

$(s, r; \mu)$ -nets and alternating forms graphs

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Abstract

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The equivalence between Bruck nets and mutually orthogonal latin squares is extended to $(s, r; \mu)$ -nets and mutually orthogonal quasi frequency squares. We investigate geometries arising from classical forms such as bilinear forms, alternating bilinear forms, hermitian forms and symmetric forms and show that $(s, r; \mu)$ -nets provide the right building blocks for each of these geometries with suitable values of μ . Toward the goal of geometric classification of distance-regular graphs, the local structure of the case of alternating forms graphs is stressed.

1. Introduction

The structure of $(s, r; \mu)$ -nets includes Bruck nets as the special case of $\mu = 1$ and their duals are transversal designs $TD_\mu[r, s]$ introduced by Hanani [6]. Indeed, $(s, r; \mu)$ -nets are equivalent to affine designs $S_r(1, s\mu, s^2\mu)$ and to orthogonal arrays $OA_\mu(s, r)$, and in this language they have been studied since around 1945. A survey on the geometric and group-theoretic aspects of $(s, r; \mu)$ -nets can be found in [11], where problems concerning existence, completion and geometric configurations are emphasized.

In Section 2, we recall the notion of $(s, r; \mu)$ -nets, the procedure of 'inflation' is used for constructing $(s, r; \mu)$ -nets from existing $(s, r; 1)$ -nets. In Section 3, the notion of quasi frequency squares is introduced and then we prove the equivalence between $(s, r; \mu)$ -nets and sets of mutually orthogonal quasi frequency squares, which includes the well-known relationship between Bruck nets and sets of mutually orthogonal latin

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squares as a special case. After reviewing how nets (with $\mu = 1$) provide the right building blocks for the lower semilattice $\mathcal{L}_q(V, W)$ [4] by using Sprague's [14] result on the characterization of d -dimensional nets, we study in Section 4 the geometries associated with classical forms (alternating bilinear forms, symmetric forms, hermitian forms and bilinear forms) and show how their local structures involve $(s, r; \mu)$ -nets (with $\mu \geq 2$). We also emphasize the relationships between these geometries and the association schemes of affine type carried by each family of the above classical forms. In the final section, we further investigate the geometric properties of the association schemes defined over alternating bilinear forms, hence covering the initial step toward the problem of characterization of their graphs by their intersection arrays.

2. $(s, r; \mu)$ -nets

In this section, first we shall recall the notion of $(s, r; \mu)$ -nets, and then a specified class of $(s, r; \mu)$ -nets which can be obtained from given $(s, r; 1)$ -nets by the procedure of 'inflation' will be studied. The diagram



is introduced for the class of duals of $(s, r; \mu)$ -nets.

A finite incidence structure $\Pi = (\mathcal{P}, \mathcal{B}, \epsilon)$ is called a $(s, r; \mu)$ -net of multiplicity μ if the block set \mathcal{B} can be partitioned into r ($r \geq 3$) block classes $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_r$ such that

- (N1) the blocks of each block class \mathcal{B}_i form a partition of \mathcal{P} ,
- (N2) any two blocks from distinct block classes meet at μ points,
- (N3) one of the block classes consists of s blocks.

Since $r \geq 3$, it follows that each block class $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_r$ consists of s blocks, each block in \mathcal{B} consists of $s\mu$ points, each point lies on exactly r blocks and, hence, $|\mathcal{P}| = s^2\mu$.

For an $(s, r; \mu)$ -net, it is known that [11] $r \leq (s^2\mu - 1)/(s - 1)$, with equality holding if and only if it is an affine 2-design. Indeed, if $r = (s^2\mu - 1)/(s - 1)$, then any two points are on precisely $\lambda = (s\mu - 1)/(s - 1)$ blocks. The sets $T(r, \mu)$ of integers s for which $(s, r; \mu)$ -nets exist were investigated by Hanani [6]; in particular, he showed that $s \in T(7, \mu)$ for every $s \geq 1$ and every $\mu \geq 2$. Recently, Ray-Chaudhuri and Singhi [13] have shown that, for given r and s , there exists an $(s, r; \mu)$ -net whenever μ is sufficiently large. $(s, r; 1)$ -nets are simply the well-studied classical Bruck nets; for example, the existence of an $(s, r; 1)$ -net is equivalent to the existence of each of the following structures:

- (1) $r - 2$ mutually orthogonal latin squares of order s ,
- (2) an orthogonal array $OA(s, r)$,
- (3) an affine design $S_r(1, s, s^2)$,
- (4) a transversal design $TD[r, s]$.

The intersections among blocks would shed light on the structures of $(s, r; \mu)$ -nets, in particular when $\mu \geq 2$. Let $\Pi = (\mathcal{P}, \mathcal{B}, \epsilon)$ be a $(s, r; \mu)$ -net; for any two blocks B and B' of Π , we write $B // B'$ if they are in the same class (so $B \cap B' = \emptyset$), and $B \# B'$ otherwise (so

$|B \cap B'| = \mu$). For $B \in \mathcal{B}_i$, and $j \neq i$, let

$$\begin{aligned}\mathcal{E}_j(B) &= \{B \cap B' \mid B' \in \mathcal{B}_j\}, \\ \mathcal{E}(B) &= \bigcup_{\substack{1 \leq j \leq r \\ j \neq i}} \mathcal{E}_j(B).\end{aligned}$$

Clearly, each $\mathcal{E}_j(B)$, $1 \leq j \leq r$ and $i \neq j$, forms a partition of B . For $1 \leq i, j \leq r$, $i \neq j$, let

$$\mathcal{E}_{ij} = \{B \cap B' \mid B \in \mathcal{B}_i, B' \in \mathcal{B}_j\},$$

and

$$\mathcal{E} = \bigcup_{\substack{1 \leq i, j \leq r \\ i \neq j}} \mathcal{E}_{ij}.$$

Then \mathcal{E}_{ij} forms a partition of \mathcal{P} , and elements of $\mathcal{E} \subseteq \binom{\mathcal{P}}{\mu}$ are called lines. For distinct pairs (i, j) and (i', j') , where $1 \leq i, i', j, j' \leq r$, $i \neq j$, $i' \neq j'$, the two partitions \mathcal{E}_{ij} and $\mathcal{E}_{i'j'}$ of the point set \mathcal{P} do not necessarily coincide. An interesting special case is when all such partitions coincide, i.e., $\mathcal{E}_{ij} = \mathcal{E}_{i'j'}$ for all $1 \leq i, j, i', j' \leq r$, with $i \neq j$, $i' \neq j'$. This happens exactly when the following condition (*) holds:

- (*) For any two intersecting blocks B and B' (say $B \in \mathcal{B}_1$, $B' \in \mathcal{B}_2$), there exist blocks $B_i \in \mathcal{B}_i$, $1 \leq i \leq r$, such that $B \cap B' = \bigcap_{1 \leq i \leq r} B_i$.

On the other hand, we may define a relation \simeq on \mathcal{P} in such a way that, for any $x, y \in \mathcal{P}$, $x \simeq y$ if and only if $x, y \in B \cap B'$ for some distinct blocks $B, B' \in \mathcal{B}$. Under condition (*), the relation \simeq is clearly an equivalence relation on \mathcal{P} , with \mathcal{E} as its family of equivalent classes. The above observations are summarized in the following theorem.

