KÄHLER STRUCTURES AND WEIGHTED ACTIONS ON THE COMPLEX TORUS

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Abstract

Let T be the compact real torus, and $T_{\mathbf{C}}$ its complexification. Fix an integral weight α, and consider the α-weighted T-action on $T_{\mathbf{C}}$. If ω is a T-invariant Kähler form on $T_{\mathbf{C}}$, it corresponds to a pre-quantum line bundle $\mathbf{L}$ over $T_{\mathbf{C}}$. Let H_ω be the square-integrable holomorphic sections of $\mathbf{L}$. The weighted T-action lifts to a unitary T-representation on the Hilbert space H_ω, and the multiplicity of its irreducible sub-representations is considered. It is shown that this is controlled by the image of the moment map, as well as the principle that 'quantization commutes with reduction'.

1. *Introduction*

Let T be the compact real n-torus, and $T_{\mathbf{C}}$ its complexification. Then T acts naturally on $T_{\mathbf{C}}$, as subgroup of $T_{\mathbf{C}}$. In [**3**], we study T-invariant Kähler structures on $T_{\mathbf{C}}$, and the corresponding geometric quantization. The present paper follows a suggestion of V. Guillemin, and considers the more general T-actions with weights.

We write $T = \mathbf{R}^n/\mathbf{Z}^n$ and $T_{\mathbf{C}} = \mathbf{C}^n/\mathbf{Z}^n$ as in [**3**], where

$$T_{\mathbf{C}} = \{z = x+\sqrt{-1}[y] : x\in\mathbf{R}^n, [y]\in\mathbf{R}^n/\mathbf{Z}^n = T\}. \tag{1.1}$$

Let $\mathfrak{t}$ be the Lie algebra of T. The notation (1.1) automatically identifies $\mathfrak{t}$, $\mathfrak{t}^*$, $\mathbf{R}^n$, $\mathbf{R}^{n*}$ with one another.

Consider now a weight $\alpha = (\alpha_1, \ldots, \alpha_n)$ in the integral lattice $\mathbf{Z}^n \subset \mathbf{R}^n = \mathfrak{t}^*$. We define the α-weighted T-action on $T_{\mathbf{C}}$ by

$$T\times T_{\mathbf{C}} \longrightarrow T_{\mathbf{C}}, \quad ([t_j])\times(x_j+\sqrt{-1}[y_j]) \longmapsto (x_j+\sqrt{-1}[y_j+\alpha_j t_j]), \tag{1.2}$$

where $t_j, x_j, y_j \in \mathbf{R}$ for all $j = 1, \ldots, n$. In particular, if $\alpha_j = 1$ for all j, then (1.2) is just the standard action of subgroup T on $T_{\mathbf{C}}$. We shall always deal with Kähler structures on $T_{\mathbf{C}}$ that are invariant under this standard action, and we call them T-*invariant*. Let D_α be the diagonal matrix with entries $\alpha_1, \ldots, \alpha_n$ along the diagonal. We shall see that a T-invariant Kähler form is necessarily invariant under the weighted action (1.2), and has the expression $\omega = \sqrt{-1}\partial\bar{\partial}F$. In fact, the weighted action preserving ω is Hamiltonian, with moment map $\Phi : T_{\mathbf{C}} \longrightarrow \mathfrak{t}^*$ given by

$$\Phi(z) = \tfrac{1}{2}D_\alpha \cdot F'(x) = \frac{1}{2}\left(\alpha_j \frac{\partial F}{\partial x_j}(x)\right)$$

for all $z = x+\sqrt{-1}[y] \in T_{\mathbf{C}}$.

Since $\omega = \sqrt{-1}\partial\bar{\partial}F$, it has to be exact, and is in particular integral. We obtain a pre-quantum line bundle $\mathbf{L}$ over $T_{\mathbf{C}}$ [**5**, **6**]. The Chern class of $\mathbf{L}$ is the cohomology class $[\omega] = 0$, so $\mathbf{L}$ is a trivial bundle. It is equipped with a connection ∇ whose

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curvature is ω, as well as an invariant Hermitian structure $(,)$. We say that a smooth section s of $\mathbf{L}$ is *holomorphic* if $\nabla_v s = 0$ for every anti-holomorphic vector field v. Let $H(\mathbf{L})$ denote the space of all holomorphic sections. The weighted action (1.2) leads to a T-representation on $H(\mathbf{L})$. Let dV be the Haar measure on $T_{\mathbf{C}}$. To obtain a unitary representation out of $H(\mathbf{L})$, let H_ω be the space of all holomorphic sections s that satisfy

$$\int_{T_{\mathbf{C}}} (s, s)\, dV < \infty. \tag{1.3}$$

Then H_ω is a unitary T-representation space. Its infinitesimal $\mathfrak{t}$-representation is written as $\xi \cdot s \in H_\omega$, for $\xi \in \mathfrak{t}$, $s \in H_\omega$. The irreducible subrepresentations of H_ω are 1-dimensional, and each is a subspace of

$$(H_\omega)_\lambda = \{s \in H_\omega : \xi \cdot s = (\lambda, \xi)\, s \text{ for all } \xi \in \mathfrak{t}\},$$

for some $\lambda \in \mathbf{Z}^n \subset \mathfrak{t}^*$. A basic question in geometric quantization is to compute the multiplicity of irreducible representations in H_ω.

Let Ω be the image of the moment map. If the weight α in (1.2) contains no zero entry, then the multiplicity problem is solved by an easy generalization of [**3**], as follows.

THEOREM 1.1. *If α has no zero entry, then the unitary representation H_ω is multiplicity-free. It contains $(H_\omega)_\lambda$ if and only if $\lambda \in \Omega$ and $\lambda_j/\alpha_j \in \mathbf{Z}$ for all j.*

We shall prove Theorem 1.1 in §2. Given a unitary representation of a Lie group, we call it a *model* if it contains every irreducible representation once. This terminology is due to I. M. Gelfand and A. Zelevinski [**4**]. From Theorem 1.1, Corollary 1.1 follows.

