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The towers of Hanoi problem with cyclic parallel moves

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Abstract

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This paper discusses a variant of the towers of Hanoi problem with cyclic parallel moves. We propose an algorithm for this problem and show that the algorithm is optimal.

Keywords: Towers of Hanoi; recurrence relations; algorithms

1. Introduction

The towers of Hanoi problem is extensively discussed in [1–5], among which Atkinson [1] especially focuses on the variant of the towers of Hanoi problem with cyclic moves and Wu and Chen [5] discuss the variant of the towers of Hanoi problem with parallel moves. This paper proposes a combined variant: the towers of Hanoi problem with *cyclic parallel moves*, as presented in the following.

Suppose there are three pegs (A, B, C), and n disks of different sizes are placed in small-on-large ordering on peg A. The object is to move all the n disks from peg A to either peg B or peg C in original order. The rules of disk moves are:

Rule 1. Every top disk can be simultaneously moved from its original peg to the next peg in clockwise direction $A \to B \to C \to A$, at a time.

Rule 2. No disk is ever placed upon a smaller one.

In other words, three types of move including *single move, consecutive move* and *circular move* can be made. These moves are presented in Fig. 1.

2. The optimal algorithm

In this section we propose an algorithm for the towers of Hanoi problem with cyclic parallel moves and prove its optimality. The notation to be used consistently throughout this paper is described as follows.

Notation:

A, B, C: A is FROM peg, B and C are TO pegs.

$A(d_1, d_2, \ldots)$: peg A with disks $d_1, d_2, \ldots$, from top to bottom; similarly for $B(d_1, d_2, \ldots)$ and $C(d_1, d_2, \ldots)$.

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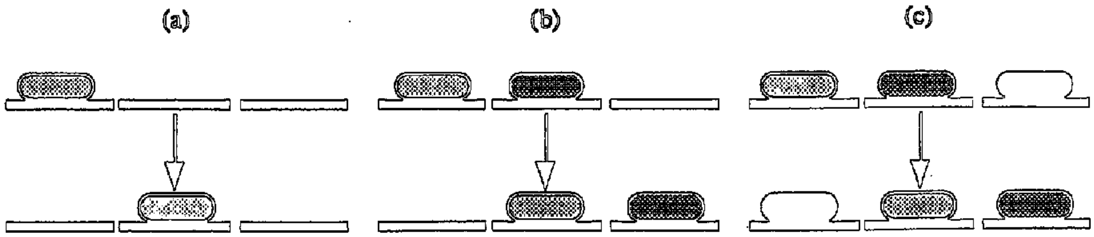
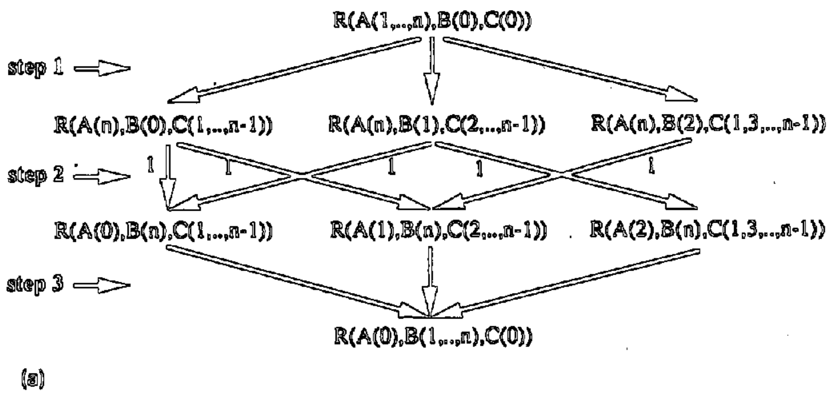


Fig. 1. Three moves: (a) single move, (b) consecutive move, (c) circular move.