Theorem 2.1. An $(s, r; \mu)$ -net $\Pi = (\mathcal{P}, \mathcal{B}, \epsilon)$ satisfies condition (*) if and only if the associated incidence structure $\Pi / \simeq = (\mathcal{E}, \mathcal{B}, \subseteq)$ is an $(s, r; 1)$ -net.

The following procedure of inflation will provide us $(s, r; \mu)$ -nets satisfying condition (*) from existing $(s, r; 1)$ -nets. Let $\Pi = (\mathcal{P}, \mathcal{B}, \epsilon)$ be a given $(s, r; 1)$ -net and μ be a given positive integer. Let

$$\tilde{\mathcal{P}} = \{(x, i) \mid x \in \mathcal{P} \text{ and } 1 \leq i \leq \mu\},$$

$$\tilde{\mathcal{B}} = \{(x, i) \mid x \in B \text{ and } 1 \leq i \leq \mu\},$$

where $B \in \mathcal{B}$, and

$$\tilde{\mathcal{B}} = \{\tilde{B} \mid B \in \mathcal{B}\}.$$

Then one says that the incidence structure $\tilde{\Pi} = (\tilde{\mathcal{P}}, \tilde{\mathcal{B}}, \epsilon)$ is obtained from $\Pi = (\mathcal{P}, \mathcal{B}, \epsilon)$ by inflation. The following can be easily verified.

Theorem 2.2. (1) The incidence structure $\tilde{\Pi} = (\tilde{\mathcal{P}}, \tilde{\mathcal{B}}, \epsilon)$ is a $(s, r; \mu)$ -net satisfying condition (*).

(2) If $\Pi = (\mathcal{P}, \mathcal{B}, \epsilon)$ is a $(s, r; \mu)$ -net satisfying (*), then $\Pi = (\mathcal{P}, \mathcal{B}, \epsilon)$ can be obtained from the $(s, r; 1)$ -net $\Pi / \simeq = (\mathcal{E}, \mathcal{B}, \subseteq)$ by inflation.

Certainly, $(s, r; \mu)$ -nets $\Pi = (\mathcal{P}, \mathcal{B}, \in)$ satisfying $(*)$ possess tighter structures than general $(s, r; \mu)$ -nets do. For instance, let $\Gamma_1(x) = \{z \mid z \in \mathcal{P} \text{ and } x, z \text{ are in a common block}\}$,

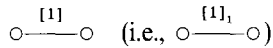
(1) If $B \in \mathcal{B}$ and x is not in B , then $\Gamma_1(x) \cap B$ consists of $r - 1$ pairwise disjoint lines, and $|\Gamma_1(x) \cap B| = \mu(r - 1)$.

(2) If $x, y \in \mathcal{P}$ are not collinear, then the blocks containing x or y can be indexed as B_i and B'_j , where $1 \leq i, j \leq r$ such that $B_i \neq B'_j$ whenever $i \neq j$ and $B_i // B'_i, 1 \leq i \leq r$.

For a prime power q , a specific class of $(q^{n-1}, q + 1; q)$ -nets satisfying $(*)$ and related to alternating bilinear forms defined over finite dimensional vector spaces over finite field $\text{GF}(q)$ will be studied in Section 5. In order to classify d -injection geometries, the diagram



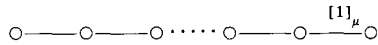
was used by Deza and Laurent [5], for a class of rank-2 geometries, i.e., d -transversal planes, where $d \geq 1$ is an integer. When $d = 1$, any incidence geometry belonging to



is simply the dual of an $(s, r; 1)$ -net, i.e., $\text{TD}(r, s)$. Following this line, the class of the duals of $(s, r; \mu)$ -nets will be denoted by the diagram



in the rest of this paper. In Section 4, we shall show that geometries associated with classical association schemes of affine type belong to diagram



with suitable choices of μ . The reader is referred to [1] for the details of association schemes and to [2] for the details of diagram geometries.

3. $(s, r; \mu)$ -nets and quasi frequency squares

In this section, relaxing the condition required for frequency squares [12], we shall introduce the notion of quasi frequency squares, which includes latin squares and frequency squares as special cases. Moreover, the well-known equivalence between $(s, r; 1)$ -nets and sets of mutually orthogonal latin squares can be generalized to $(s, r; \mu)$ -nets and sets of mutually orthogonal quasi frequency squares.

A *frequency square* $F(n; \mu_1, \mu_2, \dots, \mu_s)$ of order n (FS for short) is an $n \times n$ array with entries from the set $[1, s] = \{1, 2, \dots, s\}$ with the property that each symbol $i \in S$ occurs exactly μ_i times in each row and each column. Clearly, $n = \mu_1 + \mu_2 + \dots + \mu_s$ and an $F(n; 1, 1, \dots, 1)$ frequency square of order n is simply a latin square of order n . Two frequency squares $F(n; \mu_1, \mu_2, \dots, \mu_{s_1})$ and $F(n; \nu_1, \nu_2, \dots, \nu_{s_2})$ are said to be *orthogonal* if each ordered pair (i, j) of symbols occurs exactly $\mu_i \nu_j$ times for all $i \in [1, s_1]$ and

$j \in [1, s_2]$ when the square F_2 is superimposed on the square F_1 . A set $\{F_1, F_2, \dots, F_r\}$ of $r \geq 2$ frequency squares is said to be mutually orthogonal if F_i is orthogonal to F_j whenever $i \neq j$. In what follows, we shall consider only frequency squares with $\mu_1 = \dots = \mu_s = \mu$; a frequency square of this type is called a frequency square $F(s, \mu)$ of frequency μ on s symbols. Such frequency squares can easily be constructed from latin squares by the following inflation procedure: Take a latin square L of order s , replace each point $(x, y) \in [1, s]^2$ by the $\mu \times \mu$ matrix whose entries are all equal to $L(x, y)$; then what we obtain is a $\mu s \times \mu s$ matrix F which is obviously a FS of frequency μ on s symbols.

We need some more notations to introduce the notion of quasi frequency squares. A partition \mathcal{A} of $[1, n]^2$ is called a *row partition* if its classes are row-closed, i.e., for any class $A \in \mathcal{A}$, $(a, y) \in A$ for all $y \in [1, n]$ whenever $(a, b) \in A$ for some b . Obviously, a row partition of $[1, n]^2$ is uniquely determined by a partition of $[1, n]$. Similarly, a *column partition* is a partition \mathcal{C} of $[1, n]^2$ whose classes are column-closed. For a given $n = \mu s$, with $\mu, s \geq 1$, let \mathcal{I}_0 denote the partition of $[1, n]$ into s classes $\{[(k-1)\mu + 1, k\mu] \mid 1 \leq k \leq s\}$, where $[(k-1)\mu + 1, k\mu]$ is the set of all integers between $(k-1)\mu + 1$ and $k\mu$ (included), and let $\mathcal{A}_0, \mathcal{C}_0$ denote the row partition and the column partition, respectively, of $[1, n]^2$ determined by \mathcal{I}_0 . Hence, both \mathcal{A}_0 and \mathcal{C}_0 consist of s classes, each class consists of $\mu^2 s$ points, and any two classes $A \in \mathcal{A}_0, C \in \mathcal{C}_0$ intersect in μ^2 points.