COROLLARY 1.1. *H_ω is a model of T if and only if the moment map is surjective and $\alpha_j = \pm 1$ for all j.*

The main purpose of this paper is to consider the more complicated situation where the weight α in (1.2) contains zero entries. In this case, the multiplicity of $(H_\omega)_\lambda$ is no longer determined by the image of the moment map alone. To handle this problem, we introduce *symplectic reduction*, a process first explored by J. Marsden and A. Weinstein [**7**]. In the study of Hamiltonian group actions on symplectic manifolds, two of the central aspects are geometric quantization and symplectic reduction. A unifying theme between them is given by V. Guillemin and S. Sternberg [**5**] and is often called 'quantization commutes with reduction'. A summary of recent developments of such concepts can be found in [**8**]. We shall see that it helps to solve our multiplicity problem.

We now perform symplectic reduction. Suppose that λ is in the image Ω of the moment map Φ. Then T acts on $\Phi^{-1}(\lambda)$, and we call $B_\lambda = \Phi^{-1}(\lambda)/T$ the *reduced space*. Let

$$\iota : \Phi^{-1}(\lambda) \hookrightarrow T_{\mathbf{C}}, \quad \pi : \Phi^{-1}(\lambda) \longrightarrow B_\lambda$$

respectively denote the natural inclusion and quotient. Then B_λ is equipped with a symplectic structure ω_λ, such that $\pi^*\omega_\lambda = \iota^*\omega$. This process is called symplectic reduction, and ω_λ is called the *reduced symplectic form*. We study certain properties of the reduced space $(B_\lambda, \omega_\lambda)$ in §3.

Let k be the number of non-zero entries of the weight α, where $1 \leqslant k \leqslant n$. We may rearrange the indices and assume that $\alpha = (\alpha_1, \dots, \alpha_k, 0, \dots, 0)$, where $\alpha_1, \dots, \alpha_k$ are non-zero. Recall that $\omega = \sqrt{-1}\partial\bar{\partial}F$. Since F can be regarded as a strictly convex function on $\mathbf{R}^n$ (see [3]), the subset $X \subset \mathbf{R}^n$ defined by

$$X = \{x \in \mathbf{R}^n : \frac{1}{2}\frac{\partial F}{\partial x_j}(x) = \frac{\lambda_j}{\alpha_j} \text{ for } j = 1, \dots, k\}$$

is a smooth submanifold of dimension $(n-k)$. Let $T^{n-k} \subset T$ be the real subtorus, spanned by the last $(n-k)$ coordinates. Then $X \times T^{n-k}$ imbeds into $T_{\mathbf{C}}$, via

$$\jmath : X \times T^{n-k} \hookrightarrow \mathbf{R}^n \times T = T_{\mathbf{C}}.$$

In §3, we prove the following theorem.

THEOREM 1.2. *The reduced space $(B_\lambda, \omega_\lambda)$ is symplectomorphic to the symplectic submanifold $(X \times T^{n-k}, \jmath^*\omega)$.*

By Theorem 1.2, we identify the reduced space B_λ with the symplectic submanifold $X \times T^{n-k}$. Since ω_λ is exact, we again have the pre-quantum line bundle over B_λ, denoted $\mathbf{L}_\lambda$. Here $\mathbf{L}_\lambda$ is trivial, because its Chern class is the cohomology class $[\omega_\lambda] = 0$. It may be regarded as the restriction of $\mathbf{L}$ to B_λ, due to Theorem 1.2. The space $B_\lambda \subset T_{\mathbf{C}}$ is not complex, so there is no intrinsic polarization for an immediate definition of 'holomorphic' sections $H(\mathbf{L}_\lambda)$. We shall define $H(\mathbf{L}_\lambda)$ among the smooth sections of $\mathbf{L}_\lambda$ in §4. This coincides with the usual holomorphic sections in cases where B_λ happens to be complex.

The Haar measure of $T_{\mathbf{C}}$ restricts to a measure on $B_\lambda = X \times T^{n-k}$, still denoted by dV. We again use the Hermitian structure on $\mathbf{L}_\lambda$ to define an L^2-structure on $H(\mathbf{L}_\lambda)$, and let $H_{(\omega_\lambda)}$ denote the square-integrable sections in $H(\mathbf{L}_\lambda)$. In other words, $H_{(\omega_\lambda)}$ consists of all $s \in H(\mathbf{L}_\lambda)$ in which

$$\int_{B_\lambda} (s, s)\, dV < \infty.$$

In §4, we prove that geometric quantization commutes with reduction, as stated in Theorem 1.3.

THEOREM 1.3. *$H_{(\omega_\lambda)}$ is a Hilbert space, and $(H_\omega)_\lambda \cong H_{(\omega_\lambda)}$.*

Clearly, Theorem 1.1 is a special case of Theorem 1.3: if the weight α has no zero entry, then for all $\lambda \in \Omega$, the reduced space B_λ is just a point. Therefore $H_{(\omega_\lambda)} = \mathbf{C}$, and Theorem 1.3 implies that $(H_\omega)_\lambda$ occurs with multiplicity 1.

If the Hilbert spaces in Theorem 1.3 are infinite-dimensional, then of course the isomorphism in question is trivial. In §5, we justify the significance of this theorem by showing that their dimensions can be any of $0, 1, 2, \dots, \infty$.

2. *Geometric quantization*

Let ω be a T-invariant Kähler form on the complex torus $T_{\mathbf{C}}$. By [3],

$$\omega = d\beta = \sqrt{-1}\partial\bar{\partial}F, \tag{2.1}$$

where β and F are T-invariant. We use the standard coordinates $z = x + \sqrt{-1}[y]$

introduced in equation (1.1). Here F, being T-invariant, depends on the x-variables only. Since the weighted action defined in (1.2) acts along the $[y]$-variables, it preserves F. Therefore, the weighted action also preserves ω.

