Mathematical Methods in the Applied Sciences
Math. Meth. Appl. Sci., **23**, 777–802 (2000)
MOS subject classification: 35 B 27, 35 B 45, 35 K 65, 76 S 05, 76 T 05

Convergence of a Dual-porosity Model for Two-phase Flow in Fractured Reservoirs

Li-Ming Yeh*

Department of Applied Mathematics, National Chiao Tung University, Hsinchu, 30050, Taiwan

Communicated by B. Brosowski

A dual-porosity model describing two-phase, incompressible, immiscible flows in a fractured reservoir is considered. Indeed, relations among fracture mobilities, fracture capillary presure, matrix mobilities, and matrix capillary presure of the model are mainly concerned. Roughly speaking, proper relations for these functions are (1) Fracture mobilities go to zero slower than matrix mobilities as fracture and matrix saturations go to their limits, (2) Fracture mobilities times derivative of fracture capillary presure and matrix mobilities times derivative of matrix capillary presure are both integrable functions. Galerkin's method is used to study this problem. Under above two conditions, convergence of discretized solutions obtained by Galerkin's method is shown by using compactness and monotonicity methods. Uniqueness of solution is studied by a duality argument. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: two-phase flow; dual-porosity; fractured reservoirs; global pressure

1. Introduction

A dual-porosity model [5, 11] describing two-phase, incompressible, immiscible flow in fractured reservoirs is considered. The model considered physically corresponds to a waterflooding or unsaturated groundwater flow in a fractured reservoir. Flow in a fractured reservoir behaves as if the reservoir consisted of two superimposed continua, a continuous fracture system and a discontinuous system of matrix blocks. The fracture system has a low storativity and high conductivity while the majority of the fluids reside in matrix blocks of low conductivity; and different time scales for saturation evolution appear in fracture system and matrix blocks. If ε is the ratio between the size of one matrix block to the size of the whole reservoir, then the time scale for saturation evolution in the block will be of order ε^{-2}. If global pressure is used, equations for the fracture system can be written as [5, 11, 13], for $x \in \Omega$, $t > 0$,

$$\Phi\partial_t S - \nabla_x\left(\mathbf{K}\Lambda_w(S)\nabla_x(P - J_w) - \mathbf{K}\frac{\Lambda_w(S)\Lambda_o(S)}{\Lambda(S)}\nabla_x\mathbf{P}_c(S)\right) = q_w \tag{1.1}$$

$$-\nabla_x(\mathbf{K}(\Lambda(S)\nabla_x P - \Lambda_w(S)\nabla_x J_w - \Lambda_o(S)\nabla_x J_o)) = 0 \tag{1.2}$$

*Correspondence to: Li-Ming Yeh, Department of Applied Mathematics, National Chiao Tung University, Hsinchu, 30050, Taiwan. E-mail: liming@math.nctu.edu.tw.

Contract/grant sponsor: National Science Council of the Republic of China.

 Received 10 July 1999

$\Omega \subset \Re^3$ is the reservoir; Φ the fracture porosity; $S \in [0, 1]$ the water saturation; $\mathbf{K}$ the absolute permeability of the fracture system; and P the global pressure. $\Lambda_\alpha = \Lambda_\alpha(S)$, $\alpha = w, o$, is the phase mobility and is a monotone function. $\Lambda := \Lambda_w + \Lambda_o$. When S approaches 0, $\Lambda_w(S)$ goes to 0; while S approaches 1, $\Lambda_o(S)$ is close to 0. $\mathbf{P}_c$ represents the capillary pressure function and $d\mathbf{P}_c/dS < 0$. J_α, $\alpha = w, o$, is a given function, which depends on density of α-phase, position, and gravity. q_w is the water matrix source. If $q_w = 0$, then (1.1)–(1.2) are equations for two-phase flows in non-fractured reservoirs.