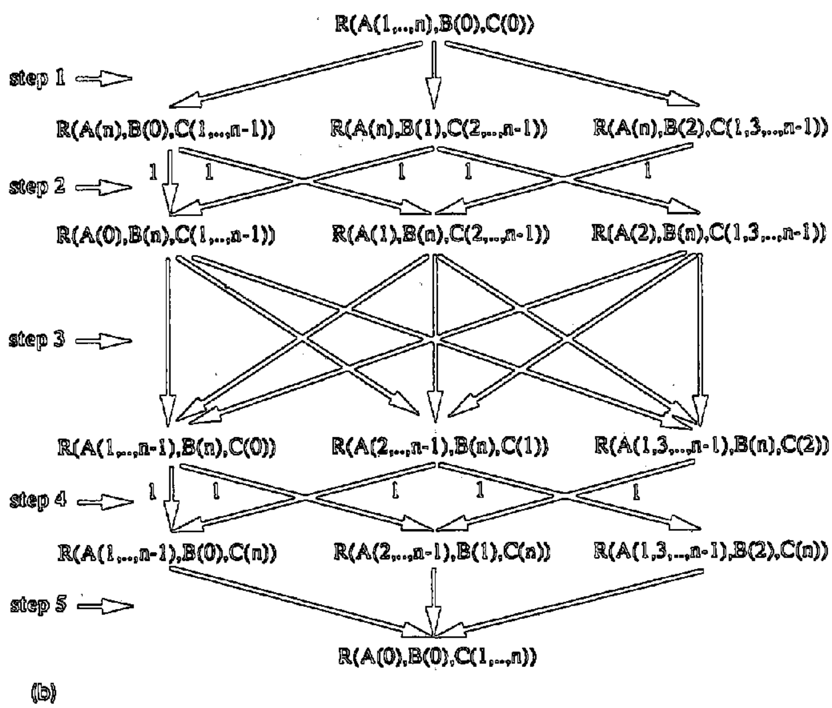


Fig. 2. Two transformation diagrams. (a) Transformation diagram from $R(A(1,\ldots,n), B(0), C(0))$ into $R(A(0), B(1,\ldots,n), C(0))$. (b) Transformation diagram from $R(A(1,\ldots,n), B(0), C(0))$ into $R(A(0), B(0), C(1,\ldots,n))$.

$A(0)$:	peg A with no disk; similarly for $B(0)$ and $C(0)$.
$R(A, B, C)$:	state of pegs A, B and C.
$c(n)$:	the minimal number of disk moves required to transfer n disks to the next position clockwise ($A \to B \to C \to A$).
$a(n)$:	the minimal number of disk moves required to transfer n disks to the next position anticlockwise ($A \to C \to B \to A$).

An optimal algorithm is derived from the following lemmas and proved to be correct.

Lemma 1. *For $n \geqslant 1$, the minimal number of disk moves for transforming $R(A(2,\dots,n), B(1), C(0))$ into $R(A(0), B(1,\dots,n), C(0))$, is $c(n)-1$.*

Proof. For $n \geqslant 1$, consider the optimal transformation of $R(A(1,\dots,n), B(0), C(0))$ into $R(A(0), B(1,\dots,n), C(0))$. The first disk move is

$$R(A(1,\dots,n), B(0), C(0)) \to R(A(2,\dots,n), B(1), C(0)).$$

Hence the minimal number of disk moves for transforming $R(A(2,\dots,n), B(1), C(0))$ into $R(A(0), B(1,\dots,n), C(0))$, is $c(n)-1$. □

Lemma 2. *For $n \geqslant 1$, the minimal numbers of disk moves are $c(n)-1$, for the following transformations,*

(1) transforming $R(A(1,\dots,n), B(0), C(0))$ into $R(A(1), B(2,\dots,n), C(0))$,

(2) transforming $R(A(2,\dots,n), B(1), C(0))$ into $R(A(0), B(1,\dots,n), C(0))$;

and $a(n)-1$, for the following transformations,

(1) transforming $R(A(1,\dots,n), B(0), C(0))$ into $R(A(0), B(1), C(2,\dots,n))$,

(2) transforming $R(A(2,\dots,n), B(1), C(0))$ into $R(A(0), B(0), C(1,\dots,n))$.