Given $\xi \in \mathfrak{t} = \mathbf{R}^n$, let $\xi^\#$ be the infinitesimal vector field on $T_{\mathbf{C}}$ induced by the weighted T-action. Hence

$$\xi^\# = \sum_j \alpha_j \xi_j \frac{\partial}{\partial y_j}.$$

In (2.1), $\beta = \frac{1}{2}\sum_j \partial F/\partial x_j \, dy_j$. Hence the moment map $\Phi: T_{\mathbf{C}} \longrightarrow \mathfrak{t}^*$ of the weighted action is given by

$$\begin{aligned}(\Phi(z), \xi) &= (\beta, \xi^\#)\\ &= \left(\tfrac{1}{2}\sum_j \frac{\partial F}{\partial x_j}(x)\, dy_j, \sum_k \alpha_k \xi_k \frac{\partial}{\partial y_k}\right)\\ &= \sum_j \tfrac{1}{2}\alpha_j \xi_j \frac{\partial F}{\partial x_j}(x),\end{aligned}$$

for all $z \in T_{\mathbf{C}}$ and $\xi \in \mathfrak{t}$ (see [**1**, Theorem 4.2.10]). Therefore, the moment map is

$$\Phi(z) = \frac{1}{2}\left(\alpha_j \frac{\partial F}{\partial x_j}(x)\right).$$

As discussed in §1, ω corresponds to a pre-quantum line bundle $\mathbf{L}$ [**5**, **6**], whose holomorphic sections are denoted by $H(\mathbf{L})$. By [**3**], there exists a non-vanishing T-invariant holomorphic section s_0 satisfying

$$(s_0, s_0) = e^{-F}. \tag{2.2}$$

For the rest of this section, we assume that in the weighted T-action (1.2), α has no zero entry. The weighted action lifts to a T-representation on $H(\mathbf{L})$. Each irreducible subrepresentation is one-dimensional, and is of the form $\mathbf{C}(e^{c\cdot z}s_0)$ for some $c \in \mathbf{Z}^n$. Since s_0 is T-invariant, the corresponding infinitesimal $\mathfrak{t}$-representation is given by

$$\begin{aligned}\xi \cdot (e^{c\cdot z}s_0) &= (\xi \cdot e^{c\cdot z})\, s_0\\ &= \frac{d}{dt}\bigg|_0 \exp\left(\sum_j c_j(x_j + \sqrt{-1}[y_j + \alpha_j t\xi_j])\right) s_0\\ &= \sum_j c_j \alpha_j \xi_j e^{c\cdot z} s_0\end{aligned} \tag{2.3}$$

for all $\xi \in \mathfrak{t}$. Therefore, if we define

$$H(\mathbf{L})_\lambda = \{s \in H(\mathbf{L}) : \xi \cdot s = (\lambda, \xi)\, s \text{ for all } \xi \in \mathfrak{t}\},$$

then (2.3) says that

$$H(\mathbf{L})_\lambda = \mathbf{C}(e^{c\cdot z}s_0) \Leftrightarrow \lambda_j = c_j \alpha_j; j = 1, \ldots, n. \tag{2.4}$$

We consider the unitary T-representation H_ω consisting of holomorphic sections which converge under the integral (1.3). Let

$$(H_\omega)_\lambda = H(\mathbf{L})_\lambda \cap H_\omega.$$

We want to consider the multiplicity of $(H_\omega)_\lambda$ in H_ω, and prove Theorem 1.1.

Proof of Theorem 1.1. By (2.4), we only need to consider the cases with $\lambda_j/\alpha_j \in \mathbf{Z}$ for all j. For such cases, let $c_j = \lambda_j/\alpha_j \in \mathbf{Z}$. Consider $s = e^{c\cdot z}s_0 \in H(\mathbf{L})_\lambda$, where s_0 is the holomorphic section in (2.2). Define $G \in C^\infty(\mathbf{R}^n)$ by $G(x) = F(x) - 2c\cdot x$. Then

$$\int_{T_{\mathbf{C}}} (e^{c\cdot z}s_0, e^{c\cdot z}s_0)\,dV = \int_{T_{\mathbf{C}}} e^{2c\cdot x}e^{-F}\,dV = \int_{T_{\mathbf{C}}} e^{-G}\,dV. \tag{2.5}$$

Since F and G have the same Hessian and F is strictly convex, so is G. According to Proposition 3.3 of [**3**], the integral (2.5) converges if and only if G has a global minimum. This is equivalent to $2c$ being contained in the image of the gradient function F'. Recall that D_α is the diagonal matrix with entries $\alpha_1, \ldots, \alpha_n$, so that the moment map is $\Phi = \frac{1}{2}D_\alpha \cdot F'$. It follows that

$$\begin{aligned} e^{c\cdot z}s_0 \in H_\omega &\Leftrightarrow 0 \in \mathrm{Image}(G') \\ &\Leftrightarrow 2c \in \mathrm{Image}(F') \\ &\Leftrightarrow c \in \mathrm{Image}(\tfrac{1}{2}F') \\ &\Leftrightarrow D_\alpha c \in \mathrm{Image}(\tfrac{1}{2}D_\alpha \cdot F') \\ &\Leftrightarrow \lambda \in \mathrm{Image}(\Phi). \end{aligned}$$

By (2.4), $\mathbf{C}(e^{c\cdot z}s_0) = H(\mathbf{L})_\lambda$, which completes the proof of Theorem 1.1. □

Since $F \in C^\infty(\mathbf{R}^n)$ is a strictly convex function [**3**], the image of $\frac{1}{2}F'$ is a convex set in $\mathbf{R}^n$. Thus the image of the moment map $\Phi = \frac{1}{2}D_\alpha \cdot F'$ is convex, and it includes all $\lambda \in \mathbf{Z}^n$ exactly when Φ is surjective. Therefore, by Theorem 1.1, $(H_\omega)_\lambda \neq 0$ for all $\lambda \in \mathbf{Z}^n$ if and only if Φ is surjective and $\alpha_j = \pm 1$ for all j. This proves Corollary 1.1.

3. *Symplectic reduction*

Let ω be a T-invariant Kähler form on $T_{\mathbf{C}}$, preserved by the α-weighted T-action (1.2). From now on, we consider the more interesting case where α has zero entries, which is the main purpose of this paper. The square-integrable holomorphic sections H_ω now have a more complicated multiplicity problem. It turns out that symplectic reduction [**7**] can handle this problem. In this section, we describe the process of symplectic reduction, and prove Theorem 1.2.

