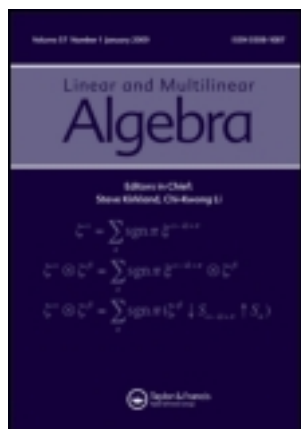


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Linear and Multilinear Algebra

Publication details, including instructions for authors and subscription information:

<http://www.tandfonline.com/loi/glma20>

Line segments and elliptic arcs on the boundary of a numerical range

Hwa-Long Gau^a & Pei Yuan Wu^b

^a Department of Mathematics, National Central University, Chungli 32001, Taiwan

^b Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan

Published online: 04 Dec 2010.

To cite this article: Hwa-Long Gau & Pei Yuan Wu (2008) Line segments and elliptic arcs on the boundary of a numerical range, *Linear and Multilinear Algebra*, 56:1-2, 131-142, DOI: [10.1080/03081080701396051](https://doi.org/10.1080/03081080701396051)

To link to this article: <http://dx.doi.org/10.1080/03081080701396051>

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Line segments and elliptic arcs on the boundary of a numerical range^{§¶}

HWA-LONG GAU^{*†} and PEI YUAN WU[‡]

[†]Department of Mathematics, National Central University, Chungli 32001, Taiwan

[‡]Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan

Communicated by C.-K. Li

(Received 10 August 2006; in final form 5 April 2007)

For an n -by- n complex matrix A , we consider the numbers of line segments and elliptic arcs on the boundary $\partial W(A)$ of its numerical range. We show that (a) if $n \geq 4$ and A has an $(n-1)$ -by- $(n-1)$ submatrix B with $W(B)$ an elliptic disc, then there can be at most $2n-2$ line segments on $\partial W(A)$, and (b) if $n \geq 3$, then $\partial W(A)$ contains at most $(n-2)$ arcs of any ellipse. Moreover, both upper bounds are sharp. For nilpotent matrices, we also obtain analogous results with sharper bounds.

Keywords: Numerical range; Nilpotent matrix

AMS Subject Classification: 15A60

1. Introduction

For an n -by- n complex matrix A , its numerical range $W(A)$ is, by definition, the subset $\{\langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\}$ of the complex plane, where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are the standard inner product and norm in \mathbb{C}^n , respectively. It is known that $W(A)$ is always a convex set. For other properties of the numerical range, the reader is referred to [5, Chapter 1].

In this article, we are concerned with the numbers of line segments and elliptic arcs on the boundary of a numerical range. The study of the former has been the topic of some recent articles [1–3]. The main result in Section 2 below says that if A is an n -by- n ($n \geq 4$) matrix which has an $(n-1)$ -by- $(n-1)$ submatrix B with numerical range an elliptic disc, then $\partial W(A)$ can have at most $2(n-2)$ line segments. On the other hand,

*Corresponding author. Email: hlgau@math.ncu.edu.tw

§Dedicated to Ky Fan on his ninety-third birthday.

¶Part of the results here was presented by the second author in the 8th WONRA at Bremen, Germany and the 13th ILAS Conference at Amsterdam, the Netherlands in July, 2006.

if $W(A)$ is only assumed to contain an elliptic disc E and $\partial W(A)$ and ∂E intersect at more than n points, then $\partial W(A)$ contains at least one and at most $(n-2)$ arcs of ∂E . These results are proved using the classical Riesz–Fejér theorem on nonnegative trigonometric polynomials via Kippenhahn’s theorem on the duality between the boundary $\partial W(A)$ and the curve given by the determinantal equation $\det(x\operatorname{Re} A + y\operatorname{Im} A + zI_n) = 0$, where $\operatorname{Re} A = (A + A^*)/2$ and $\operatorname{Im} A = (A - A^*)/(2i)$ are the real and imaginary parts of A , respectively. All the upper bounds here are shown to be sharp by various examples.

In Section 3, we restrict ourselves to nilpotent matrices. We prove the nilpotent analogues of the two above-mentioned results in Section 2, namely, for an n -by- n nilpotent matrix A , (a) if A has an $(n-1)$ -by- $(n-1)$ submatrix B with $W(B)$ a circular disc centered at the origin, then, for $3 \leq n \leq 5$ (resp., $n \geq 6$), $\partial W(A)$ has at most $n-2$ (resp., $2(n-4)$) line segments, and (b) if $W(B)$ contains a circular disc D centered at the origin and $\partial W(A)$ and ∂D intersect at more than $n-2$ points, then $W(A) = D$ if $2 \leq n \leq 4$, and $\partial W(A)$ contains at least one and at most $n-4$ arcs of ∂D if $n \geq 5$. Again, all such upper bounds are sharp.

In the following, we use \mathbb{D} to denote the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$, and J_n the n -by- n Jordan block

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}.$$

It is known that $W(J_n) = \{z \in \mathbb{C} : |z| \leq \cos(\pi/(n+1))\}$ (cf. [4, Proposition 1]).

