Torsion, compactification, and inflation

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We study possible implications of a ten-dimensional Einstein-Kalb-Ramond theory. It is speculated that the compactification process and inflationary process are closely related. It is also found that the torsion field tends to vanish after the inflation era is completed. This is in fact a general feature for the torsion Lagrangian. Hence the contribution from the Kalb-Ramond action is negligible effectively after the inflationary era. Some solutions to the field equations are also presented.

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There has been intensive study on the implications of the ten-dimensional Einstein-Kalb-Ramond action [1,2]

$$S = \int d^{10}x\sqrt{g^{(10)}}\left[-R^{(10)} - \frac{1}{6}e^{\phi}F_{MNP}F^{MNP} - \frac{1}{2}\partial_M\phi\partial^M\phi - V(\phi)\right]. \quad (1)$$

The Kalb-Ramond field strength F_{MNP} is the curvature tensor for the skew symmetric torsion field A_{MN}. Defining a three-form [3]

$$F \equiv F_{MNP}dx^M \wedge dx^N \wedge dx^P \quad (2)$$

and a two-form

$$A \equiv A_{MN}dx^M \wedge dx^N, \quad (3)$$

the formal relation between F_{MNP} and A_{MN} can be read off directly from the following definition:

$$F = dA. \quad (4)$$

Here we use capital indices $M, N, \ldots$ $(= 0, 1, 2, \ldots, 9)$ to denote the ten-dimensional space-time indices. Also lower case latin indices from the begining $(a, b, c, \ldots)$ of the alphabet will denote the four-dimensional space time indices $(a, b, c = 0, 1, 2, 3)$. Moreover, $i, j, k, l (= 1, 2, 3)$ labels the spatial 3-manifold. Finally, we will use lowercase latin indices from the middle $(m, n, \ldots)$ of the alphabet to label the six-dimensional compactified internal space. Here ϕ is the dilaton field. The F^2 term has also been studied in many articles, especially in the pointlike limit of the superstring low-energy effective theory, namely, the ten-dimensional supergravity theory [2] where the F^2 term is known as the Kalb-Ramond action.

The equation of motion for the action (1) can be shown [1,4] to be

$$D_M(e^{\phi}F^{MNP}) = 0, \quad (5)$$

$$G_{MN} = T^{\phi}_{MN} + T^{F}_{MN}, \quad (6)$$

$$D_M\partial^M\phi = \frac{1}{6}e^{\phi}F^2 + \partial_\phi V. \quad (7)$$

Here $G_{MN} \equiv \frac{1}{2}Rg_{MN} - R_{MN}$ is the Einstein tensor. We have also defined the energy-momentum tensors T^{ϕ}_{MN} and T^{F}_{MN} as

$$T^{\phi}_{MN} = \frac{1}{2}\partial_M\phi\partial_N\phi - \frac{1}{4}g_{MN}(\partial_P\phi\partial^P\phi + 2V), \quad (8)$$

$$T^{F}_{MN} = -\frac{1}{12}e^{\phi}(F^2g_{MN} - 6F_{MPQ}F_N^{PQ}), \quad (9)$$

respectively.

Recent observations [5] indicate that the large-scale structure of our four-dimensional Universe should be described by the well-known Friedmann-Robertson-Walker [6] (FRW) spaces (denoted as M^4). It is known that the FRW spaces describe all 4-spaces that can be time-foilated into homogeneous and isotropic spatial 3-spaces. Moreover, if the ten-dimensional models studied here are going to have anything to do with real physics, the ten-dimensional space M^{10} has to undergo a dimensional reduction process (as any Kaluza-Klein theory does). Therefore M^{10} must be compactified into product spaces of FRW space M^4 and some unknown six-dimensional spaces M^6, namely, $M^{10} \rightarrow M^4 \times M^6$.