The torus T has dimension n. Let k be the number of non-zero entries of the weight α, where $1 \leqslant k \leqslant n$. We may arrange the indices so that the first k entries $\alpha_1, \ldots, \alpha_k$ are non-zero. We identify $\mathbf{R}^k$ with the subspace of $\mathbf{R}^n$ spanned by the first k variables. Intuitively, we can think of it as being 'horizontal'. In this way, the horizontal k-dimensional affine subspaces $H^v \subset \mathbf{R}^n$ are defined by

$$H^v = \mathbf{R}^k + v = \{(x_1, \ldots, x_k, 0, \ldots, 0) : x_j \in \mathbf{R}\} + v, \quad v \in \mathbf{R}^n. \tag{3.1}$$

Similarly, we may regard $\mathbf{R}^{n-k} \subset \mathbf{R}^n$ as the subspace spanned by the last $(n-k)$-coordinates, and define the 'vertical' affine $(n-k)$-subspaces $V^c \subset \mathbf{R}^n$ by

$$V^c = c + \mathbf{R}^{n-k} = c + \{(0, \ldots, 0, x_{k+1}, \ldots, x_n) : x_j \in \mathbf{R}\}, \quad c \in \mathbf{R}^n. \tag{3.2}$$

Recall that ω has potential function F. Let Ω be the image of the moment map, and let D_α be the diagonal matrix with entries $\alpha_1, \ldots, \alpha_n$. Fix $\lambda \in \Omega$, and consider

$$X = (\tfrac{1}{2}D_\alpha \cdot F')^{-1}(\lambda) = \{\mathrm{x} \in \mathbf{R}^n : \frac{1}{2}\frac{\partial F}{\partial x_j}(x) = \frac{\lambda_j}{\alpha_j} \text{ for } j = 1, \ldots, k\}. \tag{3.3}$$

The space X will play an important role in our study of symplectic reduction.

If we let $\bar{X}$ denote the closure of X in $\mathbf{R}^n$, then the boundary of X is defined by $\partial X = \bar{X} \backslash X$. The following proposition gives some properties of X.

PROPOSITION 3.1. *The space* $X = (\frac{1}{2} D_\alpha \cdot F')^{-1}(\lambda)$ *is a closed, unbounded* $(n-k)$-*dimensional submanifold of* $\mathbf{R}^n$, *and* $\partial X = \emptyset$. *For each* $v \in \mathbf{R}^n$, *the horizontal affine* k-*space* H^v *intersects* X *at most once.*

Proof. Since $F \in C^\infty(\mathbf{R}^n)$ is strictly convex [3], $\frac{1}{2}F'$ maps $\mathbf{R}^n$ diffeomorphically onto a domain $U \subset \mathbf{R}^n$. Then D_α maps U onto the image Ω of the moment map. Since D_α is a diagonal matrix whose last $(n-k)$ entries vanish, $\lambda \in \Omega$ may be written as $\lambda = (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$.

Let $c = (\lambda_1/\alpha_1, \ldots, \lambda_k/\alpha_k, 0, \ldots, 0)$, and let $V^c \subset \mathbf{R}^n$ be the vertical affine $(n-k)$-space defined in (3.2). Then $D_\alpha^{-1}(\lambda) \cap U = V^c \cap U$ is $(n-k)$-dimensional. However, $\frac{1}{2}F'$ is a diffeomorphism between X and $V^c \cap U$, so X is an $(n-k)$-dimensional manifold.

Since $V^c \cap U$ is closed in U, we conclude from the diffeomorphism $\frac{1}{2}F'$ that X is closed in $\mathbf{R}^n$. Since $V^c \cap U$ is not compact, neither is X. Hence X, being non-compact and closed in $\mathbf{R}^n$, is unbounded. Also, X equals its closure $\bar{X}$ simply because X is closed, so the boundary ∂X is empty.

For $v \in \mathbf{R}^n$, let H^v be the horizontal affine k-space defined in (3.1). It remains to show that X intersects each H^v at most once. Suppose that, for some $v \in \mathbf{R}^n$, there exist distinct $p, q \in X \cap H^v$. Let $S \subset H^v \subset \mathbf{R}^n$ be the straight line joining p and q. Let $f \in C^\infty(S)$ be the restriction of F to S. Since $p, q \in X$, equation (3.3) says that

$$\frac{\partial F}{\partial x_j}(p) = \frac{\partial F}{\partial x_j}(q) = 2\frac{\lambda_j}{\alpha_j}$$

for all $j = 1, \ldots, k$. This means that $f'(t)$ has the same value at p and q, where t is a linear variable on S. This is a contradiction, because f should be strictly convex on S. Hence, for all $v \in \mathbf{R}^n$, $X \cap H^v$ contains at most one point. This proves the proposition. $\square$

Since T is abelian, the moment map $\Phi = \frac{1}{2} D_\alpha \cdot F'$ is T-invariant. By Proposition 3.1, $\Phi^{-1}(\lambda)$ is a real $(2n-k)$-submanifold of $T_{\mathbf{C}}$ given by

$$\Phi^{-1}(\lambda) = X \times T = \{x + \sqrt{-1}[y] : x \in X\}. \tag{3.4}$$

Let ι be the natural inclusion of $\Phi^{-1}(\lambda)$ into $T_{\mathbf{C}}$. The torus T acts on $\Phi^{-1}(\lambda)$, and we let $B_\lambda = \Phi^{-1}(\lambda)/T$ be the quotient space. Let π be the quotient map from $\Phi^{-1}(\lambda)$ onto B_λ. There exists a symplectic form ω_λ on B_λ, satisfying $\pi^*\omega_\lambda = \iota^*\omega$. The construction of the symplectic manifold $(B_\lambda, \omega_\lambda)$ is called *symplectic reduction.*

Proof of Theorem 1.2. When T acts on $\Phi^{-1}(\lambda)$, the weight α has k non-zero entries. We have arranged the indices so that T acts only along the first k-variables of $[y]$. Therefore, equation (3.4) says that B_λ is diffeomorphic to the product manifold $X \times T^{n-k}$, where T^{n-k} denotes the subtorus of T spanned by the last $(n-k)$ variables. To prove the theorem, it remains to check the assertion on symplectic forms. Consider the following diffeomorphism σ followed by two inclusions $\jmath$ and ι,

$$B_\lambda \xrightarrow{\sigma} X \times T^{n-k} \xhookrightarrow{\jmath} \Phi^{-1}(\lambda) \xhookrightarrow{\iota} T_{\mathbf{C}}.$$

Let π be the quotient map from $\Phi^{-1}(\lambda)$ to B_λ. By the definition of ω_λ, $\imath^*\omega = \pi^*\omega_\lambda$. Therefore, since $\pi\cdot\jmath\cdot\sigma$ is the identity function on B_λ,

$$\sigma^*\cdot\jmath^*\cdot\imath^*\omega = \sigma^*\cdot\jmath^*\cdot\pi^*\omega_\lambda = \omega_\lambda.$$

This shows that the diffeomorphism σ identifies ω_λ with the pullback of ω to $X\times T^{n-k}$. Hence Theorem 1.2 holds. □

The realization of B_λ as $X\times T^{n-k}$ has the defect that the real submanifold $X\times T^{n-k}\subset T_{\mathbf{C}}$ is generally not complex. This is because the tangent bundle of $X\times T^{n-k}$ may not be preserved by the almost complex structure of $T_{\mathbf{C}}$.