2. Matrices in general

We start by recalling Kippenhahn’s theorem [7, Satz 10], which brings algebraic geometry techniques to bear on the study of numerical ranges. It says that if A is an n -by- n matrix, then $W(A)$ equals the convex hull of the real points in the dual of the curve $p_A(x, y, z) \equiv \det(x\operatorname{Re} A + y\operatorname{Im} A + zI_n) = 0$, that is,

$$W(A) = \{a + ib \in \mathbb{C} : a, b \text{ real, } ax + by + z = 0 \text{ tangent to } p_A(x, y, z) = 0\}^\wedge.$$

Based on this, we can give an upper bound for the number of line segments on the boundary of a numerical range. This was also alluded to in [2].

LEMMA 2.1 *For an n -by- n matrix A , there can be at most $n(n-1)/2$ line segments on $\partial W(A)$. If p_A is irreducible, then there are at most $(n-1)(n-2)/2$ such line segments.*

This is because through duality every line segment on $\partial W(A)$ corresponds to a singular point of the degree- n curve $p_A = 0$, and the number of the latter is at most $n(n-1)/2$ (resp., $(n-1)(n-2)/2$ if p_A is irreducible) (cf. [8, p. 59, Example 5]).

Since the numerical range of a 2-by-2 matrix is an elliptic disc, a line segment or a singleton, the upper bound $n(n-1)/2$ ($=1$) is obviously sharp for $n=2$. The numerical

range of a 3-by-3 matrix A has been classified in [7,6]. It can be a triangular region (or its degenerate form, a line segment or a singleton) if p_A has only linear factors, an elliptic disc with or without a cone added to it if p_A has a quadratic irreducible factor, or the convex hull of a heart-shaped curve with a line segment on its boundary or an oval disc if p_A is irreducible. From this, it can be seen that for $n=3$ the upper bound $n(n-1)/2$ ($=3$) in Lemma 1.1 is again sharp. The next lemma takes care of the cases $n=4$ and 5. For $n=4$, this was given in [1, Theorem 37].

LEMMA 2.2 *If A is a 4-by-4 (resp., 5-by-5) matrix, then there can be at most 4 (resp., 6) line segments on $\partial W(A)$. Moreover, “4” and “6” are sharp in their respective cases.*

For any matrix A , let $l(A)$ denote the number of line segments on $\partial W(A)$.

Proof of Lemma 2.2 Consider a 4-by-4 matrix A . If p_A is irreducible, then $l(A) \leq 3$ by Lemma 2.1; otherwise, $l(A) \leq 3$ or 4 depending on whether p_A is the product of a linear factor and an irreducible cubic factor or the product of two (possibly reducible) quadratic factors. That “4” is sharp can be seen by the 4-by-4 diagonal matrix $\text{diag}(1, i, -1, -i)$.

Now let A be a 5-by-5 matrix. If p_A is irreducible, then $l(A) \leq 6$ by Lemma 2.1. For the reducible p_A with a linear factor, it can be easily seen using the result for the 4-by-4 case that $l(A) \leq 6$. Below we consider the case when p_A is the product of two irreducible factors $p_A = p_1 p_2$ with p_1 quadratic and p_2 cubic. Let C_j be the dual curve of $p_j = 0$, $j = 1, 2$. Then C_1 is an ellipse and C_2 is either a heart-shaped curve or an oval with an inner part. Since the degrees of p_1 and p_2 are 2 and 3, respectively, Bézout’s theorem [8, Theorem 1.3.5] says that the curves $p_1 = 0$ and $p_2 = 0$ have at most 6 intersection points. By duality, C_1 and C_2 can have at most 6 common tangent lines. If C_2 is a heart-shaped curve, then its convex hull \widehat{C}_2 has a line segment L on its boundary. In case C_1 and C_2 are in the same closed half-plane determined by L , then $W(A)$, as the convex hull of C_1 and C_2 , can have at most 5 line segments on its boundary; otherwise, $\partial W(A)$ cannot contain L and thus has at most 6 line segments. For the other case, if C_2 has an oval outer part, then obviously $l(A) \leq 6$. That “6” is attained is seen by the 5-by-5 matrix

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus \text{diag}(1 + \varepsilon, (1 + \varepsilon)i, -1 - \varepsilon)$$

with sufficiently small $\varepsilon > 0$ (cf. [1, Example 38]). ■

A classical result due to J. Anderson from the early 1970s says that if A is an n -by- n matrix with $W(A)$ contained in $\overline{\mathbb{D}}$ and $\partial W(A)$ intersecting $\partial \mathbb{D}$ at more than n points, then $W(A) = \overline{\mathbb{D}}$. Among its four known proofs, the one in [10, Lemma 6] is based on the Riesz–Fejér theorem on nonnegative trigonometric polynomials and the fundamental theorem of algebra in classical analysis. In the following, we will exploit the ideas there to deduce sharp bounds of $l(A)$ for certain finite matrices A . The next theorem is one such example. The condition on the submatrix below arises from our past experience on estimating $l(A)$ for A a companion matrix (cf. [3]).