Proof. By Lemma 1, this lemma is obviously shown. □

Lemma 3. *For $n \geqslant 3$, consider the optimal transformation of $R(A(1,\dots,n), B(0), C(0))$ into $R(A(0), B(1,\dots,n), C(0))$; the details of disk moves are shown in Fig. 2(a).*

Proof. For $n \geqslant 3$, the transformation of $R(A(1,\dots,n), B(0), C(0))$ into $R(A(0), B(1,\dots,n), C(0))$ can be divided into three steps:

Step 1 shows the transformation of $R(A(1,\dots,n, B(0), C(0))$ into the state prior to disk n being moved from peg A to peg B,

Step 2 shows the move of disk n from peg A to peg B,

Step 3 shows the transformation of the state when disk n has been moved from peg A to peg B into $R(A(0), B(1,\dots,n), C(0))$.

In the final state of Step 1, only disk n is on peg A, and disk 1 or disk 2 or none is on peg B while the others are on peg C. And in the initial state of Step 3, only disk n is on peg B, and disk 1 or disk 2 or none is on peg A while the others are on peg C. Hence the transformation diagram is shown in Fig. 2(a). □

Lemma 4. *For $n \geqslant 3$, consider the optimal transformation of $R(A(1,\dots,n), B(0), C(0))$ into $R(A(0), B(0), C(1,\dots,n))$; the details of disk moves are shown in Fig. 2(b).*

Proof. The transformation $R(A(1,\dots,n), B(0), C(0))$ into $R(A(0), B(1,\dots,n), C(0))$ can be divided into five steps:

Step 1 shows the transformation of $R(A(1,\dots,n), B(0), C(0))$ into the state prior to disk n being moved from peg A to peg B,

Step 2 shows the move of disk n from peg A to peg B,

Step 3 shows the transformation of the state when disk n has been moved from peg A to peg B into the state prior to disk n being moved from peg B to peg C,

Step 4 shows the move of disk n from peg B to peg C,

Step 5 shows the transformation of the state when disk n has been moved from peg B to peg C into $R(A(0), B(0), C(1,\dots,n))$.

First, in the final state of Step 1, only disk n is on peg A, and disk 1 or disk 2 or none is on peg B while others are on peg C. Later, in the initial state of Step 3, only disk n is on peg B, and disk

1 or disk 2 or none is on peg A while others are on peg C; and in the final state of Step 3, only disk n is on peg B, and disk 1 or disk 2 or none is on peg C while others are on peg A. Finally, in the initial state of Step 5, only disk n is on peg C, and disk 1 or disk 2 or none is on peg B while others are on peg A. Hence the transformation diagram is shown in Fig. 2(b). □

Lemma 5. *For $n \geqslant 2$, the minimal numbers of disk moves for transforming $R(A(1, 3, \ldots, n), B(2), C(0))$ into $R(A(0), B(1, \ldots, n), C(0))$ and*