A *quasi frequency square* of frequency μ on s symbols (QFS for short) is an $n \times n$ array, $n = \mu s$, with the property that each symbol $i \in [1, s]$ occurs exactly μ^2 times in each class A of \mathcal{A}_0 and in each class C of \mathcal{C}_0 . Clearly, any frequency square $F(s, \mu)$ is a quasi frequency square, and a QFS $F(s, 1)$ is simply a latin square of order s . Orthogonality between QFS can be defined similarly as before, i.e., each ordered pair (i, j) of symbols occurs exactly μ^2 times for $i, j \in [1, s]$ when one QFS is superimposed over another.

Let $\{F_1, \dots, F_r\}$ be r mutually orthogonal QFS of frequency μ on s symbols $[1, s]$. Let $n = \mu s$; we set

$$\mathcal{P} = [1, n]^2,$$

$$\mathcal{B}_i = \{[i, \alpha] \mid 1 \leq \alpha \leq s\}, \quad 1 \leq i \leq r,$$

where

$$[i, \alpha] = \{(x, y) \in \mathcal{P} \mid F_i(x, y) = \alpha\},$$

and

$$\mathcal{B} = \bigcup_{i \leq i \leq r} \mathcal{B}_i.$$

Some easy observations are as follows:

(1) Each block in \mathcal{B} consists of $\mu^2 s$ points, and each family \mathcal{B}_i consists of s blocks which form a partition of \mathcal{P} .

(2) Any two blocks from distinct families $\mathcal{B}_i, \mathcal{B}_j$ intersect in μ^2 points (by the orthogonality among $\{F_1, \dots, F_r\}$). Hence, the incidence structure

$$\Pi_{\text{QFS}} = \left([1, n]^2, \bigcup_{1 \leq i \leq r} \mathcal{B}_i, \in \right)$$

is a $(s, r; \mu^2)$ -net.

(3) By the assumption that each F_i is a QFS, one can extend the above QFS-net by adding two more block classes \mathcal{A}_0 and \mathcal{C}_0 , i.e.,

$$\Pi_0 = \left([1, n]^2, \left(\bigcup_{1 \leq i \leq r} \mathcal{B}_i \right) \cup \mathcal{A}_0 \cup \mathcal{C}_0, \epsilon \right)$$

is a $(s, r + 2; \mu^2)$ -net.

Equivalence indeed holds between sets of mutually orthogonal QFS and nets of multiplicity μ^2 as shown in the next theorem.

Theorem 3.1. *The following assertions are equivalent:*

- (1) *there exist r mutually orthogonal quasi frequency squares of frequency μ on s symbols,*
- (2) *there exists a $(s, r + 2; \mu^2)$ -net.*

Proof. It remains to show that (2) implies (1). Let $\Pi = (\mathcal{P}, \mathcal{B}, \epsilon)$ be a $(s, r + 2, \mu^2)$ -net and let $\mathcal{A}, \mathcal{C}, \mathcal{B}_1, \dots, \mathcal{B}_r$ denote its $r + 2$ block classes. Then the point set \mathcal{P} can be partitioned into the s^2 sets $A \cap C$ of size μ^2 , where $A \in \mathcal{A}$ and $C \in \mathcal{C}$. It is easy to see that this enables us to represent \mathcal{P} as $[1, n]^2$, $n = \mu s$, in such a way that \mathcal{A}, \mathcal{C} are the row partition \mathcal{A}_0 and the column partition \mathcal{C}_0 , respectively. For $1 \leq j \leq r$, let

$$\mathcal{B}_j = \{B_1^{(j)}, \dots, B_s^{(j)}\}.$$

We define an $n \times n$ matrix F_j , with entries in $\{1, 2, \dots, s\}$, by setting $F_j(x, y) = k$ if and only if $(x, y) \in B_k^{(j)}$. Since Π is a $(s, r + 2; \mu^2)$ -net,

$$|B_k^{(j)} \cap A| = |B_k^{(j)} \cap C| = \mu^2$$

for all $A \in \mathcal{A}, C \in \mathcal{C}, B_k^{(j)} \in \mathcal{B}_j$, and $|B_k^{(i)} \cap B_k^{(j)}| = \mu^2$ whenever $i \neq j$. It follows that each F_j is a QFS of frequency μ and any two F_i, F_j are orthogonal. Hence, $\{F_1, \dots, F_r\}$ is a set of r mutually orthogonal QFS of frequency μ on s symbols. \square

Corollary 3.2. *If there exist r mutually orthogonal frequency squares of frequency μ on s symbols, then there exists a $(s, r + 2; \mu^2)$ -net.*

We now turn to the question of determining whether it is possible to derive a set of r mutually orthogonal FS from a $(s, r + 2; \mu^2)$ -net. In fact, as shown in the following example, this is not true in general.

Example 3.3. Let $\mathcal{P} = [1, 6]^2$, $\mathcal{A}_0 = \{A_1, A_3, A_5\}$, $\mathcal{C}_0 = \{C_1, C_3, C_5\}$ and $\mathcal{B} = \{B_1, B_2, B_3\}$, where $A_i = \{(i, y), (i + 1, y) \mid 1 \leq y \leq 6\}$, $C_i = \{(y, i), (y, i + 1) \mid 1 \leq y \leq 6\}$ for $i = 1, 3, 5$ and

$$B_1 = \{(1, 1), (2, 3), (2, 5), (2, 6), (3, 1), (4, 3), (4, 4), (4, 6), (5, 1), (5, 2), (5, 5), (6, 3)\},$$

$$B_2 = \{(1, 2), (1, 5), (2, 1), (2, 4), (3, 4), (3, 6), (4, 2), (4, 5), (5, 3), (5, 6), (6, 2), (6, 4)\},$$

$$B_3 = \{(1, 3), (1, 4), (1, 6), (2, 2), (3, 2), (3, 3), (3, 5), (4, 1), (5, 4), (6, 1), (6, 5), (6, 6)\}.$$

Then $\Pi = (\mathcal{P}, \mathcal{A}_0 \cup \mathcal{C}_0 \cup \mathcal{B}, \epsilon)$ forms a (3, 3; 4)-net. Clearly, Π correspond to a QFS of frequency 2 on 3 symbols, but it is certainly not a frequency square as shown in the following diagram, where value k , $1 \leq k \leq 3$, appears at position (i, j) , indicating that $(i, j) \in B_k$.

1	2	3	3	2	3
2	3	1	2	1	1
1	3	3	2	3	2
3	2	1	1	2	1
1	1	2	3	1	2
3	2	1	2	3	3

Quite naturally, if we require that a $(s, r + 2; \mu^2)$ -net satisfies (*), mentioned in Section 2, i.e., be the inflation of some $(s, r + 2; 1)$ -net, then there exists a set of r mutually orthogonal FS which, in fact, are the inflation of some mutually orthogonal latin squares.