THEOREM 2.3 *If A is an n -by- n ($n \geq 4$) matrix which has an $(n-1)$ -by- $(n-1)$ submatrix B with $W(B)$ an elliptic disc, then $l(A) \leq 2(n-2)$. Moreover, “ $2(n-2)$ ” is sharp.*

For the proof of Theorem 2.3, we need a lemma from [3, Lemma 5]. It relates the line segments on $\partial W(A)$ to the numerical ranges of submatrices of A .

LEMMA 2.4 *If A is an n -by- n matrix and B is an $(n-1)$ -by- $(n-1)$ submatrix of A , then every line segment on $\partial W(A)$ intersects $\partial W(B)$.*

Proof of Theorem 1.3 After a suitable affine transformation, we may assume that $W(B) = \overline{\mathbb{D}}$. Let $p(z, u) = \det(zI_n - \operatorname{Re}(uA))$ for z in \mathbb{C} and u in $\partial\mathbb{D}$. Since $\operatorname{Re}(uB)$ is an $(n-1)$ -by- $(n-1)$ submatrix of $\operatorname{Re}(uA)$ for any u , $|u| = 1$, their eigenvalues interlace. We infer from $W(uB) = \overline{\mathbb{D}}$ that $p(1, u) \leq 0$ for all u in $\partial\mathbb{D}$. Since $-p(1, u)$ is a nonnegative trigonometric polynomial of degree at most n , by the Riesz–Fejér theorem [9, p. 77, Problem 40] there is a polynomial q of degree $\leq n$ such that $-p(1, u) = |q(u)|^2$ for all u . We now assume that there are more than $2(n-2)$ line segments on $\partial W(A)$. Since each of these lines intersects $\partial\mathbb{D}$ by Lemma 2.4, we have $p(1, u_k) = 0$ and hence $q(u_k) = 0$ for more than $2(n-2)$ many u_k 's in $\partial\mathbb{D}$. As $2(n-2) \geq n$, the fundamental theorem of algebra implies that $q \equiv 0$. Therefore $p(1, u) = 0$ for all u . We express $p(z, u)$ in the form $(\sum_{j=0}^n p_j(z)u^j) + (\sum_{j=1}^n \tilde{p}_j(z)\bar{u}^j)$, where, for each j , $p_j(z)$ is a polynomial in z with degree $\leq n-j$ and $\tilde{p}_j(z) = p_j(\bar{z})$. From $p(1, u) = 0$ for all u , we obtain $p_j(1) = 0$ and thus $z-1$ divides p_j for all j . Similarly, by considering the smallest eigenvalues of $\operatorname{Re}(uB)$ and $\operatorname{Re}(uA)$, we also obtain that $z+1$, divides p_j for all j . In particular, since $p_n(z)$ and $p_{n-1}(z)$ are divisible by z^2-1 , they are identically zero. Hence $p(z, u) = (z^2-1)q_1(z, u)$, where, for each fixed z , $q_1(z, u)$ is a trigonometric polynomial in u of degree $\leq n-2$. Now we show by some continuity argument that $q_1(1, u_k) = 0$ for all k . To this end, let $\alpha(u) = \max\{z \in \mathbb{R} : q_1(z, u) = 0\}$ and $\beta(u) = \max\{z \in \mathbb{R} : p(z, u) = 0\}$ for u in $\partial\mathbb{D}$. Since a line segment on $\partial W(A)$ is tangent to $\partial\mathbb{D}$ at u_k , we have $\beta(u_k) = 1$ and $\beta(u) > 1$ for all $u \neq u_k$ in a neighborhood of u_k in $\partial\mathbb{D}$. Hence there is a sequence v_m , $m \geq 1$, in $\partial\mathbb{D}$ such that $\beta(v_m) > 1$ for all m and $v_m \rightarrow u_k$ as $m \rightarrow \infty$. We obtain from

$$0 = p(\beta(v_m), v_m) = (\beta(v_m)^2 - 1)q_1(\beta(v_m), v_m)$$

that $q_1(\beta(v_m), v_m) = 0$ and hence $\alpha(v_m) = \beta(v_m) > 1$ for all m . We may assume that $\alpha(v_m)$ converges, say, to r . Then $r \geq 1$ and

$$q_1(r, u_k) = \lim_{m \rightarrow \infty} q_1(\alpha(v_m), v_m) = \lim_{m \rightarrow \infty} q_1(\beta(v_m), v_m) = 0.$$

It follows that $p(r, u_k) = 0$ and thus $r \leq \beta(u_k) = 1$. We conclude that $r = 1$ and hence $q_1(1, u_k) = 0$ for all k . Applying the fundamental theorem of algebra again to the polynomial $u^{n-2}q_1(1, u)$ of degree at most $2(n-2)$ with more than $2(n-2)$ zeros (that is, the u_k 's) to obtain $q_1(1, u) = 0$ for all u . In a similar fashion, we have $q_1(-1, u) = 0$ for all u . Thus $p(z, u) = (z^2-1)^2q_2(z, u)$, where, for each fixed z , $q_2(z, u)$ is a trigonometric polynomial in u of degree $\leq (n-4)$. Repeating the above arguments with q_2 replacing q_1 and so forth, we obtain eventually $p(z, u) = (z^2-1)^p$ or $(z^2-1)^p(z-1)$, $p \geq 2$. If this is the case, then the eigenvalues of $\operatorname{Re}(uA)$ are ± 1 for all u , $|u| = 1$, and thus $W(A) = \overline{\mathbb{D}}$ contradicting our assumption on the line segments of $\partial W(A)$. We conclude that $l(A) \leq 2(n-2)$.