We remark that there is a complex realization of B_λ, in terms of Reinhardt domain, with ω_λ being identified with the 'linear' Kähler structure $\omega_L = \sqrt{-1}/2\sum_j dz_j\wedge d\bar{z}_j$. However, this Kähler realization will not be used below, and we merely describe it in brief here. Recall that U is the image of the gradient function $\frac{1}{2}F'$. Consider the diffeomorphism

$$\tau: T_{\mathbf{C}}\longrightarrow U\times T,\quad \tau(x+\sqrt{-1}[y]) = \tfrac{1}{2}F'(x)+\sqrt{-1}[y].$$

By the definition (3.3) of X,

$$\tau(X\times T^{n-k}) = (V^c\cap U)\times T^{n-k}\subset V^c\times T^{n-k}, \tag{3.5}$$

where V^c is the vertical affine space (3.2) corresponding to $c = (\lambda_1/\alpha_1,\dots,\lambda_k/\alpha_k,0,\dots,0)$. Hence τ is a diffeomorphism from $X\times T^{n-k}$ to $R = (V^c\cap U)\times T^{n-k}$. In particular, R is a Reinhardt domain in a complex torus $V^c\times T^{n-k}$. Consider the standard Kähler structure on $T_{\mathbf{C}}$, $\omega_L = (\sqrt{-1}/2)\sum_j dz_j\wedge d\bar{z}_j$. It satisfies

$$\begin{aligned}\tau^*\omega_L &= \tau^*\frac{\sqrt{-1}}{2}\sum_j dz_j\wedge d\bar{z}_j\\ &= \tau^*\sum_j dx_j\wedge dy_j\\ &= \sum_j d(\tau^*x_j)\wedge dy_j\\ &= \frac{1}{2}\sum_{i,j}\frac{\partial^2 F}{\partial x_i\,\partial x_j}dx_i\wedge dy_j\\ &= \omega.\end{aligned}$$

Hence, by (3.5), τ identifies ω_λ with the pullback of ω_L to R. In other words, τ is a symplectomorphism between the pullback of ω to $X\times T^{n-k}$ and the pullback of ω_L to R. Unfortunately, the Kähler realization ω_L does not reflect the geometry of the original Kähler form ω, and its quantization is not so interesting. Hence we will always stick to the other realization, $(X\times T^{n-k},\omega_\lambda)$. From now on, we think of B_λ as the symplectic submanifold $X\times T^{n-k}\subset T_{\mathbf{C}}$, and regard the reduced symplectic form ω_λ as the pullback of ω to B_λ.

4. *Quantization commutes with reduction*

Recall that T acts on $T_{\mathbf{C}}$ by mapping (1.2), preserving $\omega = \sqrt{-1}\partial\bar{\partial}F$. We have assumed the weight of this action to be $\alpha = (\alpha_1,\dots,\alpha_k,0\dots,0)$, where $\alpha_1,\dots,\alpha_k$ are non-zero. In this way, every integral point in the image of the moment map is of the form $\lambda = (\lambda_1,\dots,\lambda_k,0,\dots,0)\in\Omega$. In the previous section, we performed symplectic reduction to λ, and obtained the reduced symplectic manifold $(B_\lambda,\omega_\lambda)$.

In this section, we apply geometric quantization to B_λ, and prove Theorem 1.3. Using Theorem 1.2, we shall always identify B_λ with the real submanifold $X \times T^{n-k} \subset T_{\mathbf{C}}$. In this way, the reduced symplectic form ω_λ is just the pullback of ω to B_λ. Since ω is exact, so is ω_λ. We let $\mathbf{L}_\lambda$ be the pre-quantum line bundle over B_λ, with Chern class $[\omega_\lambda] = 0$. Hence $\mathbf{L}_\lambda$ is a trivial bundle, and in fact is the restriction of $\mathbf{L}$ to B_λ.

As remarked at the end of the previous section, the submanifold $B_\lambda \subset T_{\mathbf{C}}$ is not complex. Therefore, there is no intrinsic polarization on the space of smooth sections $C^\infty(\mathbf{L}_\lambda)$. To overcome this problem, consider

$$\mathcal{O} = \{x + \sqrt{-1}[y] \in T_{\mathbf{C}} : H^x \text{ intersects } X\}, \tag{4.1}$$

where H^x is the horizontal k-space introduced in (3.1). Clearly, $\mathcal{O} \subset T_{\mathbf{C}}$ is open. In fact, since T acts along the first k variables of $[y]$, $\mathcal{O}$ is the smallest complex submanifold of $T_{\mathbf{C}}$ which contains B_λ and is preserved by the weighted T-action. Therefore, we can define $H(\mathcal{O}, \mathbf{L})_\lambda$ to be the holomorphic sections over $\mathcal{O}$ which transform by the weight λ under the T-action. Consequently, among the smooth sections $C^\infty(\mathbf{L}_\lambda)$ over B_λ, we can define

$$H(\mathbf{L}_\lambda) = \{s \in C^\infty(\mathbf{L}_\lambda) : s \text{ extendable to } H(\mathcal{O}, \mathbf{L})_\lambda\}. \tag{4.2}$$

In other words, $H(\mathbf{L}_\lambda)$ consists of all smooth sections of $\mathbf{L}_\lambda$ obtained from the restriction of $H(\mathcal{O}, \mathbf{L})_\lambda$ to B_λ.

We restrict the Haar measure dV to B_λ, and use the Hermitian structure of $\mathbf{L}_\lambda$ to define an L^2-structure on $H(\mathbf{L}_\lambda)$. Let $H_{(\omega_\lambda)}$ be the corresponding square-integrable sections:

$$H_{(\omega_\lambda)} = \left\{ s \in H(\mathbf{L}_\lambda) : \int_{B_\lambda} (s, s)\, dV < \infty \right\}. \tag{4.3}$$

In this way, $H_{(\omega_\lambda)}$ is a complex inner product space.