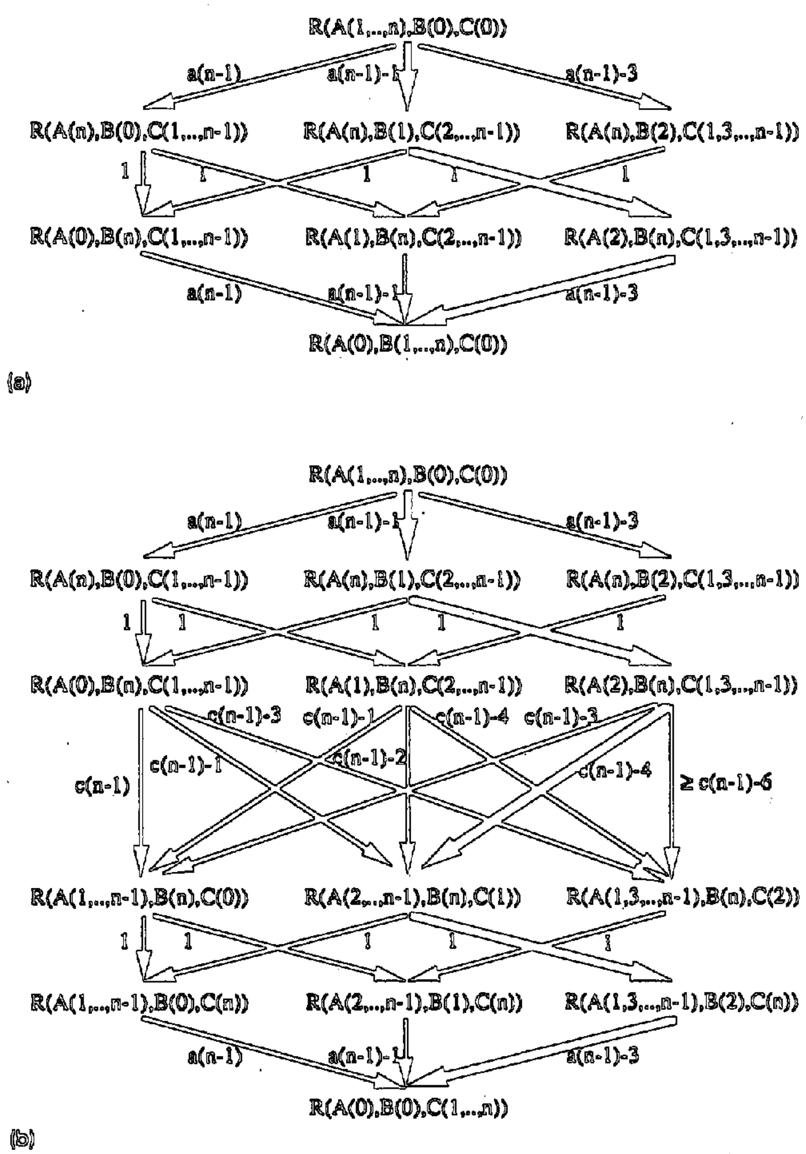


Fig. 3. Two optimal transformation diagrams. (a) Optimal transformation of $R(A(1, \ldots, n), B(0), C(0))$ into $R(A(0), B(1, \ldots, n), C(0))$. (b) Optimal transformation of $R(A(1, \ldots, n, B(0), C(0))$ into $R(A(0), B(0), C(1, \ldots, n))$.

$R(A(0), B(0), C(1,\ldots,n))$, are $c(n)-3$ and $a(n)-3$, respectively.

Proof. We want to prove this lemma by induction. It is clearly true for $n=2$. We assume this lemma is true for $n-1$. This assumption implies

(1) the minimal numbers of disk moves for transforming $R(A(1, 3,\ldots,n-1), B(2), C(0))$ into $R(A(0), B(1,\ldots,n-1), C(0))$ and $R(A(0), B(0), C(1,\ldots,n-1))$, are $c(n-1)-3$ and $a(n-1)-3$, respectively;

(2) the minimal numbers of disk moves for transforming $R(A(1,\ldots,n-1), B(0), C(0))$ into $R(A(2), B(1, 3,\ldots,n-1), C(0))$ and $R(A(0), B(2), C(1, 3,\ldots,n-1))$, are $c(n-1)-3$ and $a(n-1)-3$, respectively.

By Lemmas 1 and 2 and the above two results, we obtain

(3) the minimal numbers of disk moves for transforming $R(A(2,\ldots,n-1), B(1), C(0))$ into $R(A(2), B(1, 3,\ldots,n-1), C(0))$ and $R(A(1, 3,\ldots,n-1), B(2), C(0))$ into $R(A(1), B(2,\ldots,n-1), C(0))$, are both $c(n-1)-4$.