Now we show the sharpness of “ $2(n - 2)$ ”. For $n \geq 4$, let $A = A_1 \oplus A_2$, where

$$A_1 = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

with $2 \cos(\pi/(n - 2)) < a < 2$ and A_2 is the $(n - 2)$ -by- $(n - 2)$ matrix

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{bmatrix}.$$

Then A has the $(n - 1)$ -by- $(n - 1)$ submatrix $B = A_1 \oplus J_{n-3}$ with $W(B) = \{z \in \mathbb{C} : |z| \leq a/2\}$. On the other hand, since $W(A_1) = \{z \in \mathbb{C} : |z| \leq a/2\}$ and $W(A_2)$ is the polygonal region bounded by the regular $(n - 2)$ -gon with vertices $e^{2\pi ij/(n-2)}$, $0 \leq j \leq n - 3$, which circumscribes the circle $|z| = \cos(\pi/(n - 2))$, we deduce from $\cos(\pi/(n - 2)) < a/2 < 1$ that $W(A_2)$ has $2(n - 2)$ line segments on its boundary. This completes the proof. ■

It was suspected in [1] that “ $2(n - 2)$ ” might be the sharp upper bound of $l(A)$ for all n -by- n ($n \geq 4$) matrices A . The preceding theorem gives another piece of evidence.

An analogous argument as above can be used to give an upper bound for the number of elliptic arcs on the boundary of a numerical range.

THEOREM 2.5 *Let A be an n -by- n ($n \geq 3$) matrix. Then*

- (a) *the boundary $\partial W(A)$ contains at most $n - 2$ arcs of any ellipse, and*
- (b) *if $W(A)$ contains an elliptic disc E and $\partial W(A)$ and ∂E intersect at more than n points, then $\partial W(A)$ contains at least one arc of ∂E .*

In these cases, both “ $n - 2$ ” and “ n ” are sharp.

Proof To prove (a), we may assume otherwise that $\partial W(A)$ contains arcs α_j of the unit circle $\partial \mathbb{D}$ with endpoints u_{2j-1} and u_{2j} , $1 \leq j \leq n - 1$. Consider $p(z, u) \equiv \det(zI_n - \operatorname{Re}(uA))$ for z in \mathbb{C} and u in $\partial \mathbb{D}$ as a trigonometric polynomial in u . Since $p(\pm 1, u) = 0$ for infinitely many values of u , the coefficients of $p(\pm 1, u)$ are all zero. Thus $(z^2 - 1)$ divides all such coefficients and hence $p(z, u) = (z^2 - 1)q_1(z, u)$ for some trigonometric polynomial q_1 in u of degree $\leq (n - 2)$. As in the proof of Theorem 2.3, consider $\alpha(u) = \max\{z \in \mathbb{R} : q_1(z, u) = 0\}$ and $\beta(u) = \max\{z \in \mathbb{R} : p(z, u) = 0\}$ for $|u| = 1$. Then for each k , $1 \leq k \leq 2(n - 1)$, $\beta(u_k) = 1$ and $\beta(u) > 1$ for all $u \neq u_k$ in a one-sided neighborhood of u_k in $\partial \mathbb{D}$. Hence there is a sequence of points v_m , $m \geq 1$, in $\partial \mathbb{D}$ such that $\beta(v_m) > 1$ for all m and $v_m \rightarrow u_k$ as $m \rightarrow \infty$. We may argue as before that (a subsequence of) $\{\alpha(v_m)\}_{m=1}^\infty$ converges to 1 and hence $q_1(1, u_k) = 0$ for all k . Applying the fundamental theorem of algebra to the polynomial $u^{n-2}q_1(1, u)$ with degree $\leq 2(n - 2)$ yields that $q_1(1, u) = 0$ for all u , $|u| = 1$. Similarly, we obtain $q_1(-1, u) = 0$ for all u . Thus $p(z, u) = (z^2 - 1)^2 q_2(z, u)$, where, for each fixed z , $q_2(z, u)$ is a trigonometric polynomial in u of degree $\leq n - 4$. Repeating these with q_2 replacing q_1 and so forth, we eventually obtain that $p(z, u) = (z^2 - 1)^p$ or $(z^2 - 1)^p(z - 1)$, $p \geq 1$. Thus $W(A) = \overline{\mathbb{D}}$ contradicting our assumption. We conclude that there can be at most $(n - 2)$ arcs of ∂E on $\partial W(A)$.