Theorem 3.4. *If there exists a $(s, r + 2; \mu^2)$ -net which satisfies condition (*), then there exist r mutually orthogonal frequency squares of frequency μ on s symbols.*

Proof. Following the notation used in the proof of Theorem 3.1, it only remains to verify that, for all $1 \leq j \leq r$ and $B \in \mathcal{B}_j$, B contains exactly μ elements in each row and in each column. Fix $A \in \mathcal{A}$, $B = B_k^{(j)} \in \mathcal{B}_j$. By condition (*), we can find $C \in \mathcal{C}$ such that $A \cap B = A \cap B \cap C = A \cap C$; this implies that $F_j(x, y) = k$ for all $(x, y) \in A \cap C$ and, hence, F_j is indeed the inflation of a latin square of order s . \square

Of course, we would like to obtain a set of mutually orthogonal frequency squares from a $(s, r + 2; \mu^2)$ -net by using conditions milder than (*). One step towards this direction is to observe that, for a given set of r mutually orthogonal FS, one can sometimes derive a $(s, r'; \mu^2)$ -net with $r' \geq r + 3$, i.e., one can define additional block classes besides the families $\mathcal{A}_0, \mathcal{C}_0, \mathcal{B}_1, \dots, \mathcal{B}_r$ constructed so far. This can be done if one can define row or column partitions other than \mathcal{A}_0 and \mathcal{C}_0 . Recall that \mathcal{I}_0 denotes the initial partition of $[1, n]$ which determines $\mathcal{A}_0, \mathcal{C}_0$; suppose that $\mathcal{I}_1 \neq \mathcal{I}_0$ is another partition of $[1, n]$ into s subsets of equal size μ and $\mathcal{A}_1, \mathcal{C}_1$ denote its associated row and column partitions. We can extend the $(s, r + 2; \mu^2)$ -net obtained from Corollary 3.2 by adding the two additional block classes $\mathcal{A}_1, \mathcal{C}_1$ if and only if $|A_0 \cap A_1| = |C_0 \cap C_1| = \mu^2$ for $A_i \in \mathcal{A}_1$ and $C_i \in \mathcal{C}_1, i = 0, 1$, i.e., $|I_0 \cap I_1| = \mu/s$ for $I_0 \in \mathcal{I}_0$ and $I_1 \in \mathcal{I}_1$; hence, s must be a divisor of μ . Therefore, no additional row or column partition of $[1, n]^2$ can be added to the list $\{\mathcal{A}_0, \mathcal{C}_0, \mathcal{B}_1, \dots, \mathcal{B}_r\}$ to form a $(s, r'; \mu)$ -net with $r' \geq r + 3$ whenever s is not a divisor of μ and, in particular, if $\mu < s$.

We now assume that s is a divisor of $\mu, \mu = \lambda s$, with $\lambda \geq 1$. Based on the above observations, two remarks are in order:

(1) Suppose $\rho \geq 3$ and there exists a $(s, \rho; \lambda)$ -net $\Pi = ([1, \mu s], \bigcup_{1 \leq i \leq \rho} \mathcal{I}_i, \epsilon)$ on $[1, \mu s]$ with block classes $\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_\rho$; so, each \mathcal{I}_i is a partition of $[1, \mu s]$ into

s equal parts. For $i \in [1, \rho]$, let $\mathcal{A}_i, \mathcal{C}_i$ denote the row and the column partition of $[1, \mu s]^2$, respectively, determined by \mathcal{I}_i . Then

$$\Pi_G = \left([1, \mu s]^2, \left(\bigcup_{1 \leq i \leq \rho} \mathcal{A}_i \right) \cup \left(\bigcup_{1 \leq i \leq \rho} \mathcal{C}_i \right), \epsilon \right)$$

is a $(s, 2\rho; \mu^2)$ -net on $[1, \mu s]^2$, called the grid net determined by Π .

(2) The $(s, r; \mu^2)$ -net

$$\Pi_{\text{QFS}} = \left([1, \mu s]^2, \bigcup_{1 \leq i \leq r} \mathcal{B}_i, \epsilon \right)$$

determined by the set of r mutually orthogonal frequency squares is always extendible by the grid net Π_G , i.e.,

$$\Pi_{\text{QFS}} * \Pi_G = \left([1, \mu s]^2, \left(\bigcup_{1 \leq i \leq \rho} \mathcal{A}_i \right) \cup \left(\bigcup_{1 \leq i \leq \rho} \mathcal{C}_i \right) \cup \left(\bigcup_{1 \leq i \leq r} \mathcal{B}_i \right), \epsilon \right)$$

is a $(s, r + 2\rho; \mu^2)$ -net. Therefore, the next theorem follows immediately.

Theorem 3.5. *Let $\mu = \lambda s$ and $\rho \geq 3$ be the largest integer for which there exists a $(s, \rho; \lambda)$ -net on $[1, \mu s]$. If there exists a set of r mutually orthogonal frequency squares of frequency μ on s symbols, then there exists a $(s, r + 2\rho; \mu^2)$ -net which is an extension of the grid net associated with the given $(s, \rho; \lambda)$ -net.*

Moreover, if there exists a $(s, s + 1; 1)$ -net, i.e., an affine plane of order s , and $\mu = s$, then Theorem 3.5 can be strengthened as Theorem 3.6.

Theorem 3.6. *If there exists a $(s, s + 1; 1)$ -net, then the following assertions are equivalent:*

(1) *There exists a set of r mutually orthogonal frequency squares of frequency s on s symbols.*

(2) *There exists a $(s, r + 2(s + 1); s^2)$ -net which is an extension of the grid net associated with the given $(s, s + 1; 1)$ -net.*

Proof. It remains to show that (2) implies (1). Let $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_s$ denote the block classes of the given $(s, s + 1; 1)$ -net defined on $[1, s^2]$. By assumption, the block classes of the $(s, r + 2(s + 1); s^2)$ -net Π can be denoted by $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_s, \mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_s$, and $\mathcal{B}_1, \dots, \mathcal{B}_r$ in such a way that $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_s$ are the row partitions and $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_s$ are the column partitions of $[1, s^2]^2$ determined by $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_s$. Let F_1, F_2, \dots, F_r be the set of r mutually orthogonal quasi frequency squares obtained from $\mathcal{B}_1, \dots, \mathcal{B}_r$ by Theorem 3.1. To show that each F_j is indeed a frequency square is equivalent to showing that each $B \in \mathcal{B}_j$ contains exactly s elements in each row and in each column. Let $B \in \mathcal{B}_i$ and α_i be the number of elements of B occurring in the i th row of $[1, s^2]^2$; hence, $\alpha_1 + \dots + \alpha_{s^2} = s^3$. Next we claim that $\alpha_i = s$ for all $1 \leq i \leq s^2$.

Let $I_0 \in \mathcal{I}_0, \dots, I_s \in \mathcal{I}_s$ be the $s + 1$ blocks of the given $(s, s + 1; 1)$ -net which contain element 1 of $[1, s^2]$; then the sets $\{1\}, I_0 - \{1\}, \dots, I_s - \{1\}$ form a partition of $[1, s^2]$. Let $A_i \in \mathcal{A}_i$ be the block of Π formed by the rows indexed by I_i for $0 \leq i \leq s$. Since Π is a $(s, r + 2(s + 1); s^2)$ -net, we have that $|B \cap A_i| = s^2$, i.e., $\sum_{j \in I_i} \alpha_j = s^2$ for $0 \leq i \leq s$. Summing the above $s + 1$ equations, we obtain that $(s + 1)\alpha_1 + \sum_{1 \leq j \leq s^2} \alpha_j - \alpha_1 = s^2(s + 1)$, and, since $\sum_{1 \leq j \leq s^2} \alpha_j = s^3$, it follows that $\alpha_1 = s$. The other cases $\alpha_i = s, 2 \leq i \leq s^2$, can be proved similarly. \square

4. (s, r; μ)-nets and association schemes of affine type

In this section, we shall introduce some geometries related to classical forms and then show that their local structures can be described as the dual of $(s, r; \mu)$ -nets with suitable parameters.