Because the minimal numbers of disk moves for transforming $R(A(1,\ldots,n-1), B(0), C(0))$ into $R(A(1, 3,\ldots,n-1), B(2), C(0))$ and $R(A(2), B(1, 3,\ldots,n-1), C(0))$ into $R(A(0), B(1,\ldots,n-1), C(0))$, are both 3, it is obvious that

(4) the minimal number of disk moves for transforming $R(A(1, 3,\ldots,n-1), B(2), C(0))$ into $R(A(2), B(1, 3,\ldots,n-1), C(0))$ is at least $c(n-1)-6$.

By Lemma 3 and 4 and the above four results, we obtain Fig. 3. In Fig. 3(a), there is the optimal transformation of $R(A(1,\ldots,n), B(0), C(0))$ into $R(A(0), B(1,\ldots,n), C(0))$:

$$\begin{aligned}&R(A(1,\ldots,n), B(0), C(0))\\ &\quad\to R(A(n), B(1), C(2,\ldots,n-1))\\ &\quad\to R(A(2), B(n), C(1,3,\ldots,n-1))\\ &\quad\to R(A(0), B(1,\ldots,n), C(0)).\end{aligned}$$

Because the minimal numbers of disk moves for transforming $R(A(1,\ldots,n), B(0), C(0))$ into $R(A(1, 3,\ldots,n), B(2), C(0))$ and $R(A(1, 3,\ldots,n), B(2), C(0))$ into $R(A(n), B(1), C(2,\ldots,n-1))$ are 3 and $a(n-1)-4$, respectively, the optimal transformation can be modified as:

$$\begin{aligned}&R(A(1,\ldots,n), B(0), C(0))\\ &\quad\to R(A(1,3,\ldots,n), B(2), C(0))\\ &\quad\to R(A(n), B(1), C(2,\ldots,n-1)) \to \cdots\\ &\quad\to R(A(0), B(1,\ldots,n), C(0)).\end{aligned}$$

Hence we have proven that the minimal number of disk moves for transforming $R(A(1, 3,\ldots,n), B(2), C(0))$ into $R(A(0), B(1,\ldots,n), C(0))$ is $c(n)-3$. By the same method, we also prove that the minimal number of disk moves for transforming $R(A(1, 3,\ldots,n), B(2), C(0))$ into $R(A(0), B(0), C(1,\ldots,n))$ is $a(n)-3$ in Fig. 3(b). □

Theorem 6. *For $n \geqslant 3$, $c(n)=2\cdot a(n-1)-3$ and $a(n)=2\cdot a(n-1)+c(n-1)-6$.*

Proof. By Lemma 5 and Fig. 3, we obtain $c(n)=[a(n-1)-1]+1+[a(n-1)-3]=2\cdot a(n-1)-3$ and $a(n)=[a(n-1)-1]+1+[c(n-1)-4]+1+[a(n-1)-3]=2\cdot a(n-1)+c(n-1)-6$, for $n \geqslant 3$. □

Finally, applying the two recurrence relations in Theorem 6, we obtain $c(n)$ and $a(n)$ as follows.

Theorem 7. *For $n \geqslant 3$,*

$$c(n)=\frac{\left[(1+\sqrt{3})^{n-1}+(1-\sqrt{3})^{n-1}\right]}{2}+3,$$

$$a(n)=\frac{\left[(1+\sqrt{3})^{n}+(1-\sqrt{3})^{n}\right]}{4}+3.$$

Proof. It is trivial that $c(1)$, $a(1)$, $c(2)$ and $a(2)$ are 1, 2, 4, and 5, respectively. By recurrence relations in Theorem 1, $c(n)=2\cdot a(n-1)-3$ and $a(n)=2\cdot a(n-1)=c(n-1)-6$, we have $c(n)=[(1+\sqrt{3})^{n-1}+(1-\sqrt{3})^{n-1}]/2+3$ and $a(n)=[(1+\sqrt{3})^{n}+(1-\sqrt{3})^{n}]/4+3$, for $n \geqslant 3$. □

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