $\bar{r}_j - \xi$, and set the corresponding $\Delta\underline{r}_j$ or $\Delta\bar{r}_j$ to zeros, and let the other $\Delta\underline{r}_i = \Delta r$ and $\Delta\bar{r}_i = \Delta r$ $(i = 1, \ldots, n)$; reset $I = J = 0$.
Step 7: If every $\Delta\underline{r}_i$ and $\Delta\bar{r}_i$ $(i = 1, \ldots, n)$ are zeros or $\max(\underline{r}_1, \ldots, \underline{r}_n, \bar{r}_1, \ldots, \bar{r}_n) \geqslant \alpha$, then stop and print out $\underline{\delta}_i$ and $\bar{\delta}_i$; $i = 1, \ldots, n$.
Step 8: Otherwise, go to step 3.

4. EXAMPLE

In this section, we give a comparison between using μ-analysis and the interval method in determining the stability margins of a real uncertainty problem.

Assume the 5×5 transfer matrix

$$\mathbf{M}(s) =$$

$$\begin{bmatrix} \frac{1}{s^2+3s+2} & \frac{1}{s^2+8s+17} & \frac{1}{s^2+3s+2} & \frac{1}{s^2+7s+12} & \frac{1}{s^2+3s+2} \\ \frac{1}{s^2+12s+61} & \frac{s-7}{s^2+9s+22} & \frac{1}{s^2+20s+96} & \frac{1}{s^2+18s+82} & \frac{1}{s^2+27s+170} \\ \frac{s+1}{s^3+6s^2+10s+8} & \frac{1}{s^2+8s+15} & \frac{s-1}{s^2+12s+11} & \frac{s+5}{s^3+6s^2+11s+6} & \frac{s+4}{s^3+11s^2+43s+65} \\ \frac{1}{s+14} & \frac{1}{s^2+16s+15} & \frac{1}{s^2+2s+10} & \frac{1}{s+7} & \frac{1}{s^2+9s+8} \\ \frac{1}{s^2+4s+29} & \frac{s+1}{s^2+2s+17} & \frac{1}{s^2+10s+74} & \frac{1}{s+3} & \frac{1}{s^2+3s+2} \end{bmatrix},$$

and the corresponding real uncertainty matrices

$$\Delta = \begin{bmatrix} \delta_1 & 0 & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & 0 & 0 \\ 0 & 0 & \delta_2 & 0 & 0 \\ 0 & 0 & 0 & \delta_3 & 0 \\ 0 & 0 & 0 & 0 & \delta_4 \end{bmatrix}, \quad \delta_i \in \mathbb{R},\ i = 1, \ldots, 4.$$

Because $\mu_\Delta(\mathbf{M}(s)) = 0.8471$, the closed-loop system consists of $\mathbf{M}(s)$ and Δ being stable in $\|\Delta\|_\infty \leqslant (\mu_\Delta(\mathbf{M}(s)))^{-1} = 1.1804$. So, the stability margin yielded by μ-analysis is $|\delta_1| \leqslant 1.1804$, $|\delta_2| \leqslant 1.1804$, $|\delta_3| \leqslant 1.1804$ and $|\delta_4| \leqslant 1.1804$. But, if we use the interval algorithm described in the preceding section to compute the stability margin of δ_1, δ_2, δ_3 and δ_4. We can obtain the maximum uncertain bounds $-3.1049 \leqslant \delta_1 \leqslant 1.1804$, $-3.1049 \leqslant \delta_2 \leqslant 1.1804$, $-3.1049 \leqslant \delta_3 \leqslant 1.1804$ and $-\alpha \leqslant \delta_4 \leqslant 1.1804$, where $\alpha = 10000$, within which the system is stable. So, the stability margin yielded by μ-analysis is more conservative than that obtained by using the interval algorithm.

5. CONCLUSIONS

Recently, the following two important problems have attracted a lot of attention [11–13].

Problem 1 (*Stability radii problem*). For given matrices $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times m}$, $\mathbf{C} \in \mathbb{C}^{p\times n}$ and a nontrival partition of the complex plane $\mathbb{C} = \mathbb{C}_g \cup \mathbb{C}_b$, where $\mathbb{C}_g$ is open region, and $\Delta \in \mathscr{K}$ is an unknown disturbance matrix belonging to a given perturbation set $\mathscr{K}$ measure the distance of the stable matrix $\mathbf{A}$ to instability, i.e. find

$$\Upsilon_{\mathscr{K}}(\mathbf{A};\ \mathbf{B},\ \mathbf{C};\ \mathbb{C}_b) := \inf\{\|\Delta\|;\ \Delta \in \mathscr{K}^{m\times p},\ \sigma(\mathbf{A} + \mathbf{B}\Delta\mathbf{C}) \cap \mathbb{C}_b \neq \emptyset\}$$

where $\|\Delta\|$ is any operator norm.

Problem 2. For the given stable matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ find the largest interval matrix $\Delta^{\mathbf{I}}$ with elements belonging to a given perturbation set $\mathscr{K}$ such that the interval matrix $\mathbf{A}(\Delta^{\mathbf{I}}) = \mathbf{A} + \mathbf{B}\Delta^{\mathbf{I}}\mathbf{C}$ is stable.