4.1. The lower semilattice $\mathcal{L}_q(U, V)$.

First, we shall concentrate on the lower semilattice $\mathcal{L}_q(U, V)$ of bilinear forms ($\mu = 1$). Let U, V be vector spaces of dimensions d and n , respectively, ($d \leq n$) over a finite field $\text{GF}(q)$, and $U + V$ be the direct sum of U and V . Let

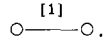
$$\mathcal{L}_i = \{A \mid A \subseteq U + V \text{ is an } i\text{-subspace and } A \cap V \text{ is trivial}\},$$

where $0 \leq i \leq d$. Note that the condition $A \cap V = \{0\}$ is equivalent to that of $\dim(\pi_1(A)) = \dim(A)$, where $\pi_1(A)$ is the projection of A onto the first summand U . Furthermore, if $A \in \mathcal{L}_d$, then $\dim(\pi_1(A)) = \dim(U)$ and, hence, there exists a linear transformation $f: U \rightarrow V$ such that $A = \{(x, f(x)) \mid x \in U\}$. It follows that, with respect to fixed bases of U and V , the set \mathcal{L}_d of roofs of the geometry

$$\mathcal{L} = (\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_d)$$

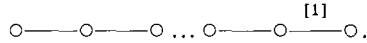
can be identified with the set $M_{d \times n}(\text{GF}(q))$ of all $d \times n$ matrices over $\text{GF}(q)$. Similarly, each other level \mathcal{L}_i can be identified with $\begin{bmatrix} U \\ i \end{bmatrix} \times M_{i \times n}(\text{GF}(q))$, $1 \leq i \leq d$, where $\begin{bmatrix} U \\ i \end{bmatrix}$ denotes the family of all i -dimensional subspaces of U . In other words, $\mathcal{L}_i = \{(f, Y) \mid Y \in \begin{bmatrix} U \\ i \end{bmatrix} \text{ and } f: Y \rightarrow V \text{ is linear}\}$. For (f, Y) and $(g, Z) \in \bigcup_{i=0}^d \mathcal{L}_i$, we define $(f, Y) \leq (g, Z)$ if and only if $Y \subseteq Z$ and the restriction $g|_Y$ of g on Y is identical with f ; (f, Y) and $(g, Z) \in \bigcup_{0 \leq i \leq d} \mathcal{L}_i$ are called comparable if either $(f, Y) \leq (g, Z)$ or $(g, Z) \leq (f, Y)$. Clearly, $\bigcup_{0 \leq i \leq d} \mathcal{L}_i$ is the lower semilattice $\mathcal{L}_q(U, V)$ introduced by Delsarte [4]. Any flag \mathcal{F} of type $\{0, 1, 2, \dots, d - 2\}$ in the geometry $\mathcal{L} = (\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_d)$ contains a unique $(f_{d-2}, U_{d-2}) \in \mathcal{L}_{d-2}$, where $U_{d-2} \in \begin{bmatrix} U \\ d-2 \end{bmatrix}$ and $f: U_{d-2} \rightarrow V$ is a linear transformation. The residue $\text{Res}(\mathcal{F})$ of \mathcal{F} in the geometry $\mathcal{L} = (\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_d)$ is defined to be $\mathcal{P} \cup \mathcal{B}$, where $\mathcal{P} = \{(f, A) \mid A \in \begin{bmatrix} U \\ d-1 \end{bmatrix}, f: A \rightarrow V \text{ is linear}, (f_{d-2}, U_{d-2}) \leq (f, A)\}$, and $\mathcal{B} = \{(g, U) \mid g: U \rightarrow V \text{ is linear}, (f_{d-2}, U_{d-2}) \leq (g, U)\}$. Elements in \mathcal{P} and \mathcal{B} are called points and blocks, respectively.

Theorem 4.1 (Sprague [14]). (1) *The incidence structure $\Pi=(\mathcal{P}, \mathcal{B}, \epsilon)$ associated with the residue $\text{Res}(\mathcal{F})$ of a flag \mathcal{F} of type $\{0, 1, 2, \dots, d-2\}$ in the geometry $\mathcal{L}=(\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_d)$ belongs to the diagram*



More specifically, the dual of $\text{Res}(\mathcal{F})$ is a $(q^n, q+1; 1)$ -net.

(2) *The geometry $\mathcal{L}=(\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_d)$ belongs to the diagram*



Any incidence structure Π isomorphic to the semilinear incidence structure $(\mathcal{L}_d, \mathcal{L}_{d-1}, \supseteq)$ is called an $(n, q; d)$ -attenuated space [14], or a d -attenuated space in short. These are examples of the following specific class of incidence structure with $(s, r; 1)$ -nets as their planes. A d -dimensional net is a connected semilinear incidence structure Π such that the following conditions hold:

- (D1) every plane is a $(s, r; 1)$ -net,
- (D2) the intersection of two subspaces is connected,
- (D3) if two planes in a 3-space of Π have a point in common, then they have a second point in common, and
- (D4) the minimum number of points which generate Π is $d+1$.

Sprague [14] proved that every finite d -dimensional net ($d \geq 3$) is an $(n, q; d)$ attenuated space for some finite field $\text{GF}(q)$. Sprague also characterized d -dimensional nets as the duals of those incidence structures belonging to the diagram in Theorem 4.1.

As mentioned before, the set \mathcal{L}_d of roofs of the geometry $\mathcal{L}=(\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_d)$ can be identified with the set $M_{d \times n}(\text{GF}(q))$ of all $d \times n$ matrices over $\text{GF}(q)$. Furthermore, it is worth mentioning here that it also carries the structure of $(P\&Q)$ -association schemes and, hence, distance-regular graphs. Set

$$\text{LR}_i = \{(A, B) \mid A, B \in M_{d \times n}(\text{GF}(q)), \text{rank}(A - B) = i\}, \quad 0 \leq i \leq d.$$

Then

- (i) LR_0 is the diagonal of $(M_{d \times n}(\text{GF}(q)))^2$, and $\{\text{LR}_i \mid 0 \leq i \leq d\}$ forms a partition of $(M_{d \times n}(\text{GF}(q)))^2$,
- (ii) the transpose LR_i^T of LR_i is identical with LR_i , $0 \leq i \leq d$.
- (iii) if $(A, B) \in \text{LR}_k$, then $|\{C \mid C \in M_{d \times n}(\text{GF}(q)), (A, C) \in \text{LR}_i \text{ and } (C, B) \in \text{LR}_j\}|$ is a constant p_{ij}^k which is independent of the choice of A and B .

In other words, $(M_{d \times n}(\text{GF}(q)), \{\text{LR}_i \mid 0 \leq i \leq d\})$ forms a symmetric association scheme of d classes. The reader is referred to [1] for more details about association schemes. Furthermore, $(M_{d \times n}(\text{GF}(q)), \text{LR}_1)$ turns out to be a distance-regular graph with LR_1 as its edge set, denoted by $H_q(d, n)$, and, indeed, a distance-transitive graph of diameter d .