The equation for the matrix block $\Omega_x \subset \Re^3$ suspended topologically over $x \in \Omega$ is given by, for $x \in \Omega$, $y \in \Omega_x$, $t > 0$,

$$\phi \partial_t s + \nabla_y \left(\mathbf{k} \frac{\lambda_w(s)\lambda_o(s)}{\lambda(s)} \nabla_y \mathbf{p}_c(s) \right) = 0 \tag{1.3}$$

where each lower case symbol denotes the quantity on Ω_x corresponding to that denoted by an upper case symbol in the fracture system equations (see Fig. 1).

The water matrix source is

$$q_w(x, t) = \frac{-1}{|\Omega_x|} \int_{\Omega_x} \phi \partial_t s(x, y, t)\, dy \quad \text{for } x \in \Omega, \quad t > 0 \tag{1.4}$$

where $|\Omega_x|$ is the volume of Ω_x. Boundary of Ω includes two parts: Υ_1, Υ_2. $\Upsilon_1 \cap \Upsilon_2 = \emptyset$, $\partial\Omega = \Upsilon_1 \cup \Upsilon_2$. Boundary conditions for fracture system are

$$\text{B.C.} \begin{cases} S = S_b & \text{for } x \in \Upsilon_1, \\ P = P_b & \text{for } x \in \Upsilon_1, \\ \mathbf{K}\left(\Lambda_w(S)\nabla_x(P - J_w) - \dfrac{\Lambda_w(S)\Lambda_o(S)}{\Lambda(S)} \nabla_x \mathbf{P}_c(S) \right) \cdot \vec{n} = 0 & \text{for } x \in \Upsilon_2, \\ \mathbf{K}(\Lambda(S)\nabla_x P - \Lambda_w(S)\nabla_x J_w - \Lambda_o(S)\nabla_x J_o) \cdot \vec{n} = 0 & \text{for } x \in \Upsilon_2, \end{cases} \quad t > 0 \tag{1.5}$$

where $\vec{n}$ is the unit vector outward normal to $\partial\Omega$, and for each matrix block require

$$\text{B.C.} \quad \mathbf{p}_c(s)(x, y, t) = \mathbf{P}_c(S)(x, t) \quad \text{for } x \in \Omega, \quad y \in \partial\Omega_x, \quad t > 0 \tag{1.6}$$

Initial equilibrium gives

$$\text{I.C.} \begin{cases} S(x, 0) = S_0(x) & \text{for } x \in \Omega, \\ s(x, y, 0) = s_0(x) & \text{for } x \in \Omega, \quad y \in \Omega_x \end{cases} \tag{1.7}$$

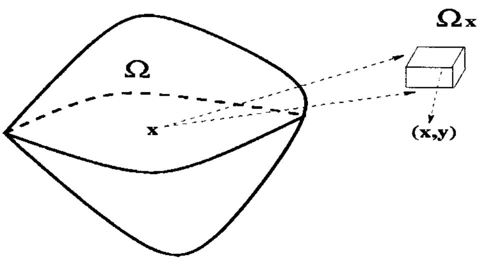

Fig. 1. Domains for fracture system Ω and matrix block Ω_x

By (1.6), $\mathbf{p}_c(s_0)(x) = \mathbf{P}_c(S_0)(x)$, $x \in \Omega$. Also note functions S, P, q_w are defined on fracture system depending on x, t; while s is defined on matrix blocks and depending on x, y, t. The second-order derivative term for s in (1.3) only takes derivative with respect to y variable. More physical background of this model can be found in [5, 11, 13].