PROPOSITION 4.1. *$H_{(\omega_\lambda)}$ is a Hilbert space.*

Proof. The only thing to check is completeness. We do this by constructing an inner product space isomorphism between $H_{(\omega_\lambda)}$ and a Hilbert space.

From Proposition 3.1, $X \subset \mathbf{R}^n$ is an $(n-k)$-submanifold which intersects every horizontal affine k-space H^v (3.1) at most once. Define

$$W = \{v \in \mathbf{R}^{n-k} \subset \mathbf{R}^n : H^v \text{ intersects } X\}. \tag{4.4}$$

Then we get a diffeomorphism

$$\psi : X \longrightarrow W, \quad x \longmapsto (0, \ldots, 0, x_{k+1}, \ldots, x_n). \tag{4.5}$$

Let dV be the restriction of the Lebesgue measure of $\mathbf{R}^{n-k}$ to its open set $W = \psi(X)$, and dV_λ the restriction of the Lebesgue measure of $\mathbf{R}^n$ to X. Also, let $c_j = \lambda_j/\alpha_j \in \mathbf{Z}$ for $j = 1, \ldots, k$, and let $c \cdot x$ denote $\sum_1^k c_j x_j$. Recall that F is the potential function of ω. Since ψ is a diffeomorphism, it has a Jacobian $J_\psi \in C^\infty(W)$ between the volume forms $e^{2c \cdot x - F(x)}\, dV_\lambda$ on X and dV on W. In other words,

$$e^{2c \cdot x - F(x)}\, dV_\lambda = \psi^*(J_\psi\, dV). \tag{4.6}$$

Let $W_{\mathbf{C}} \subset T_{\mathbf{C}}^{n-k}$ be the Reinhardt domain

$$W_{\mathbf{C}} = \{x + \sqrt{-1}[y] \in T_{\mathbf{C}}^{n-k} : x \in W\}. \tag{4.7}$$

The function J_ψ extends naturally to $W_{\mathbf{C}}$ by T^{n-k}-invariance. Let $B(W_{\mathbf{C}}, J_\psi)$ be the Bergman space of J_ψ-weighted L^2-holomorphic functions on the Reinhardt domain $W_{\mathbf{C}}$. In other words, it is the Hilbert space defined by

$$B(W_{\mathbf{C}}, J_\psi) = \left\{ h \in C^\infty(W_{\mathbf{C}}) : h \text{ holomorphic}, \int_{W_{\mathbf{C}}} h\bar{h} J_\psi \, dV < \infty \right\}.$$

We now check that the inner product space $H_{(\omega_\lambda)}$ is isomorphic to the weighted Bergman space $B(W_{\mathbf{C}}, J_\psi)$, which implies that $H_{(\omega_\lambda)}$ is a Hilbert space.

From the definition (4.2) of $H(\mathbf{L}_\lambda)$, we have the natural restriction map

$$\kappa : H(\mathcal{O}, \mathbf{L})_\lambda \longrightarrow H(\mathbf{L}_\lambda).$$

Let s_0 be the T-invariant holomorphic section of (2.2), restricted to $\mathcal{O}$. Pick $s \in H_{(\omega_\lambda)}$. By the definition of $H(\mathbf{L}_\lambda)$, s is of the form $s = \kappa(he^{c \cdot z} s_0)$, where h is a holomorphic function on $\mathcal{O}$ and depends only on the variables $z_{k+1}, \dots, z_n$ because the section $he^{c \cdot z} s_0 \in H(\mathcal{O}, \mathbf{L})_\lambda$ needs to transform by λ. Define

$$L : H_{(\omega_\lambda)} \longrightarrow B(W_{\mathbf{C}}, J_\psi), \quad \kappa(he^{c \cdot z} s_0) \longmapsto h|_{W_{\mathbf{C}}}. \tag{4.8}$$

We claim that L is an inner product space isomorphism.

Since h is independent of the first k variables, the function $h\bar{h}$ satisfies

$$(h\bar{h})(q) = (h\bar{h})(\psi(q)) \tag{4.9}$$

for all $q \in X$. We let $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the norms of $H_{(\omega_\lambda)}$ and $B(W_{\mathbf{C}}, J_\psi)$ respectively. For $s = \kappa(he^{c \cdot z} s_0) \in H_{(\omega_\lambda)}$,

$$\begin{aligned}
\|s\|_1^2 &= \int_X h\bar{h} e^{2c \cdot x - F(x)} \, dV_\lambda && \text{by (2.2)} \\
&= \int_X h\bar{h} \psi^*(J_\psi \, dV) && \text{by (4.6)} \\
&= \int_X \psi^*(h\bar{h} J_\psi \, dV) && \text{by (4.9)} \\
&= \int_W h\bar{h} J_\psi \, dV && \text{by (4.5)} \\
&= \int_{W_{\mathbf{C}}} h\bar{h} J_\psi \, dV \\
&= \|L(s)\|_2^2. && \text{by (4.8)}
\end{aligned} \tag{4.10}$$

Let $T_{\mathbf{C}}^k$ be the complex subtorus spanned by the first k variables. It follows from the definitions (4.1), (4.4) and (4.7) that $\mathcal{O} = T_{\mathbf{C}}^k \times W_{\mathbf{C}}$. Hence the operation $h \longmapsto h|_{W_{\mathbf{C}}}$ in (4.8) is bijective, because a holomorphic function h on $\mathcal{O}$ independent of $z_1, \dots, z_k$ is equivalent to a holomorphic function on $W_{\mathbf{C}}$. Therefore, L is a bijection. Then (4.10) says that L is an isomorphism of inner product spaces from $H_{(\omega_\lambda)}$ to $B(W_{\mathbf{C}}, J_\psi)$. Therefore $H_{(\omega_\lambda)}$ is a Hilbert space. □

Recall that $(H_\omega)_\lambda = H(\mathbf{L})_\lambda \cap H_\omega$. Let $c_j = \lambda_j / \alpha_j$ for $j = 1, \dots, k$. From an argument similar to the one leading to (2.4), we know that, if any c_j is not an integer, then $(H_\omega)_\lambda$

vanishes. Assuming $c_j \in \mathbf{Z}$ from now on, our goal is to construct a natural Hilbert space isomorphism $(H_\omega)_\lambda \cong H_{(\omega_\lambda)}$, and prove Theorem 1.3. In order to compare these two Hilbert spaces, the next two propositions provide integrability conditions.