The d -shadow $\sigma_d(g, A)$ of $(g, A) \in \mathcal{L}_r$ is defined to be $\{(f, U) \mid (f, U) \in \mathcal{L}_d \text{ and } (g, A) \leq (f, U)\}$. It is easy to see that $\sigma_d(g, A)$ is a $\{0, 1, \dots, d-r\}$ -clique in the graph $H_q(d, n)$, i.e., $\text{rank}(A - B) \leq d-r$ for $A, B \in \sigma_d(g, A)$. Indeed, Huang [9] proved

that each maximum $\{0, 1, \dots, d-r\}$ -clique of $H_q(d, n)$ is of the above form as d -shadow of some $(g, A) \in \mathcal{L}_r$ whenever $n \geq d+1$ and $(n, q) \neq (d+1, 2)$. In particular, $\{\sigma_d(g, A) \mid (g, A) \in \mathcal{L}_{d-1}\} \subseteq \mathcal{L}_d$ can be identified with the set of all maximum cliques of $H_q(d, n)$.

Toward the program of classifying distance-regular graphs, Huang [8] proved that the above-mentioned distance-regular graph $H_q(d, n)$ defined on $\mathcal{L}_d = M_{d \times n}(\text{GF}(q))$ is uniquely determined by its intersection array, subject to some extra conditions by using Sprague's characterization of d -dimensional nets. These extra conditions were modified by Cuypers [3] recently. Following similar approaches, Yokoyama [16] proved similar results in the context of distance-transitive graphs which includes even the square case $d = n$.

4.2. Geometries for classical forms

In the second half of this section, we concentrate on square matrices as well as some examples of $(s, r; \mu)$ -nets with $\mu \geq 2$. Let V be a vector space of dimension n over a finite field $\text{GF}(q)$, where $q = p^m$ is a prime power, $U \subseteq V$ be a subspace of dimension i . Let $\text{Bil}(U)$, $\text{Alt}(U)$, $\text{Her}(U)$ and $\text{Sym}(U)$ be, respectively, the set of all bilinear forms, alternating bilinear forms, hermitian forms and symmetric forms defined on U . We assume that $p \neq 2$ in the case of alternating forms and $m = 2r$ is even in the case of hermitian forms. Then $\text{Bil}(U)$, $\text{Alt}(U)$, $\text{Sym}(U)$ are vector spaces of dimensions i^2 , $i(i-1)/2$, $i(i+1)/2$, respectively, over $\text{GF}(q)$, and $\text{Her}(U)$ is a vector space of dimension i^2 over $\text{GF}(p^r)$. $\text{Bil}(U)$, $\text{Alt}(U)$, $\text{Sym}(U)$ and $\text{Her}(U)$ will be denoted by $\text{Bil}(i, q)$, $\text{Alt}(i, q)$, $\text{Sym}(i, q)$ and $\text{Her}(i, q)$, respectively when there is no confusion. Let

$$\mathcal{A}_i = \{(f, U) \mid U \in [i]_V \text{ and } f \in \text{Alt}(U)\},$$

$$\mathcal{H}_i = \{(f, U) \mid U \in [i]_V \text{ and } f \in \text{Her}(U)\},$$

$$\mathcal{S}_i = \{(f, U) \mid U \in [i]_V \text{ and } f \in \text{Sym}(U)\},$$

and

$$\mathcal{B}_i = \{(f, U) \mid U \in [i]_V \text{ and } f \in \text{Bil}(U)\},$$

where $0 \leq i \leq n$. Let \mathcal{A} denote the geometry $(\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n)$ and the other geometries \mathcal{S} , \mathcal{B} and \mathcal{H} are defined similarly.

As mentioned for the geometry $\mathcal{L} = (\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_n)$, the notions of comparability, flags and residues can be similarly defined for geometries \mathcal{A} , \mathcal{S} , \mathcal{B} and \mathcal{H} . For example, for the geometry $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n)$, $(f, U) \leq (g, W)$ if and only if $U \subseteq W$ and the restriction $g|_U$ of g on U is identical with f whenever $(f, U) \in \mathcal{A}_i$, $(g, W) \in \mathcal{A}_j$. Any $(f, U), (g, W) \in \mathcal{A}_i \cup \mathcal{A}_j$ are comparable if either $(f, U) \leq (g, W)$ or $(g, W) \leq (f, U)$. The residue of a flag \mathcal{F} of type $\{1, 2, \dots, n-2\}$ is uniquely determined by some $(f_{n-2}, U_{n-2}) \in \mathcal{A}_{n-2}$, where $U_{n-2} \subseteq V$ is a subspace of dimension $n-2$ and $f_{n-2} \in \text{Alt}(U_{n-2})$; more specifically, the residue $\text{Res}(\mathcal{F})$ of the flag \mathcal{F} is defined to be $\mathcal{P} \cup \mathcal{B}$, where $\mathcal{P} = \{(f, U) \mid U \in [n-1]_V, f \in \text{Alt}(U), (f_{n-2}, U_{n-2}) \leq (f, U)\}$ and $\mathcal{B} = \{(g, V) \mid g \in \text{Alt}(V), (f_{n-2}, U_{n-2}) \leq (g, V)\}$. Elements in \mathcal{P} and \mathcal{B} are called points

and blocks, respectively. With respect to the induced incidence relation, we shall show that the incidence structure $\Pi = (\mathcal{P}, \mathcal{B}, \epsilon)$ associated with the residue $\text{Res}(\mathcal{F})$ of a flag \mathcal{F} of type $\{0, 1, \dots, n-2\}$ in the geometries $\mathcal{A}, \mathcal{H}, \mathcal{S}$ and \mathcal{B} belongs to the diagram

$$\circ \xrightarrow{[1]_{\mu}} \circ$$

with a suitable choice of μ for each case.

Theorem 4.2. (1) *The residue $\text{Res}(\mathcal{F})$ of a flag \mathcal{F} of type $\{0, 1, 2, \dots, n-2\}$ of the geometries $\mathcal{A}, \mathcal{H}, \mathcal{S}$ and \mathcal{B} belongs to the diagram*

$$\circ \xrightarrow{[1]_{q^i}} \circ,$$

where $i = 1, 2, 3$ and 4 , respectively. More specifically, the dual of the residue $\text{Res}(\mathcal{F})$ in the geometry \mathcal{A} is a $(q^{n-2}, q+1; q)$ -net satisfying $(*)$.

(2) *The geometries $\mathcal{A}, \mathcal{H}, \mathcal{S}$ and \mathcal{B} belong to the diagram*

$$\circ \text{---} \circ \text{---} \circ \dots \circ \text{---} \circ \xrightarrow{[1]_{q^i}} \circ$$

where $i = 1, 2, 3$ and 4 , respectively.

Proof. Only the proof for the geometry $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n)$ is given. Similar arguments work for the other geometries \mathcal{H}, \mathcal{S} and \mathcal{B} . Consider the partition

$$\left\{ \{(f, U) \mid f \in \text{Alt}(U) \text{ and } f|_{U_{n-2}} = f_{n-2}\} \mid U \in \begin{bmatrix} V \\ n-1 \end{bmatrix}, \text{ with } U_{n-2} \subseteq U \right\}$$

of the point set of $\text{Res}(\mathcal{F})$. Obviously, any block (f, V) is incident with exactly one point of each of the above classes.