For non-fractured reservoir case (i.e. $q_w = 0$), existence of solution had been extensively studied. We refer readers to [3, 4, 8, 12, 15] and references therein. But for fractured reservoir case, many questions still need to be answered. Numerical simulations for (1.1)–(1.7) had been conducted in [11, 13]. Different time scales for saturation evolution in fracture system and matrix blocks can be observed from numerical results in [13]. Convergence analysis of a numerical scheme for (1.1)–(1.7) for 'very small' matrix block case can be found in [10]. Existence results of (1.1)–(1.7) but for linearized matrix phase mobilities λ_α, $\alpha = w, o$ and for 'very small' matrix block cases were shown in [6, 9]. An existence result for a model close to (1.1)–(1.3) but with different fracture–matrix interface condition (1.6) was considered in [7]. In this work, we will consider the general model (1.1)–(1.7). Actually, we address the relations among fracture mobilities, fracture capillary presure, matrix mobilities, and matrix capillary presure for (1.1)–(1.7). Galerkin's method and monotonicity method will be used to study this problem. Time-discretization for (1.1) and (1.3) is backward Euler method. Resulting equations, counting for the interaction between matrix blocks and fracture system, are non-linear and they are expressed by a variational formulation. We first prove the discretized method is solvable, and then show a subsequence of the solutions for the discretized method converges to a weak solution of (1.1)–(1.7). To obtain convergence of discretized solutions of the differential equations, relations between mobilities and capillary presures in (1.1)–(1.7) are (1) $\Lambda_w\Lambda_o(S)$ goes to 0 slower than $\lambda_w\lambda_o(s)$ as S, s approach their limits respectively, (2) $\Lambda_w\Lambda_o|\mathrm{d}\mathbf{P}_c/\mathrm{d}S|$ and $\lambda_w\lambda_o|\mathrm{d}\mathbf{p}_c/\mathrm{d}s|$ are integrable functions (see **A7, 8** below). Then we consider the uniqueness of (1.1)–(1.7), which will be analysed by a duality argument.

The following sections are organized as: In section 2, we state our problems, which include four subsections section 2.1–section 2.4. In section 2.1, we give notation and assumption; In section 2.2, we introduce a discretized scheme and a regularized system for (1.1)–(1.7), and we claim a subsequence of solutions of the discretized scheme converges to a solution of the regularized problem of (1.1)–(1.7); In section 2.3, we claim a subsequence of solutions of the regularized problem in section 2.2 converges to a solution of a weak solution of (1.1)–(1.7); In section 2.4, we state a uniqueness result for (1.1)–(1.7). In section 3., 4., 5., we prove the results claimed in section 2.2, 2.3, 2.4, respectively.

2. Statement of the problem

2.1. *Notation and assumption*

Let $\Omega \subset \mathfrak{R}^3$ be open, bounded, and connected with Lipschitz boundary. For every $x \in \Omega, \Omega_x$ is a bounded region contained in $\mathfrak{R}^3$. Identify the product space $\Pi_{x\in\Omega}\Omega_x := \mathcal{Q}$ as a subset of $\mathfrak{R}^6$. We require $\mathcal{Q}$ be a measurable subset of $\mathfrak{R}^6$. We will assume all matrix blocks are identical. So $\mathcal{Q} = \Omega \times \mathcal{B}$, where $\mathcal{B}$ is a bounded measurable subset in $\Re^3$ and its boundary, $\partial\mathscr{B}$, is piecewise C^1. Set $\Omega^t := \Omega \times [0, t]$; $\mathscr{Q}^t := \mathscr{Q} \times [0, t]$; $\bar{\partial}\mathscr{Q} := \Omega \times \partial\mathscr{B}$; $\bar{\partial}\mathscr{Q}^t := \Omega \times \partial\mathscr{B} \times [0, t]$; and $\Upsilon_i^t := \Upsilon_i \times [0, t]$, $i = 1, 2$.