Let s_0 be the holomorphic section of equation (2.2). Given $b = (b_{k+1}, \dots, b_n) \in \mathbf{Z}^{n-k}$, we define a one-dimensional subspace

$$S_b = \left\{ a \exp\left(\sum_1^k c_j z_j \right) \exp\left(\sum_{k+1}^n b_j z_j \right) s_0 : a \in \mathbf{C} \right\} \subset H(\mathbf{L})_\lambda. \tag{4.11}$$

Proposition 4.2 follows, from [**3**].

Proposition 4.2 [**3**]. *Let $0 \neq s \in S_b$. Then $\int_{T_\mathbf{C}} (s,s)\, dV$ converges if and only if $(c_1, \dots, c_k, b_{k+1}, \dots, b_n)$ is in the image of $\frac{1}{2}F'$.*

Recall that Ω is the image of the moment map. Following Proposition 4.2, it is clear that

$$(H_\omega)_\lambda \neq 0 \Leftrightarrow \lambda \in \Omega, \frac{\lambda_j}{\alpha_j} \in \mathbf{Z} \qquad \text{for } j = 1, \dots, k. \tag{4.12}$$

Assume that $\lambda \in \Omega$, so that we have the reduced space B_λ. Let $s \in S_b$. In Proposition 4.2, we have given a necessary and sufficient condition for $(s,s)\, dV$ to be integrable over $T_\mathbf{C}$. We now restrict it to $B_\lambda \subset T_\mathbf{C}$, but for simplicity, we still denote it by $(s,s)\, dV$. The next proposition considers its integrability over B_λ.

Proposition 4.3. *Let $0 \neq s \in S_b$. Then $\int_{B_\lambda} (s,s)\, dV$ converges if and only if $(c_1, \dots, c_k, b_{k+1}, \dots, b_n)$ is in the image of $\frac{1}{2}F'$.*

Proof. Let $s \in S_b$. Suppose that $(c_1, \dots, c_k, b_{k+1}, \dots, b_n)$ is in the image of $\frac{1}{2}F'$. By Proposition 4.2, $\int_{T_\mathbf{C}} (s,s)\, dV$ converges. Therefore, when restricted to $B_\lambda \subset T_\mathbf{C}$, $\int_{B_\lambda} (s,s)\, dV$ also converges.

Therefore, it only remains to prove the converse. Suppose that $\int_{B_\lambda} (s,s)\, dV$ converges for all $s \in S_b$. Define $G \in C^\infty(\mathbf{R}^n)$ by

$$G(x) = F(x) - 2 \sum_1^k c_i x_i - 2 \sum_{k+1}^n b_j x_j. \tag{4.13}$$

Since F and G have the same Hessian, G is strictly convex. Since $(s_0, s_0) = e^{-F}$, equations (4.11) and (4.13) imply that up to a positive constant, $e^{-G} = (s,s)$. Hence

$$\int_X e^{-G}\, dV_\lambda = \int_{B_\lambda} (s,s)\, dV < \infty, \tag{4.14}$$

where dV_λ is the restriction of the Lebesgue measure to $X \subset \mathbf{R}^n$. By Proposition 3.1, X is unbounded and has no boundary. Thus (4.14) implies that e^{-G} approaches 0 along every direction of X, in the sense that, for any $\epsilon > 0$, there exists a compact subset of X such that $e^{-G(x)} < \epsilon$ for x outside this compact set. This means that e^{-G} acquires a maximum point in X. Equivalently, G has a minimum point p in X:

$$G(p) \leqslant G(x), \quad x \in X. \tag{4.15}$$

Recall the notions of horizontal and vertical affine spaces, defined in equations (3.1) and (3.2). Let $G|_{V^p}$ denote the restricted function on the vertical space V^p. We want to show that p is the global minimum of $G|_{V^p}$. However, since $G|_{V^p}$ is strictly convex, it suffices to show that p is a local minimum of $G|_{V^p}$. By Proposition 3.1, X intersects H^p exactly once, at p. Hence, for each $v \in V^p$ that is sufficiently near p,

$$X \cap H^v = \{q_v\} \tag{4.16}$$

for some q_v. By the definition of X (3.3) and the definition of G in equation (4.13),

$$\frac{\partial G}{\partial x_j}(q_v) = 0, \quad j = 1, \ldots, k. \tag{4.17}$$

Since the restriction of G to H^v is strictly convex, equation (4.17) says that q_v is the global minimum of the restriction of G to H^v. In particular, since $v \in H^v$, it gives

$$G(q_v) \leqslant G(v). \tag{4.18}$$

Since $q_v \in X$, equations (4.15) and (4.18) imply that $G(p) \leqslant G(v)$ whenever $v \in V^p$ is sufficiently near p. This proves that p is a local minimum of $G|_{V^p}$. However, $G|_{V^p}$ is strictly convex, so p is a global minimum of it. Therefore,

$$\frac{\partial G}{\partial x_j}(p) = 0, \quad j = k+1, \ldots, n. \tag{4.19}$$

Set $v = p$ in (4.16), so that $q_v = p$. Then (4.17) becomes

$$\frac{\partial G}{\partial x_j}(p) = 0, \quad j = 1, \ldots, k. \tag{4.20}$$

Using equations (4.19) and (4.20), we conclude that p is the global minimum of G, and so 0 is in the image of G'. Then (4.13) implies that $(c_1, \ldots, c_k, b_{k+1}, \ldots, b_n)$ is in the image of $\frac{1}{2}F'$. Hence the proposition holds. □

If $s \in H(\mathbf{L})_\lambda$, we let $\rho(s)$ be its restriction to B_λ. By the definition (4.2) of $H(\mathbf{L}_\lambda)$, $\rho(s) \in H(\mathbf{L}_\lambda)$. Therefore, we have the restriction map

$$\rho : H(\mathbf{L})_\lambda \longrightarrow H(\mathbf{L}_\lambda).$$

We apply ρ to the one-dimensional spaces S_b of (4.11). Recall that $c_j = \lambda_j/\alpha_j \in \mathbf{Z}$, for $j = 1, \ldots, k$. Let

$$I = \{(b_{k+1}, \ldots, b_n) \in \mathbf{Z}^{n-k} : (c_1, \ldots, c_k, b_{k+1}, \ldots, b_n) \in \operatorname{Image} \tfrac{1}{2}F'\}. \tag{4.21}$$