It seems impossible to pick up the right vaccum merely from the equations of motion. The choice will then rely on the choice of some appropriate boundary conditions. Since our four-dimensional space-time is observed to be very close to a FRW space with tremendous symmetries inherent, it is a natural choice to impose a similar symmetrical constraint on the compactified M^6, namely, one should take M^6 to be totally homogeneous and isotropic too. It can be shown that the hypothetical $M^4 \times M^6$ space is just the generalized FRW space [1] described by the metric

$$ds^2 \equiv g_{MN}dz^M dz^N = -dt^2 + a^2(t)h_{ij}(x)dx^i dx^j + d^2(t)h_{mn}(y)dy^m dy^n. \quad (10)$$

Here $h_{ij}(x)$ and $h_{mn}(y)$ denote the homogeous and isotropic 3- and 6-space, respectively. To be more specific, $h_{ij}dx^i dx^j \equiv [1/(1-k_1r^2)]dr^2 + r^2d\Omega_3$ and $h_{mn}dx^m dx^n \equiv [1/(1-k_2s^2)]ds^2 + s^2d\Omega_6$. Here k_1 and k_2 are constants (0, or ±1) denoting different topologies associated with different FRW spaces. Furthermore, $d\Omega_n$ denotes n-dimensional solid angles.

In order to solve Eq. (5), we will accept the proposed ansatz

$$F_{abc}(z) = F_{abc}(x), \tag{11}$$

$$F_{mnp}(z) = F_{mnp}(y), \tag{12}$$

while setting all other cross space index components to zero, i.e., $F_{abm} = F_{amn} = 0$. One can then show that F_{mnp} has to vanish identically obeying the equations of motion (5)–(7). Therefore, (5) reduces to a four-dimensional equation

$$\partial_a(\sqrt{g} d^6 e^{\phi} F^{abc}) = 0. \tag{13}$$

Note that we have written $g^{(4)}$ as g for convenience. In fact, we will omit most superscripts $^{(4)}$ from now on. Furthermore, the superscript $^{(10)}$ will also be omitted unless there are possible confusions that cannot be avoided by reading the definition domains specified by the equations.

Note that the d^6 factor [1] has been overlooked in many previous works. As we will show later, this error will not affect the result in the FRW spaces dependent only on t, except for some modifications on the vanishing torsion field. This oversight is, however, inappropriate and will have nontrivial impact in certain models. Note that this equation can in fact be integrated rather straightforwardly. Indeed, this can be solved by observing that the compactified Kalb-Ramond field strength F_{abc} is a totally skew-symmetric type $T(0,3)$ tensor. Therefore one can map it to some type $T(1,0)$ contravariant vector T^d with the help of the totally skew-symmetric type $T(0,4)$ Levi-Civita tensor ϵ_{abcd}. Specifically, there exists a contravariant vector T^d such that

$$F_{abc} = \epsilon_{abcd} T^d \tag{14}$$

for every totally skew symmetric type $T(0,3)$ tensor F_{abc} defined on our four-dimensional base manifolds. Indeed, one notes that both sides of equation (14) have exactly the same symmetry among their indices such that all degrees of freedom have been taken into account. Hence one can write (13) as

$$\epsilon^{abcd} \partial_a (d^6 e^{\phi} T_d) = 0. \tag{15}$$

By introducing a one-form $T \equiv T_a dx^a$, (15) can be written as a two-form equation

$$^* d(d^6 e^{\phi} T) = 0 \tag{16}$$

after multiplying (15) with $dx^b \wedge dx^c$ such that (15) becomes $\epsilon^{abcd}\, \partial_a (d^6 e^{\phi} T_d)\, dx_b \wedge dx_c = 0$. Here * is the Hodge star operator [3] which maps a differential n-form into its dual $(d-n)$-form in d-dimensional spaces. Therefore, one has

$$d(d^6 e^{\phi} T) = 0 \tag{17}$$

due to the involutive property of the Hodge star operator, namely, $^{**} = \mathbb{1}$ the identity map. If we live in a (pseudo-)Riemannian space M that has trivial first cohomology group [7], namely, $H^1(M) = 0$ such that all closed one-forms defined on M are exact, there exists a scalar field χ that satisfies

$$d^6 e^{\phi} T = d\chi. \tag{18}$$

For example, all simply connected spaces [i.e., $\Pi_1(M) = 0$] and contractible spaces belong to the class $H^1(M) = 0$.