To prove (b), assume that $E = \overline{\mathbb{D}}$ and that $\partial W(A)$ contains no arc of $\partial E = \partial \mathbb{D}$. For any $\varepsilon > 0$, let

$$A_\varepsilon = \begin{bmatrix} 0 & 2(1 + \varepsilon) \\ 0 & 0 \end{bmatrix} \oplus A.$$

Since $\partial W(A)$ intersects $\partial \mathbb{D}$ at more than n (isolated) points, $\partial W(A_\varepsilon)$ has more than n arcs of the circle $|z| = 1 + \varepsilon$ if ε is sufficiently small. As A_ε is an $(n + 2)$ -by- $(n + 2)$ matrix with $W(A_\varepsilon)$ containing the circular disc

$$W\left(\begin{bmatrix} 0 & 2(1 + \varepsilon) \\ 0 & 0 \end{bmatrix}\right) = \{z \in \mathbb{C} : |z| \leq 1 + \varepsilon\}$$

and $\partial W(A_\varepsilon)$ containing more than $(n + 2) - 2$ arcs of its boundary, this contradicts what we proved in (a). Hence $\partial W(A)$ must contain at least one arc of ∂E .

To show the sharpness of “ $n - 2$ ”, let

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus \text{diag}(r, re^{i\theta_0}, re^{2i\theta_0}, \dots, re^{(n-3)i\theta_0}),$$

where $1 < r < \sec(\pi/(n - 2))$ and $\theta_0 = 2\pi/(n - 2)$. Then $W(A)$ contains

$$\overline{\mathbb{D}} = W\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right)$$

and $\partial W(A)$ contains $(n - 2)$ arcs of $\partial \mathbb{D}$. For the sharpness of “ n ”, consider $A = \text{diag}(1, e^{2\pi i/n}, e^{4\pi i/n}, \dots, e^{2\pi(n-1)i/n})$. Then $W(A)$ is a polygonal region which contains $E = \{z \in \mathbb{C} : |z| \leq \cos(\pi/n)\}$, and $\partial W(A)$ is a regular n -gon which intersects ∂E at exactly n points. But, obviously, $\partial W(A)$ contains no arc of ∂E . ■

Note that Theorem 2.5 (b) can be contrasted with Anderson’s theorem: if the numerical range $W(A)$ of an n -by- n matrix A is assumed to be contained in a closed elliptic disc E with $\partial W(A)$ and ∂E intersecting at more than n points, then the stronger conclusion that $W(A) = E$ holds.

3. Nilpotent matrices

In this section, we restrict ourselves to nilpotent matrices. We will improve the results in section 2 to obtain sharper bounds for the numbers of line segments and circular arcs on the boundary of their numerical ranges.

Our first proposition is the nilpotent analogue of Anderson’s theorem. Its proof is essentially the same as the one for [10, Lemma 6], so we omit the proof.

PROPOSITION 3.1 *Let A be an n -by- n nilpotent matrix. If $W(A)$ is contained in a closed circular disc D centered at the origin and $\partial W(A)$ intersects ∂D at more than $n - 2$ points, then $W(A) = D$. In particular, in this case if A is unitarily equivalent to*

$$\begin{bmatrix} 0 & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ & 0 & \cdot & & & \cdot \\ & & \cdot & \cdot & & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & a_{n-1n} \\ & & & & & 0 \end{bmatrix},$$

then at least one of $a_{12}, \dots, a_{n-1n}, a_{1n}$ is zero.

The next proposition provides an example of a nilpotent matrix which is useful in proving the sharpness for the results in this section. In particular, it shows the sharpness of “ $n - 2$ ” in Proposition 3.1.

PROPOSITION 3.2 *For $n \geq 3$, let A_n be the n -by- n matrix*

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ & 0 & 1 & & & 0 \\ & & \cdot & \cdot & & \vdots \\ & & & \cdot & \cdot & 0 \\ & & & & \cdot & 1 \\ 0 & & & & & 0 \end{bmatrix}.$$

Then (a) $W(A_n) \not\subseteq \overline{\mathbb{D}}$, (b) $\partial W(A_n)$ intersects $\partial \mathbb{D}$ at exactly $(n - 2)$ points, and (c) there are $n - 2$ line segments on $\partial W(A_n)$.

Proof (a) For any unit vector $x = [x_1, \dots, x_n]^T$ in \mathbb{C}^n , we have

$$\begin{aligned} |\langle A_n x, x \rangle| &= |(x_2 + x_n)\bar{x}_1 + x_3\bar{x}_2 + \cdots + x_n\bar{x}_{n-1}| \\ &\leq |x_2||x_1| + |x_3||x_2| + \cdots + |x_n||x_{n-1}| + |x_1||x_n| \\ &= \langle U_n |x|, |x| \rangle \leq 1, \end{aligned}$$

where U_n is the n -by- n unitary matrix

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & \cdot & \cdot & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ 1 & & & & 0 \end{bmatrix}$$

and $|x|$ denotes the unit vector $[|x_1|, \dots, |x_n|]^T$. This shows that $W(A_n) \subseteq \overline{\mathbb{D}}$. That $W(A_n) \neq \overline{\mathbb{D}}$ follows from the second assertion of Proposition 2.1.