$C^m([a, b])$ is the space of functions with all the continuous derivatives of order $\leqslant m$ on $[a, b]$. For $r \geqslant 1$ and $m \in \mathbf{N}$, $L^r(E)$, $H^m(E)$, $W^{m,r}(E)$, $L^r(\Omega, H^m(\mathscr{B}))$, $L^r(\Omega, L^r(\partial\mathscr{B}))$, $L^r(0, T;X)$, $H^1(0, T;X)$ are Sobolev spaces [1] where $E \subset \mathscr{Q}^t$ is a measurable set and X is a Banach space. X^* denotes the dual space of X. $\mathscr{H} := \{\zeta \in H^1(\Omega) : \zeta|_{\Upsilon_1} = 0\}$; $\mathscr{W} := \{\eta \in L^2(\mathscr{Q}) : \nabla_y \eta \in L^2(\mathscr{Q})\}$ with norm

$$|||\eta|||_{\mathscr{W}} := (\|\eta\|^2_{L^2(\mathscr{Q})} + \|\nabla_y \eta\|^2_{L^2(\mathscr{Q})})^{1/2}$$

Note $\mathscr{W}$ is contained in $L^2(\Omega, H^1(\mathscr{B}))$. Let $\mathscr{T}_x$ be usual trace map of $H^1(\mathscr{B})$ into $L^2(\partial\mathscr{B})$, and define the distributed trace $\mathscr{T} : \mathscr{W} \to L^2(\Omega, L^2(\partial\mathscr{B}))$ by $\mathscr{T}\eta(x, y) = (\mathscr{T}_x \eta(x))(y)$. $\mathscr{W}_0 := \{\eta \in \mathscr{W} : \mathscr{T}\eta = 0\}$; $\mathscr{Z} := \mathscr{H} \times \mathscr{H} \times \mathscr{W}_0$.

Define $\mathscr{L} : \mathscr{H} \to L^2(\Omega, H^1(\mathscr{B}))$ by

$$\mathscr{L}\zeta\,(x, y) = \zeta(x) 1_y, \quad x \in \Omega, \quad y \in \mathscr{B} \tag{2.1}$$

where $\zeta\,(x)1_y$ is constant in $\mathscr{B}$ with value $\zeta(x)$. For a function η, $\mathscr{X}_\eta$ is a characteristic function satisfying

$$\mathscr{X}_\eta(x, y, t) = \begin{cases} 1 & \text{for } \eta(x, y, t) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

Let $\mathbf{P}_c^{-1}$, $\mathbf{p}_c^{-1}$ be the inverse functions of $\mathbf{P}_c$, $\mathbf{p}_c$, respectively. $s_{\min}$ and $1 - s_{\max}$ are the residual matrix water and oil saturations, and assume $\mathbf{p}_c^{-1}(\mathbf{P}_c(0.5)) \in (s_{\min}, s_{\max}) \subset (0, 1)$. Define time difference operator by $\partial^h \zeta(t) := (\zeta(t + h) - \zeta(t))/h$; $\Re_0^+ := \Re^+ \cup \{0\}$; ∇ represents ∇_x.

Next we make the following assumptions:

A1. $\Omega, \mathscr{Q}, \mathscr{B}, \Upsilon_1, \Upsilon_2$ are defined at the beginning of this subsection and $\Upsilon_1 \neq \emptyset$.

A2. $\Phi(x), \mathbf{K}(x), \phi(x), \mathbf{k}(x) \in [\varepsilon_0, \varepsilon_1]$ for $x \in \Omega$.

A3. Λ_w (resp. Λ_o)$: [0, 1] \to [0, 1]$ is continuous and strictly increasing (resp. decreasing); $\Lambda_w(0) = \Lambda_o(1) = 0$; λ_w (resp. λ_o)$: [s_{\min}, s_{\max}] \to [0, 1]$ is continuous and strictly increasing (resp. decreasing); $\lambda_w(s_{\min}) = \lambda_o(s_{\max}) = 0$; $0 < \varepsilon_2 \leqslant \min\{\Lambda_w(1), \lambda_w(s_{\max}), \Lambda_o(0), \lambda_o(s_{\min})\}$; $\varepsilon_2 \leqslant \inf_{z \in [0,1]} \Lambda(z)$; $\varepsilon_2 \leqslant \inf_{z \in [s_{\min}, s_{\max}]} \lambda(z)$.