Therefore, one has

$$T_a = e^{-\phi} d^{-6} \partial_a \chi, \tag{19}$$

if $H^1(M^4) = 0$. Note that the d^{-6} factor can be absorbed into χ by defining $\partial_a \chi' \equiv d^{-6} \partial_a \chi$ if d and χ are both functions of t only. This is basically what we meant by the minor error mentioned earlier. The replacement is, however, illegal if spatial dependence is present. Consequently, one can write

$$T^F_{ab} = T^\chi_{ab} = e^{-\phi} d^{-12} \left(\partial_a \chi \partial_b \chi - \frac{1}{2} g_{ab} \partial_c \chi \partial^c \chi \right), \tag{20}$$

$$T^F_{mn} = T^\chi_{mn} = \frac{1}{2} e^{-\phi} d^{-10} h_{mn} \partial_a \chi \partial^a \chi. \tag{21}$$

Note that we have used the identity $F^2 = -6e^{-2\phi} d^{-12} \partial_a \chi \partial^a \chi$ in deriving (20) and (21). Moreover, from the Bianchi identity $D^M G_{MN} = 0$, one has $D^M T^\phi_{MN} + D^M T^F_{MN} = 0$. This equation can be shown to be

$$\partial_a \chi (D_b \partial^b \chi - 6 \partial^b \beta \partial_b \chi - \partial_b \phi \partial^b \chi) = 0 \tag{22}$$

after some algebra. Here we have assumed that $\chi = \chi(t)$, a function of t only. We also write $d(t) = e^{\beta(t)}$ and $a(t) = e^{\alpha(t)}$ for convenience.

Note that (22) is valid for any form of $V(\phi)$. In fact, one can also show that Eq. (22) is still valid in the presence of the spin-1 gauge fields and spin-$\frac{1}{2}$ matter fields. Indeed, the contributions from gauge fields and matter fields in (6) can be cast into the energy-momentum tensors associated with the gauge fields A_M and matter fields ψ, i.e., $T_{MN} \to T^\phi_{MN} + T^F_{MN} + T^{A_P}_{MN} + T^\psi_{MN}$. Additional terms in the energy momentum tensor are conserved separately due to their equations of motion. Hence (22) is indeed valid in the presence of the gauge and matter fields. Moreover, (22) is in fact also true in the induced gravity model [4,8] that has $\phi^2 R$ couplings to ensure that all coupling constants are made dimensionless.

Note also that torsion field is the only variable that hides some of its dynamics implicitly in the Bianchi relation, namely, the field dependence cannot be read off after we have used up its field equation (13). This is a very unique property of the torsion field.

Furthermore, (22) can be written more explicitly as

$$\partial_t \chi (\partial_t + 3\partial_t \alpha - 6 \partial_t \beta - \partial_t \phi) \partial_t \chi = 0. \tag{23}$$

Hence if $\partial_t \chi \neq 0$, one finds

$$\partial_t \chi = k \exp(-3\alpha + \phi + 6\beta). \tag{24}$$

Here k is a constant. This indicates that the χ field will be decreasing badly if the three-space scale factor $a(t)$ inflates. In fact the Kalb-Ramond term $e^{\phi} F^2$ becomes $6k^2 e^{-6\alpha + \phi}$ after substituting the solution (24). Therefore if action (5) is assumed to induce the inflationary process, the torsion field contribution to the post-inflation era is negligible.

For clarification, we will study more explicitly a special case of this model, namely the case when $\partial_t\phi = k_1 = k_2 = 0$. Note that, by computing the ten-dimensional Einstein tensor induced by the 10D FRW metric given by (10), one has $G^{(10)}_{ab} = G^{(4)}_{ab} + t_{ab}$ and $G^{(10)}_{mn} = \frac{1}{2}d^2(t)h_{mn}G$ where t_{ab} and G are defined as

$$t_{ab} = 3g_{ab}(2\partial^2\beta + 7\partial_c\beta\partial^c\beta - 5k_1e^{-2\beta}) -6(D_a\partial_b\beta + \partial_a\beta\partial_b\beta), \tag{25}$$

$$G = R + 10\partial^2\beta + 30\partial_a\beta\partial^a\beta - 20k_1e^{-2\beta}. \tag{26}$$

By computing $\mathrm{Tr}(G^{(4)}_{ab})$ given in (6) with the help of (25), one has an equation for the expression of the 4-scalar curvature R. Furthermore, one can also get an expression of R by computing the mn equation of (6) with the help of equation (26). Eliminating both R obtained above, one has the

$$\partial^2\beta + 6\partial_a\beta\partial^a\beta - 5k_2e^{-2\beta} + \frac{1}{4}e^{-\phi-12\beta}\partial_a\chi\partial^a\chi + \frac{V}{8} = 0. \tag{27}$$