(b) If x_0 is the unit vector $[1/\sqrt{n}, \dots, 1/\sqrt{n}]^T$, then $\langle A_n x_0, x_0 \rangle = 1$. This shows that $\partial W(A_n)$ intersects $\partial \mathbb{D}$ at the point 1. To prove that they intersect at no less than $n - 2$ points, let $U = \text{diag}(1, e^{i\theta_0}, e^{2i\theta_0}, \dots, e^{(n-1)i\theta_0})$, where $\theta_0 = 2\pi/(n - 2)$. Then it is easily

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seen that $U^* A_n U = e^{i\theta_0} A_n$. Hence, in particular, $W(A_n)$ has an $(n - 2)$ -symmetry. Thus $\partial W(A_n)$ and $\partial \mathbb{D}$ intersect at the points $1, e^{i\theta_0}, \dots,$ and $e^{(n-3)i\theta_0}$. Since $W(A_n) \neq \overline{\mathbb{D}}$, the first assertion in Proposition 3.1 implies that $\partial W(A_n)$ and $\partial \mathbb{D}$ can intersect at no more than $n - 2$ points. This proves our assertion.

(c) This is proved in [2, Theorem 6] as a consequence of more general results. ■

The next corollary was essentially proved in [6, Theorem 4.1] through computations. Here we give an alternative more conceptual proof.

COROLLARY 3.3 *Let A be a 3-by-3 nilpotent matrix. Then*

- (a) $\partial W(A)$ contains at most one line segment, and
- (b) $\partial W(A)$ has one line segment if and only if A is unitarily equivalent to a matrix of the form

$$\begin{bmatrix} 0 & a & a \\ & 0 & a \\ & & 0 \end{bmatrix}$$

for some $a \neq 0$.

Proof Since 0 is the only eigenvalue of A , (a) follows easily from the classification of the numerical ranges of 3-by-3 matrices.

To prove (b), assume that $\partial W(A)$ has one line segment L and

$$A = \begin{bmatrix} 0 & a & b \\ & 0 & c \\ & & 0 \end{bmatrix}.$$

We deduce from Lemma 2.4 that L is tangent to

$$\partial W\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right), \quad \partial W\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) \quad \text{and} \quad \partial W\left(\begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}\right).$$

Since these are circles with center the origin and radii $|a|/2, |b|/2$ and $|c|/2$, respectively, we obtain, in particular, that $|a| = |b| = |c|$. Note that a, b and c are not zero for otherwise $W(A) = \{0\}$ has no line segment on its boundary, contradicting our assumption. Let $b = ae^{i\theta_1}$ and $c = ae^{i\theta_2}$ with real θ_1 and θ_2 , and let $U = \text{diag}(1, e^{i(\theta_2 - \theta_1)}, e^{i(\theta_2 - 2\theta_1)})$. A computation shows that

$$U^* A U = \begin{bmatrix} 0 & a' & a' \\ & 0 & a' \\ & & 0 \end{bmatrix},$$

where $a' = ac/b$. This proves one direction. The converse follows from Proposition 3.2 (c). ■

For a 4-by-4 nilpotent matrix A , there is not much about $W(A)$ which we know for sure. For one thing, we suspect that $l(A) \leq 2$, but are not able to prove it.

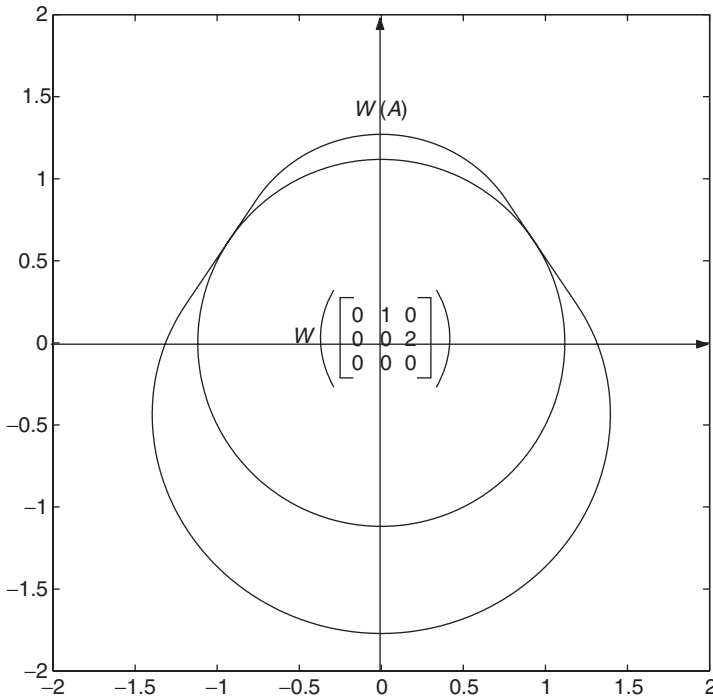


Figure 1.

Moreover, there seems to be no easy characterization of those A 's with $l(A) = 2$. For example, besides A_4 of Proposition 3.2 the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & -2 \\ & 0 & 2 & i \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

has also had two (nonparallel) line segments on $\partial W(A)$ (cf. Figure 1).

The next theorem gives the analogue of Theorem 2.3 for nilpotent matrices. It also generalizes part of Proposition 3.2 (c).