This two above problems can also be solved by interval method. The real stability radii problem ($\mathscr{K} = \mathbb{R}$) $\Upsilon_{\mathscr{K}}(\mathbf{A}; \mathbf{B}, \mathbf{C}; \mathbb{C}_b)$ can be represented as follows:

$$\begin{aligned} \Upsilon_{\mathscr{K}}(\mathbf{A}; \mathbf{B}, \mathbf{C}; \mathbb{C}_b) &:= \inf\{\|\Delta\|;\ \Delta \in \mathscr{K}^{m\times p},\ \sigma(\mathbf{A} + \mathbf{B}\Delta\mathbf{C}) \cap \mathbb{C}_b \neq \emptyset\} \\ &= \sup\{r : \sigma(\mathbf{A} + \mathbf{B}\Delta\mathbf{C}) \cap \mathbb{C}_b = \emptyset,\ \forall \Delta \in \Delta_r\} \end{aligned}$$

where Δ_r is defined by

$$\Delta_r := \{\Delta : \Delta \in \mathscr{K}^{m\times p},\ \|\Delta\| \leqslant r\},\ r \in \mathbb{R}^+$$

Let $\Delta = [\delta_{ij}]$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant p)$, and $\det(\lambda\mathbf{I} - \mathbf{A}^*(\Delta)\mathbf{A}(\Delta)) = D(\Delta, \lambda) = D(\delta_{i,j}, \lambda) = D(\delta_{i,j}, \sigma^2)$, where $\sigma \in \mathbb{C}_b \cap \mathbb{R}^+$ and $\mathbf{A}(\Delta) := \mathbf{A} + \mathbf{B}\Delta\mathbf{C}$.

If the following two conditions hold:

(a) the intersection $\mathbb{C}_b \cap \mathbb{R}^+$ can be formulated in interval-form;
(b) $\|\Delta\| \leqslant r$ can also be formulated as $\delta_{i,j} \in [\underline{\delta}_{i,j}, \bar{\delta}_{i,j}]$, where Δ are the real perturbation block, $\Delta = [\delta_{i,j}]$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant p)$; then the interval method can solve real stability radii problem.

In addition, because of **A**, **B** and **C** are stable matrices for Problem 2, then the necessary and sufficient condition for $\mathbf{A}(\Delta^{\mathbf{I}})$ to be stable is $\det(\mathbf{I} - \mathbf{A}^*(\Delta)\mathbf{A}(\Delta)) \neq 0$, $\forall \Delta \in \Delta^{\mathbf{I}}$. Let $\Delta = [\delta_{i,j}]$, $\delta_{i,j} \in [\underline{\delta}_{i,j}, \bar{\delta}_{i,j}]$, it follows that $\det(\mathbf{I} - \mathbf{A}^*(\Delta)\mathbf{A}(\Delta)) = f_{\mathbf{R}}(\delta_{i,j}) + f_{\mathbf{I}}(\delta_{i,j})\ j$, where $f_{\mathbf{R}}$ and $f_{\mathbf{I}}$ are the real and imaginary parts of $\det(\mathbf{I} - \mathbf{A}^*(\Delta)\mathbf{A}(\Delta))$ depend on $\delta_{i,j}$ $(1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant p)$.

Therefore, Problem 2 can be transformed into as a problem of systems of equations as Equation (1). We can use the interval method (illustrated in Figure 3) to determine the maximum uncertainty bound $\delta_{i,j}$ at which $\det(\mathbf{I} - \mathbf{A}^*(\Delta)\mathbf{A}(\Delta)) \neq 0$, $\forall \Delta \in \Delta^{\mathbf{I}}$. Therefore, the largest interval matrix $\Delta^{\mathbf{I}}$ can be determined by the interval method within which $\mathbf{A}(\Delta^{\mathbf{I}})$ is stable. The above results dealing with Problems 1 and 2 are obvious. The reader can easily get the results as the derivation in Section 2. Hereby, we will not derive them in further detail.

APPENDIX: NOMENCLATURE

det	determinant
diag{ · }	diagonal matrix
I	unit matrix
sup	supremum
inf	infimum

Greek letters

Δ	structured uncertainty matrix
μ_Δ	structured singular value with structured uncertainty matrix Δ

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REFERENCES

1. Moore RE. *Interval Analysis*, Prentice-Hall: Englewood Cliffs, NJ, 1966.
2. Hansen E. *Global Optimization Using Interval Analysis*, Marcel Dekker: New York, 1992.
3. Neumaier A. *Interval Methods for Systems of Equations*, Cambridge University Press: New York, 1990.
4. Barmish BR. New tools for robustness analysis. *Proceedings of the IEEE Conference on Decision and Control*, Austin, 1988; 1399–1404.
5. de Gaston RRE, Safonov MG. Exact calculation of the multiloop stability margin. *IEEE Transactions on Automatic Control* 1988; **33**: 156–171.
6. Cheng Y, DeMoor B. A multidimensional realization algorithm for parametric uncertainty modelling and multi-parameter margin problem. *International Journal of Control* 1994; **60**:789–807.
7. Krawczyk R. A remark about the convergence of interval sequences. *Computing* 1983; **31**:255–259.

8. Alefeld G. On the convergence of some interval arithmetic modifications of Newton's method. *SIAM Journal on Numerical Analysis* 1984; **21**:363–372.
9. Ge ZM, Chu LW. An interval method for computing the structured singular value of a transfer matrix with real uncertainty. *JSME International Journal*, *Series C*, 1998; **41**:741–750.
10. Moore RE. (ed.). *Reliability in Computing*, Academic Press: San Diego, 1988; 269–286.
11. Qiu L, Bernhardsson B, Rantzer A, Davison EJ, Young PM, Doyle JC. A formula for computation of the real structured stability radius. *Automatica* 1995; **31**:879–890.
12. Son NK, Hinrichsen D. Robust stability of positive linear systems. *Proceedings of the 34th Conference on Decision and Control*, New Orleans, 1994; 2985–2987.
13. Rohn J. System of linear interval equations. *Linear Algebra and its Applications* 1989; **126**:39–78.