For two distinct points $(f, U), (f', U')$ from distinct classes, $U \cap U' = U_{n-2}$ and $f|_{U_{n-2}} = f'|_{U_{n-2}} = f_{n-2}$. Fix a base $\{v_1, \dots, v_n\}$ of V such that $\{v_3, \dots, v_n\}$ is a base of U_{n-2} , $v_1 \in U - U_{n-2}$ and $v_2 \in U' - U_{n-2}$; those blocks (h, V) which go through (f, U) and (f', U') can be uniquely represented by matrices of the form

$$\begin{bmatrix} 0 & \alpha & \alpha_3 & \cdots & \alpha_n \\ -\alpha & 0 & \alpha'_3 & \cdots & \alpha'_n \\ -\alpha_3 & -\alpha'_3 & & & \\ \vdots & \vdots & & f_{n-2} & \\ -\alpha_n & -\alpha'_n & & & \end{bmatrix},$$

where $\alpha_j, \alpha'_j, 3 \leq j \leq n$, are uniquely determined by f_{n-2}, f and f' , respectively. Since there are q choices for $\alpha = h(v_1, v_2)$, it follows that there are q blocks which go through both (f, U) and (f', U') , as required. \square

Remark. The proof of the second part of (1) can be found in [10].

With respect to a fixed base of V , each member in the sets of roofs $B(n, q)$, $A(n, q)$, $H(n, q)$, $S(n, q)$ of the geometries \mathcal{B} , \mathcal{A} , \mathcal{H} and \mathcal{S} , respectively, can be expressed as an $n \times n$ matrix, $n \times n$ antisymmetric matrix with zero diagonal, $n \times n$ hermitian matrix and an $n \times n$ symmetric matrix, respectively. Furthermore, each set of roofs also carries the structure of $(P\&Q)$ -association scheme, as we mentioned before for $\mathcal{L}_q(U, W)$. Let

$$AR_i = \{(A, B) \mid A, B \in \text{Alt}(n, q) \text{ and } \text{rank}(A - B) = 2i\},$$

$$SR_i = \{(A, B) \mid A, B \in \text{Sym}(n, q) \text{ and } \text{rank}(A - B) = 2i - 1, 2i\},$$

where $0 \leq i \leq \lfloor n/2 \rfloor$ ($=d$), and

$$BR_i = \{(A, B) \mid A, B \in \text{Bil}(n, q) \text{ and } \text{rank}(A - B) = i\},$$

$$HR_i = \{(A, B) \mid A, B \in \text{Her}(n, q) \text{ and } \text{rank}(A - B) = i\},$$

where $0 \leq i \leq n$. Both $(\text{Bil}(n, q), \{\text{BR}_i \mid 0 \leq i \leq n\})$ and $(\text{Her}(n, q), \{\text{HR}_i \mid 0 \leq i \leq n\})$ are $(P\&Q)$ -polynomial association schemes of n -classes and, indeed, $B\Gamma = (\text{Bil}(n, q), \text{BR}_1)$ and $H\Gamma = (\text{Her}(n, q), \text{HR}_1)$ turn out to be distance-regular graphs of diameter n , respectively. Similarly, $(\text{Alt}(n, q), \{\text{R}_i \mid 0 \leq i \leq d\})$ and $(\text{Sym}(n, q), \{\text{R}_i \mid 0 \leq i \leq d\})$ are also $(P\&Q)$ -polynomial association schemes of d -classes; moreover, $A\Gamma = (\text{Alt}(n, q), \text{AR}_1)$ and $S\Gamma = (\text{Sym}(n, q), \text{SR}_1)$ turn out to be distance-regular graphs of diameter d and, surprisingly, they share the same intersection array, i.e.,

$$b_i = q^{4i}(q-1)(q^{n-2i}-1)(q^{n-2i}-q)/(q^2-1)(q^2-q), \quad 0 \leq i \leq d-1$$

and

$$c_i = q^{2i-2}(q^{2i}-1)/(q^2-1), \quad 1 \leq i \leq d.$$

Recall that, for a distance-regular graph Γ of diameter d , $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$ for $0 \leq i \leq d-1$ and $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$ for $1 \leq i \leq d$, where $x, y \in V(\Gamma)$ are at distance i and $\Gamma_i(x) = \{z \mid z \in V(\Gamma) \text{ and } d(x, z) = i\}$.

The above observations show that the distance-regular graphs defined over $\text{Alt}(n, q)$ and $\text{Sym}(n, q)$, respectively, are not characterized by their intersection arrays, nor by the diagram geometries they belong to.

5. Alternating-forms graphs

Toward the goal of a geometric classification of the family of distance-regular graphs $\text{Alt}(n, q)$, in this section, we shall provide more detailed analysis of the local structure of $\text{Alt}(n, q)$. The notion of pseudo-alternating incidence structures is introduced in the hope that, in addition to its diagram, the geometry $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n)$ and its adjacency graph $\text{Alt}(n, q)$ could be characterized in terms of it.

Let us start from the maximal cliques of the graph $\text{Alt}(n, q)$. Since it is distance-transitive, we are concerned only about those maximal cliques which contain the zero form.

Theorem 5.1 (Hemmeter [7]). *If \mathcal{C} is a maximal clique in $\text{Alt}(n, q)$ which contains the zero form, then either $\bigcup_{f \in \mathcal{C}} \text{Rad}(f)$ is contained in a hyperplane of V or $\bigcap_{f \in \mathcal{C}} \text{Rad}(f)$ is an $(n-3)$ -dimensional subspace of V .*

Hence, up to isomorphism, there are two types of maximal cliques in $\text{Alt}(n, q)$; cliques of the first type are of size q^{n-1} , and the others are of size q^3 . If $(f_{n-1}, U_{n-1}) \in \mathcal{A}_{n-1}$, i.e., $U_{n-1} \subseteq V$ is a subspace of dimension $n-1$ and $f_{n-1} \in \text{Alt}(U_{n-1})$, without loss of generality, we may assume that f_{n-1} is the zero form. The n th shadow $\sigma_n(f_{n-1}, U_{n-1})$ of (f_{n-1}, U_{n-1}) is defined to be $\{(f, V) \mid f \in \text{Alt}(n, q) \text{ and } f|_{U_{n-1}} = f_{n-1}\}$. An immediate consequence is the following corollary.

Corollary 5.2. *For each $(f_{n-1}, U_{n-1}) \in \mathcal{A}_{n-1}$, its shadow $\sigma_n(f_{n-1}, U_{n-1}) \subseteq \text{Alt}(n, q)$ is a maximum clique of $\text{Alt}(n, q)$ of size q^{n-1} , and vice versa.*

Remark. A similar result holds for $\mathcal{L}_q(U, V)$ and the distance-regular graphs $H_q(d, n)$, as mentioned in Section 4.

Before studying incidence structures related to $\text{Alt}(n, q)$, it will be convenient to recall some notions about incidence structures $(\mathcal{P}, \mathcal{B}, \epsilon)$ with the property that $|B \cap B'| = 0$ or μ for distinct $B, B' \in \mathcal{B}$. Let $\mathcal{L} = \{B \cap B' \mid B, B' \in \mathcal{B} \text{ are distinct and } B \cap B' \neq \emptyset\}$, then $\mathcal{L} \subseteq \binom{\mathcal{P}}{\mu}$ and elements of \mathcal{L} will be called lines. A subset $S \subseteq \mathcal{P}$ is called block-closed if $|S \cap B| \geq \mu + 1$ for block $B \in \mathcal{B}$ implies that $B \subseteq S$. Similarly, a subset $S \subseteq \mathcal{P}$ is called line-closed if $|S \cap L| \geq 2$ for line $L \in \mathcal{L}$ implies that $L \subseteq S$. A subset $S \subseteq \mathcal{P}$ is called connected if any two points x, y of S are in a common block. A subset $S \subseteq \mathcal{P}$ is called a 2-subspace of Π if S is a smallest connected block-closed subset of \mathcal{P} which contains two intersecting blocks. For any point y , $\Gamma_1(y)$ is defined to be $\{z \mid z \in \mathcal{P} \text{ and } y, z \text{ are in a common block}\}$. For $Y \subseteq \mathcal{P}$, the set of common neighbors of Y is defined to be $\bigcap_{y \in Y} \Gamma_1(y)$.