THEOREM 3.4. *If A is an n -by- n ($n \geq 3$) nilpotent matrix which has an $(n - 1)$ -by- $(n - 1)$ submatrix B with $W(B)$ a circular disc centered at the origin, then $l(A) \leq n - 2$ for $n = 3, 4, 5$, and $l(A) \leq 2(n - 4)$ for $n \geq 6$. Moreover, these upper bounds are sharp.*

Proof The assertions for $n=3$ are easily seen to be true. For $n \geq 4$, the proof is analogous to the one for Theorem 2.3. Here we give a brief sketch of it. Assuming that $W(B) = \overline{\mathbb{D}}$, we obtain from some simple computations with the upper triangular form of A that $p(z, u) \equiv \det(zI_n - \operatorname{Re}(uA))$ ($z \in \mathbb{C}$ and $u \in \partial\overline{\mathbb{D}}$) is of the form $(\sum_{j=0}^{n-2} p_j(z)u^j) + (\sum_{j=1}^{n-2} \tilde{p}_j(z)\bar{u}^j)$, where $p_0(z)$ is a polynomial of degree n , and $p_j(z)$ is of degree no more than $n - j - 2$ and $\tilde{p}_j(z) = \overline{p_j(\bar{z})}$ for $1 \leq j \leq n - 2$. Under the assumption $l(A) > n - 2$ (resp., $l(A) > 2(n - 4)$) for $n = 4, 5$ (resp., $n \geq 6$), we deduce using the

Riesz–Fejér theorem and the fundamental theorem of algebra from $p(1, u) \leq 0$ for all u , $|u| = 1$, and $p(1, u) = 0$ for more than $n - 2$ values of u that $p(1, u) = 0$ for all u . This implies that $p_j(1) = 0$ for all j . Similarly, we have $p_j(-1) = 0$, and thus $z^2 - 1$ divides $p_j(z)$ for all j . If $n = 4$, then $p(z, u) = p_0(z)$, which is independent of u . Thus the same is true for the largest eigenvalue of $\operatorname{Re}(uA)$. This shows that $W(A) = \overline{\mathbb{D}}$, contradicting our assumption that $l(A) > 2$. For $n \geq 5$, we have $p(z, u) = (z^2 - 1)q_1(z, u)$ for some polynomial q_1 such that $q_1(z, u)$ is a trigonometric polynomial in u of degree $\leq n - 4$ for each fixed z . Using the continuity argument as in the proof of Theorem 2.3, we infer that $q_1(\pm 1, u) = 0$ for more than 3 (resp., $2(n - 4)$) values of u for $n = 5$ (resp., $n \geq 6$). Since $u^{n-4}q_1(\pm 1, u)$ is a polynomial of degree $\leq 2(n - 4)$, the fundamental theorem of algebra again implies that $q_1(\pm 1, u) = 0$ for all u , $|u| = 1$. We then obtain $q_1(z, u) = (z^2 - 1)q_2(z, u)$ and hence $p(z, u) = (z^2 - 1)^2q_2(z, u)$. Proceeding in this fashion, we eventually reach a contradiction. Therefore, $l(A) \leq 3$ (resp., $l(A) \leq 2(n - 4)$) for $n = 5$ (resp., $n \geq 6$).

We now show the sharpness of “ $n - 2$ ” and “ $2(n - 4)$ ” for $3 \leq n \leq 5$ and $n \geq 6$, respectively. If $3 \leq n \leq 5$, then the matrix A_n in Proposition 3.2 has the $(n - 1)$ -by- $(n - 1)$ submatrix J_{n-1} with $W(J_{n-1}) = \{z \in \mathbb{C} : |z| \leq \cos(\pi/n)\}$, and has $n - 2$ line segments on $\partial W(A_n)$. On the other hand, for $n \geq 6$, let

$$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \oplus A_{n-2},$$

where a satisfies $2 \cos(\pi/(n - 2)) < a < 2$. Then A has the $(n - 1)$ -by- $(n - 1)$ submatrix

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \oplus J_{n-3}$$

with numerical range $\{z \in \mathbb{C} : |z| \leq a/2\}$. Since $W(A)$ is the convex hull of

$$W\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right)$$

and $W(A_{n-2})$, and the line segments on $\partial W(A_{n-2})$ are all at distance $\cos(\pi/(n - 2))$ from the origin, we infer that $\partial W(A)$ has $2(n - 4)$ line segments, completing the proof. ■

We remark that it is unknown whether “ $n - 2$ ” (for $n = 4, 5$) and “ $2(n - 4)$ ” (for $n \geq 6$) are upper bounds of $l(A)$ for a general nilpotent matrix A . One approach may be to show that every such A has a submatrix B with the property in Theorem 3.4.

Our final theorem gives the nilpotent analogue of Theorem 1.5. Again its second assertion is comparable to Proposition 3.1.

THEOREM 3.5 *Let A be an n -by- n nilpotent matrix.*

- (a) *If $n \geq 5$, then $\partial W(A)$ contains at most $n - 4$ arcs of any circle centered at the origin.*
- (b) *If $W(A)$ contains a closed circular disc D centered at the origin and $\partial W(A)$ and ∂D intersect at more than $n - 2$ points, then $W(A) = D$ if $2 \leq n \leq 4$, and $\partial W(A)$ contains at least one arc of ∂D if $n \geq 5$.*

In these cases, both “ $n - 4$ ” and “ $n - 2$ ” are sharp.