A4. $\mathbf{P}_c : (0, 1] \to \Re_0^+$ and $\mathbf{p}_c : (s_{\min}, s_{\max}) \to \Re_0^+$ are onto and C^1 functions; $\mathrm{d}\mathbf{P}_c/\mathrm{d}S, \mathrm{d}\mathbf{p}_c/\mathrm{d}s \leqslant -\varepsilon_3 < 0$; $\mathbf{P}_c(1) = \mathbf{p}_c(s_{\max}) = 0$.

A5. $P_b, \mathbf{P}_c(S_b), J_w, J_o \in L^2(0, T; H^1(\Omega))$; $\partial_t \mathbf{P}_c(S_b) \in L^1(\Omega^T)$.

A6. $\mathbf{P}_c(S_0) \in L^1(\Omega)$; $\mathbf{p}_c(s_0(x)) = \mathbf{P}_c(S_0(x))$, $x \in \Omega$.

A7. $\lambda_w \lambda_o(\mathbf{p}_c^{-1}(\mathbf{P}_c(z))) \leqslant \varepsilon_4 \Lambda_w \Lambda_o(z)$ for $z \in (0, 1]$.

A8. $\Lambda_w \Lambda_o |\mathrm{d}\mathbf{P}_c/\mathrm{d}S|(z) \in L^1((0, 1])$; $\lambda_w \lambda_o |\mathrm{d}\mathbf{p}_c/\mathrm{d}s|(z) \in L^1((s_{\min}, s_{\max}])$,

where ε_i, $i = 0, 1, 2, 3, 4$ are some positive constants.

Define $\widehat{\mathbf{P}_c} : [0, 1] \to [s_{\min}, s_{\max}]$ by

$$\widehat{\mathbf{P}_c}(z) := \begin{cases} \mathbf{p}_c^{-1}(\mathbf{P}_c(z)) & \text{for } z \in (0, 1] \\ s_{\min} & \text{for } z = 0 \end{cases}$$

then $\widehat{\mathbf{P}_c}$ is a continuous and strictly increasing function. Let $\widehat{\mathbf{P}_c}^{-1}$ be the inverse function of $\widehat{\mathbf{P}_c}$. **A7,8** are restrictions for functions $\Lambda_\alpha, \lambda_\alpha (\alpha = w, o)$, $\mathbf{P}_c$, $\mathbf{p}_c$. Basically, **A7** means $\Lambda_w \Lambda_o(z)$ goes to 0 slower than $\lambda_w \lambda_o(\xi)$ as $z, \xi (= \widehat{\mathbf{P}_c}(z))$ approach their limits respectively. Under **A3, 4, 7, 8**, $\Lambda_w \Lambda_o |\mathrm{d}\mathbf{P}_c/\mathrm{d}S|(z)$ may go to 0 or ∞ as z closes to 0 or 1. So fracture water saturation may have singular behaviours as it closes to its limits. Similar situation as fracture saturation also happens in matrix saturation.

2.2. *Discretized problem*

We first find approximation functions of $\Lambda_\alpha, \lambda_\alpha, \mathbf{P}_c, \mathbf{p}_c, S_0, s_0, S_b, P_b$; then derive a discretized scheme and a regularized system for (1.1)–(1.7). Let δ be a small positive number. By **A4**, $[\widehat{\mathbf{P}}_c(\delta), \widehat{\mathbf{P}}_c(1-\delta)] \subset (s_{\min}, s_{\max})$. We extend $\Lambda_\alpha, \lambda_\alpha, \alpha = w, o$, to $\mathfrak{R}$ continuously and constantly. By **A3**, we may find continuous monotone functions $\Lambda_\alpha^\delta, \lambda_\alpha^\delta, \alpha = w, o$, in $\mathfrak{R}$ such that

$$\begin{cases} 0 < c_1(\delta) \leqslant \inf_{z\in\mathfrak{R}}\{\Lambda_\alpha^\delta(z), \lambda_\alpha^\delta(z)\}, & \sup_{z\in\mathfrak{R}}\{\Lambda_\alpha^\delta(z), \lambda_\alpha^\delta(z)\} \leqslant 1 \\ \Lambda_\alpha^\delta(z) = \Lambda_\alpha(z) \text{ and } \lambda_\alpha^\delta(\widehat{\mathbf{P}}_c(z)) = \lambda_\alpha(\widehat{\mathbf{P}}_c(z)) & \text{if } z \in [\delta, 1-\delta] \end{cases} \tag{2.3}$$