In fact, (27) can be written more explicitly as

$$(\partial_t + 3\partial_t\alpha + 6\partial_t\beta)\partial_t\beta = -\frac{k^2}{4}e^{-6\alpha+\phi} - 5k_2e^{-2\beta} + \frac{V}{8}. \tag{28}$$

Furthermore, Eq. (7) and the tt equation of (6) give

$$(\partial_t + 3\partial_t\alpha + 6\partial_t\beta)\partial_t\phi = -k^2e^{-6\alpha+\phi} - \partial_\phi V, \tag{29}$$

$$(\partial_t\alpha)^2 + 6\partial_t\alpha\partial_t\beta + 5(\partial_t\beta)^2 + \frac{1}{12}(\partial_t\phi)^2 = \frac{k^2}{6}e^{-6\alpha+\phi} - k_1e^{-2\alpha} - 5k_2e^{-2\beta} + \frac{V}{6}. \tag{30}$$

Note that we have replaced the torsion contributions in (28)–(30). One may also note that (28) and (29) can be written more compactly as

$$\partial_t(e^{3\alpha+6\beta}\partial_t\beta) = -\frac{k^2}{4}e^{-3\alpha+6\beta+\phi} - 5k_2e^{3\alpha+4\beta} + \frac{V}{8}e^{3\alpha+6\beta}, \tag{31}$$

$$\partial_t(e^{3\alpha+6\beta}\partial_t\phi) = -k^2e^{-3\alpha+6\beta+\phi} - \partial_\phi Ve^{3\alpha+6\beta}. \tag{32}$$

Note that Eq. (32) indicates that $e^{3\alpha+6\beta}\partial_t\phi$ has to be decreasing monotonically as long as $\partial_\phi V \geq 0$. If $a(t)$ inflates and $d(t)$ contracts smoothly such that ad^2 increases, $\partial_t\phi$ must also be a decreasing function in order to remain consistent.

Now we are left with three equations [4] (30)–(32) for three unknowns. If $\partial_t\phi = k_1 = k_2 = 0$, (32) also gives $k = 0$. Therefore, (31) can be solved to give

$$\partial_t\beta = \text{const} \times e^{-3\alpha-6\beta}. \tag{33}$$

Equation (33) indicates further that if $a(t)$ inflates, $d(t)$ tends to become static very quickly. In fact we are able to solve this special case exactly. Indeed, (30) becomes $(\partial_t\alpha)^2 + 6\partial_t\alpha\partial_t\beta + 5(\partial_t\beta)^2 = 0$. This gives $\beta = -k_3\alpha +$ const accordingly. Here k_3 can be either 1 or $\frac{1}{5}$. Hence (33) can be integrated to obtain

$$a(t) = (a_0 + a_1t)^p, \tag{34}$$

$$d(t) = d_0(a_0 + a_1t)^q, \tag{35}$$

after some algebra. Here p and q are constants such that $(p,\, q) = (-\frac{1}{3},\, \frac{1}{3})$ if $k_3 = 1$ or $(p,\, q) = (\frac{5}{9},\, -\frac{1}{9})$ if $k_3 = \frac{1}{5}$. Also d_0, a_0, and a_1 are constants to be fixed by imposing appropriate boundary conditions.

In what follows, we will also assume $k_1 = k_2 = 0$ for simplicity. It is hard to find an inflationary solution without the help of the potential term. Indeed, if one expects a slow-rollover [4,8] inflationary solution which requires $(\partial_t\phi)^2 \ll (\partial_t\alpha)^2$, it is easy to show that Eqs. (28) and (30) are inconsistent with the first order approximation of α and β, namely $\partial_t\alpha \simeq \alpha_1$ and $\partial_t\beta \simeq \beta_1$. In order to obtain an inflationary solution, one will assume the broken symmetric potential $V(\phi) = \frac{\lambda}{8}(\phi^2 - v^2)^2$.