Proof (a) The proof proceeds as in that of Theorem 2.5 except that this time, for $p(z, u) \equiv \det(zI_n - \operatorname{Re}(uA))$ ($z \in \mathbb{C}$ and $|u| = 1$) the trigonometric polynomial $p(\pm 1, u)$ is of degree at most $n - 2$. Under the assumption that there are more than $n - 4$ arcs α_j of $\partial\mathbb{D}$ on $\partial W(A)$ for $n \geq 5$, we have $p(z, u) = (z^2 - 1)q_1(z, u)$ for some trigonometric polynomial q_1 in u of degree at most $(n - 4)$. A continuity argument as before yields that $q_1(\pm 1, u_k) = 0$ for all the more than $2(n - 4)$ endpoints u_k of the α_j 's. Applying the fundamental theorem of algebra to the polynomial $u^{n-4}q_1(\pm 1, u)$ of degree $\leq 2(n - 4)$ then gives $q_1(\pm 1, u) = 0$ for all u , $|u| = 1$, and thus $p(z, u) = (z^2 - 1)^2q_2(z, u)$ for some trigonometric polynomial q_2 in u of degree at most $n - 6$. Proceeding in this fashion, we eventually reach a contradiction. This proves that there can be at most $n - 4$ arcs of $\partial\mathbb{D}$ on $\partial W(A)$ for $n \geq 5$.

(b) Assume that $D = \overline{\mathbb{D}}$ and that $\partial W(A)$ and $\partial\mathbb{D}$ intersect at more than $n - 2$ points. If $n = 2$, then $W(A)$ is itself a circular disc centered at the origin and thus our assumption implies that $W(A) = \overline{\mathbb{D}}$. For $n \geq 3$, we assume further that $\partial W(A)$ contains no arc of $\partial\mathbb{D}$. Consider the $(n + 2)$ -by- $(n + 2)$ matrix

$$A_\varepsilon = \begin{bmatrix} 0 & 2 + \varepsilon \\ 0 & 0 \end{bmatrix} \oplus A$$

for sufficiently small $\varepsilon > 0$. Then $\partial W(A_\varepsilon)$ has more than $n - 2$ arcs of $\partial\mathbb{D}$, one in the vicinity of each of the more than $n - 2$ (isolated) intersection points of $\partial W(A)$ and $\partial\mathbb{D}$. This contradicts (a). Hence $\partial W(A)$ must contain some arc of $\partial\mathbb{D}$. We now show that, for $n = 3$ or 4 , this implies that $W(A) = \overline{\mathbb{D}}$. Indeed, in this case, we have $p(z, u) = (z^2 - 1)q_1(z, u)$ as before. Then the coefficient of u^j , $j = 1, 2$, in $p(z, u)$ should all vanish. This shows that $p(z, u)$ is a degree- n polynomial in z for any value of u , $|u| = 1$. Hence $W(A) = \overline{\mathbb{D}}$ as asserted.

For the sharpness of “ $n - 4$ ” ($n \geq 5$), consider

$$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \oplus A_{n-2},$$

where a satisfies $2 \cos(\pi/(n - 2)) < a < 2$ and A_{n-2} is the matrix defined in Proposition 3.2. Then $\partial W(A)$ contains $n - 4$ arcs of

$$\partial W\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right) = \left\{z \in \mathbb{C} : |z| = \frac{a}{2}\right\},$$

since each of the $n - 4$ line segments on $\partial W(A_{n-2})$ is tangent to the circle $\partial W(J_{n-3}) = \{z \in \mathbb{C} : |z| = \cos(\pi/(n - 2))\}$ by Lemma 2.4. To show the sharpness of “ $n - 2$ ” ($n \geq 3$), let $A = A_n$ as in Proposition 3.2 and let $D = \{z \in \mathbb{C} : |z| \leq \cos(\pi/n)\}$. Then, as proved in Proposition 3.2, $D \subseteq W(A)$ and $\partial W(A)$ and ∂D intersect at exactly $n - 2$ points. In particular, this shows that there can be no arc of ∂D on $\partial W(A)$, completing the proof. ■

We conclude this article by reiterating, in light of Theorems 2.3 and 3.4, the two problems concerning $l(A)$, the number of line segments on the boundary of the numerical range $W(A)$ of an n -by- n matrix A :

- (a) Is it true that $l(A) \leq 2(n - 2)$ for any n -by- n ($n \geq 6$) matrix A ?
- (b) Is it true that $l(A) \leq n - 2$ ($n = 4, 5$) and $l(A) \leq 2(n - 4)$ ($n \geq 6$) for any n -by- n nilpotent matrix A ?

Acknowledgements

The research was partially supported by the National Science Council of the Republic of China under projects NSC-94-2115-M-008-010 and NSC-94-2115-M-009-007, respectively, and, for the second author, by the MOE-ATU project.

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