Let $\Pi = (\text{Alt}(n, q), \mathcal{B}, \epsilon)$, where \mathcal{B} is the set of all maximum cliques of the alternating forms graph $\text{Alt}(n, q)$, i.e.,

$$\mathcal{B} = \{\sigma_n(f_{n-1}, U_{n-1}) \mid (f_{n-1}, U_{n-1}) \in \mathcal{A}_{n-1}\} \subseteq \binom{\text{Alt}(n, q)}{q^{n-1}}$$

If \mathcal{F} is a flag of type $\{0, 1, \dots, n-2\}$ in the geometry $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n)$ and $(f_{n-2}, U_{n-2}) \in \mathcal{A}_{n-2} \cap \mathcal{F}$, then the dual of $\text{Res}(\mathcal{F})$ is a $(q^{n-2}, q+1; q)$ -net satisfying $(*)$, as mentioned in Section 4. Furthermore, any two intersecting blocks of \mathcal{B} uniquely determine some $(f_{n-2}, U_{n-2}) \in \mathcal{A}_{n-2}$; it follows that [10] the smallest connected block-closed subsets of $\text{Alt}(n, q)$ which contain these two intersecting blocks is $\text{Res}(\mathcal{F})$, i.e., its dual is a $(q^{n-2}, q+1; q)$ -net satisfying $(*)$, where \mathcal{F} is a flag of type $\{0, 1, \dots, n-2\}$, with $(f_{n-2}, U_{n-2}) \in \mathcal{F} \cap \mathcal{A}_{n-2}$. More specifically, if B, B' are two blocks meeting at a line A , then there exist $B_i \in \mathcal{B}_i$ ($0 \leq i \leq q$), $B = B_j$, $B' = B_k$ for some $0 \leq j, k \leq q$ such that $\bigcap_{0 \leq i \leq q} B_i = A$. Let y be a point in B_0 but not in A ; then $\Gamma_1(y) \cap B_i$ consists of q pairwise disjoint lines and $|\Gamma_1(y) \cap B_i| = q^2$. Furthermore,

$(\bigcup_{1 \leq i \leq q} \Gamma_1(y) \cap B_i) \cup (\Gamma_1(z) \cap B_0)$ forms a second-type maximal clique of size q^3 in $\text{Alt}(n, q)$, where z is a point in $\Gamma_1(y) \cap B_1$. Therefore, some features of the incidence structure $\Pi = (\text{Alt}(n, q), \mathcal{B}, \epsilon)$ can be summarized as follows:

- (1) each line consists of q^{n-1} points and each point lies on $(q^n - 1)/(q - 1)$ blocks,
- (2) for any two distinct blocks $B, B' \in \mathcal{B}$, $|B \cap B'| = 0$ or q ,
- (3) any 2-subspace is a $(q^{n-2}, q + 1; q)$ -net satisfying (*),
- (4) if $B, B' \in \mathcal{B}$ are two blocks meeting at a line A , and x is a point in B' but not in A , then the common neighbors of elements of $A \cup \{x\}$ form a clique of q^3 points in $\text{Alt}(n, q)$ (i.e., an analog of the dual of the Pasch axiom holds).

The conditions for the following class of incidence structures are abstracted from the above observations; $(\text{Alt}(n, q), \mathcal{B}, \epsilon)$ provides such an example with $s = q^{n-2}$ and $\mu = q$. An incidence structure $\Pi = (\mathcal{P}, \mathcal{B}, \epsilon)$ is called *pseudo-alternating* if the following conditions are satisfied:

- (PA1) Each point lies on exactly $(s\mu^2 - 1)/(\mu - 1)$ blocks.
- (PA2) For any two distinct blocks B and B' in \mathcal{B} , $|B \cap B'| = 0$ or μ .
- (PA3) If $x, y \in B_0 \cap B_1$, then there exist $B_i, 2 \leq i \leq \mu$, such that $x, y \in \bigcap_{0 \leq i \leq \mu} B_i$, $|\bigcap_{0 \leq i \leq \mu} B_i| = \mu$ and $\bigcup_{0 \leq i \leq \mu} B_i - \{x, y\}$ consists of all common neighbors of x and y .
- (PA4) Any 2-subspace of Π is a $(s, \mu + 1; \mu)$ -net satisfying (*) such that, for any two intersecting blocks B, B' meeting at a line $A \in \mathcal{L}$ and $x \in B' - A$, the common neighbors of $A \cup \{x\}$ form a clique.

Some consequences of the existence of such incidence structures are:

- (1) $\mu - 1$ must be a divisor of $s\mu - 1$.
- (2) Each block consists of $s\mu$ points, and each block can be partitioned into s lines.
- (3) Each line is contained in $\mu + 1$ blocks of \mathcal{B} , and the intersection of those $\mu + 1$ blocks is the line itself.

It seems worthwhile to state other properties of pseudo-alternating incidence structures as a formal proposition.

Proposition 5.3. (1) For each incident point-block pair (x, B) , $B - \{x\}$ can be partitioned into $(s\mu - 1)/(\mu - 1)$ subsets such that the union of $\{x\}$ with each such subset is a line in \mathcal{L}

(2) For each block B , the induced incidence structure $\Pi_B = (B, \mathcal{L}(B), \epsilon)$ is linear, where $\mathcal{L}(B) = \{B \cap B' \mid B' \in \mathcal{B} \text{ and } B \cap B' \neq \emptyset\}$.

Proof. By (PA3), for each line A with $x \in A$ and $A \subseteq B$, in addition to B , there are exactly another μ blocks which contain A . Hence, (1) follows from (PA1), and (2) follows from (PA2) and (1). □

The next proposition treats the possible structures over 2-subspaces of pseudo-alternating incidence structures.

Proposition 5.4. If $\Pi_2 = (\mathcal{X}, \mathcal{B}, \epsilon)$ is a 2-subspace of $\Pi = (\mathcal{P}, \mathcal{B}, \epsilon)$, then μ is a prime power, Π_2 / \simeq is a 2-attenuated space and $s = \mu^{n-1}$ for some integer n .

Proof. Since Π_2/\simeq is a $(s, r; 1)$ -net which satisfies the dual Pasch axiom, the proposition follows immediately from [15]. \square

We conclude this paper by mentioning the following result without proof: If $\mu \geq 5$, $B \in \mathcal{B}$ is a block and $\Gamma_1(x) \cap B$ is either empty or line-closed in $\Pi_B = (B, \mathcal{L}(B), \epsilon)$ for $x \in \mathcal{P}$ not in B , then μ is a prime power and $\Pi_B = (B, \mathcal{L}(B), \epsilon)$ is isomorphic to the affine space $AG(n-1, q)$ for some integer n .

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