Once again, the slow-rollover assumption $(\partial_t\phi)^2 \ll (\partial_t\alpha)^2$ will be adopted during the inflationary period. Moreover, we will also assume that $\phi(t=0) \equiv \phi_0 \ll v$ as well as the first-order appproximation in α and β, i.e., $\partial_t\alpha \simeq \alpha_1$ and $\partial_t\beta \simeq \beta_1$. Consequently, (28) and (30) become

$$\alpha_1^2 + 6\alpha_1\beta_1 + 5\beta_1^2 \simeq \frac{V_0}{6}, \tag{36}$$

$$3\alpha_1\beta_1 + 6\beta_1^2 \simeq \frac{V_0}{8}. \tag{37}$$

Here we have assumed $V \simeq V_0 \simeq \frac{\lambda}{8}v^4$ during the slow-rollover period. Eliminating V_0 from (36) and (37), one finds either $\alpha_1 = \beta_1$ or $\alpha_1 = -3\beta_1$.

Furthermore, $\alpha_1 = -3\beta_1$ can be excluded due to the positivity of V_0. Therefore, one has $\alpha_1 = \beta_1$ for all slow-rollover models. Note that $\alpha_1 = \beta_1$ implies that α and β will inflate or decrease all together. Thus these models will not be helpful in explaining the smallness of the compactified scale. We have to assume here that compactification must have been completed before inflation. Note also that considering $D \neq 6$ models will not help resolve this compactification problem with the slow-rollover approach. The other non-slow-rollover process is, however, not excluded. Nonetheless, one can obtain the result

$$\alpha \simeq \alpha_0 + \frac{\sqrt{\lambda}}{24}v^2t. \tag{38}$$

Moreover, by assuming $\partial_t^2\phi \ll (\partial_t\phi)^2$, Eq. (29) can be shown to give

$$\phi \simeq \phi_0e^{\frac{4}{3}\sqrt{\lambda}t}. \tag{39}$$

There are a number of inequalities to be checked for consistency. Firstly, the inequalities $(\partial_t\phi)^2 \ll (\partial_t\alpha)^2$ and $\partial_t^2\phi \ll (\partial_t\phi)^2$ imply

$$\frac{v^2}{32} \gg \phi \gg \frac{4}{3}\sqrt{\lambda}. \tag{40}$$

Moreover, V_0 should not exceed the Planck energy to exclude quantum effect. This will simply imply that $\lambda v^4 \leq 8$. Finally, a factor e^{60} can be achieved only when $\lambda v^4 \geq 2 \times 10^{-12}$. Note that the inflationary interval is around 10^8 in Planck units. Therefore, it is possible to choose appropriate initial conditions and parameters such that an inflationary solution can be made relevant to the physical universe.

One notes also that when ϕ approaches the physical vaccum $\phi = v$, equation (29) indicates a damping oscillatory solution which will be used to slow down the inflationary process as well as to reheat the physical universe. Indeed, once ϕ approaches v, (29) reduces to

$$\partial_t^2\varphi + \lambda v^2\varphi = 0, \tag{41}$$

by keeping only terms linear in φ. Note that damping is due to higher-order terms. Moreover, one finds $\partial_t\alpha \simeq \partial_t\beta \simeq \frac{1}{12}\sqrt{\lambda}v\varphi$. Here we have assumed $\phi = v + \varphi$ and $\partial_t^2\beta \ll (\partial_t\beta)^2$. In this inflationary solution, the torsion contribution is hence damped as soon as the inflationary process is completed.

In summary, we have shown that the inflationary process is expected to eliminate the contributions from the torsion field. There are also a number of related $(4+D)$-dimensional models with torsion and perhaps with additional scalar fields coupled to the system. Equations similar to Eq. (24) for these systems can also be obtained such that the torsion field tends to be negligible in the post-inflationary era. This appears to be a general feature of the torsion coupled models. We have also shown that an inflationary solution can be obtained if we consider a broken symmetric potential. Similar chaotic inflationary solutions can also obtained by taking $v = 0$.

We are, however, unable to obtain an inflationary solution that also induces the compactification scale simultaneously in this model, although the possibility is not excluded. This dual effect can be obtained by considering an induced gravity model [8] such that the compacitified scale factor β decreases as α increases.

We also remark here that there has been confusion in taking dilaton-free models by imposing a $\phi =$ const constraint directly in action (1). This is in fact inappropriate from our analysis above. Indeed, eliminating the ϕ field from (1) means we will lose the ϕ equation (31) which imposes a strong constraint that the torsion field has to vanish identically accordingly unless $\alpha =$ const. Therefore, one should not overlook the nontrivial constraint imposed by the dilaton field. The $d(t)$ factor in (13) that has been overlooked previously has also been restored and studied carefully. After slightly redefining the χ field, all previous results are still valid.

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