By Propositions 4.2 and 4.3,

$$S_b \subset (H_\omega)_\lambda \Leftrightarrow \rho(S_b) \subset H_{(\omega_\lambda)} \Leftrightarrow b \in I. \tag{4.22}$$

It follows from definition (4.11) that, if $0 \neq s \in S_b$, then s is non-vanishing, so in particular its restriction $\rho(s)$ is non-zero. Thus ρ is injective on each S_b. Therefore, since each S_b is one-dimensional, we obtain a constant $m_b > 0$ for each $b \in I$ by

$$\|s\| = m_b \|\rho(s)\|, \quad s \in S_b. \tag{4.23}$$

Here $\|\cdot\|$ denotes the norms of both $(H_\omega)_\lambda$ and $H_{(\omega_\lambda)}$. For $b \in I$, define $\tilde{\rho}$ on S_b by

$$\tilde{\rho}(s) = m_b \rho(s), \quad s \in S_b. \tag{4.24}$$

Proof of Theorem 1.3. If λ is not in the image Ω of the moment map, then there is no reduced space B_λ or $H_{(\omega_\lambda)}$. Also, $(H_\omega)_\lambda = 0$ by (4.12), and there is nothing to prove. Therefore, we may assume that $\lambda \in \Omega$.

Proposition 4.1 says that $H_{(\omega_\lambda)}$ is a Hilbert space. To complete the proof, we show that $\tilde{\rho}$ gives the desired Hilbert space isomorphism. It follows from (4.22), (4.23) and (4.24) that

$$\|\tilde{\rho}(s)\| = \|s\| < \infty \quad \text{for} \quad s \in S_b,\, b \in I. \tag{4.25}$$

Consider the standard subgroup action of T^{n-k} on $T_{\mathbf{C}}$, which lifts to a T^{n-k}-representation on $H(\mathbf{L})$. This restricts to a T^{n-k}-representation on $H(\mathbf{L})_\lambda$, because the standard T^{n-k}-action commutes with the weighted T-action defined in (1.2). Since T^{n-k} preserves the L^2-structure (1.3) on $H(\mathbf{L})_\lambda$, we get a unitary representation

$$\pi_1 : T^{n-k} \longrightarrow \mathrm{Aut}(H_\omega)_\lambda.$$

Since T^{n-k} also acts on B_λ by acting on its toral component, we similarly get a T^{n-k}-representation on $H(\mathbf{L}_\lambda)$. It preserves the L^2-structure (4.3) on $H(\mathbf{L}_\lambda)$, so we get a unitary representation

$$\pi_2 : T^{n-k} \longrightarrow \mathrm{Aut}\, H_{(\omega_\lambda)}.$$

By the definition (4.11) and property (4.22) of S_b, the irreducible subrepresentations of π_1, π_2 are given by $\{S_b\}_{b\in I}$ and $\{\rho(S_b)\}_{b\in I}$ respectively. Apply the Peter–Weyl theorem [**2**, Chapter III] to these subrepresentations. It says that $\{S_b\}_{b\in I}$ and $\{\rho(S_b)\}_{b\in I}$ are collections of mutually orthogonal subspaces in $(H_\omega)_\lambda$ and $H_{(\omega_\lambda)}$ respectively, and their linear spans are dense in these Hilbert spaces. Let

$$S \subset (H_\omega)_\lambda, \quad R \subset H_{(\omega_\lambda)}$$

be the dense subsets given by their linear spans. Since both $\{S_b\}_{b\in I}$ and $\{\rho(S_b)\}_{b\in I}$ are collections of mutually orthogonal subspaces, definition (4.25) says that $\tilde{\rho}$ is an isometry from S to R.

If I is finite, then the Hilbert spaces are finite-dimensional, and $S = (H_\omega)_\lambda$, $R = H_{(\omega_\lambda)}$. Thus $\tilde{\rho}$ is the required isomorphism. Suppose that I is infinite. Since $\tilde{\rho}$ is an isometry between the dense subsets S and R, it extends continuously to a Hilbert space isomorphism $\tilde{\rho} : (H_\omega)_\lambda \longrightarrow H_{(\omega_\lambda)}$. This proves Theorem 1.3. □

5. *Open cones*

In this section, we give some simple examples to show that the Hilbert spaces of Theorem 1.3 can have any dimension. It suffices to consider a torus of dimension 2.

From the previous section, we see that the dimension of $(H_\omega)_\lambda$ is the cardinality of the index set I of (4.21). Consider $\omega = \sqrt{-1}\partial\bar{\partial}F$, invariant under the T-action (1.2) with weight $\alpha = (1, 0)$. Let $\lambda = (0, 0) \in \mathbf{R}^2$. Then the set I of (4.21) becomes

$$I = \{b \in \mathbf{Z} : (0, b) \in \mathrm{Image}\, \tfrac{1}{2}F'\}. \tag{5.1}$$

We now show that $|I| = \dim(H_\omega)_\lambda$ can be any of $0, 1, 2, \ldots, \infty$.

Let $v_1, v_2 \in \mathbf{R}^2$ be a basis. Define $G \in C^\infty(\mathbf{R}^2)$ by

$$G(x) = \exp(v_1 \cdot x) + \exp(v_2 \cdot x),$$

where $v_i \cdot x$ is the usual dot product. Then G is strictly convex. The image of $\frac{1}{2}G'$ is the open cone consisting of all positive linear combinations of v_1, v_2. By adjusting v_1 and v_2, we get all the open cones of $\mathbf{R}^2$ that emit from the origin.

In fact, if G is any strictly convex function and $w \in \mathbf{R}^2$, then $F(x) = G(x) + w \cdot x$ is also strictly convex. Further, the images of $\frac{1}{2}F'$ and $\frac{1}{2}G'$ differ by an affine translation of $\frac{1}{2}w$. From this observation, we consider the strictly convex function

$$F(x) = \exp(v_1 \cdot x) + \exp(v_2 \cdot x) + w \cdot x.$$

By choosing different parameters $v_1, v_2, w \in \mathbf{R}^2$ for F, the image of $\frac{1}{2}F'$ can be any given open cone $C \subset \mathbf{R}^2$. For every $s = 0, 1, 2, \ldots, \infty$, we can always find an open cone C whose intersection with the y-axis contains s integral points, so that I in (5.1) has s elements.

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