Then we define, for $\alpha = w, o$,

$$\begin{cases} \Lambda^\delta := \Lambda_w^\delta + \Lambda_o^\delta, & \lambda^\delta := \lambda_w^\delta + \lambda_o^\delta \\ \tilde{\Lambda}_\alpha^\delta(z) := \Lambda_\alpha(0.5(z-\delta/0.5-\delta)) & \tilde{\Lambda}_\delta := \tilde{\Lambda}_w^\delta + \tilde{\Lambda}_o^\delta \end{cases} \tag{2.4}$$

By **A4**, we find C^1 and decreasing functions $\mathbf{P}_c^\delta, \mathbf{p}_c^\delta$ defined in $\mathfrak{R}$ such that

$$\begin{cases} 0 < \dfrac{\varepsilon_3}{2} \leqslant \inf_{z\in\mathfrak{R}}\left\{\left|\dfrac{\mathrm{d}\mathbf{P}_c^\delta}{\mathrm{d}S}\right|(z), \left|\dfrac{\mathrm{d}\mathbf{p}_c^\delta}{\mathrm{d}s}\right|(z)\right\}, \quad \sup_{z\in\mathfrak{R}}\left\{\left|\dfrac{\mathrm{d}\mathbf{P}_c^\delta}{\mathrm{d}S}\right|(z), \left|\dfrac{\mathrm{d}\mathbf{p}_c^\delta}{\mathrm{d}s}\right|(z)\right\} \leqslant c_2(\delta) < \infty \\ \mathbf{P}_c^\delta(z) = \mathbf{P}_c(z) \text{ and } \mathbf{p}_c^\delta(\widehat{\mathbf{P}}_c(z)) = \mathbf{p}_c(\widehat{\mathbf{P}}_c(z)) \quad \text{if } z \in [\delta, 1-\delta] \\ \mathbf{P}_c^\delta, \mathbf{p}_c^\delta \text{ have inverse functions } \mathbf{P}_c^{\delta,-1}, \quad \mathbf{p}_c^{\delta,-1} \text{ in } \mathfrak{R} \\ \widehat{\mathbf{P}}_c{}^\delta(z) := \mathbf{p}_c^{\delta,-1}(\mathbf{P}_c^\delta(z)) \text{ is linear in } (-\infty, \delta)\cup(1-\delta, \infty) \text{ and has inverse } \widehat{\mathbf{P}}_c{}^{\delta,-1} \end{cases} \tag{2.5}$$

One way to get $\mathbf{P}_c^\delta, \mathbf{p}_c^\delta$ is as follows: Let $\mathbf{P}_c^\delta$ in $(-\infty, \delta)$ be the tangent line of $\mathbf{P}_c$ at δ and in $(1-\delta, \infty)$ be the tangent line of $\mathbf{P}_c$ at $1-\delta$; while $\mathbf{p}_c^\delta$ in $(-\infty, \widehat{\mathbf{P}}_c(\delta))$ be the tangent line of $\mathbf{p}_c$ at $\widehat{\mathbf{P}}_c(\delta)$ and in $(\widehat{\mathbf{P}}_c(1-\delta), \infty)$ be the tangent line of $\mathbf{p}_c$ at $\widehat{\mathbf{P}}_c(1-\delta)$. By **A5, 6**, there are smooth functions G_0^δ, G_b^δ such that

$$\begin{cases} G_0^\delta \to -\mathbf{P}_c(S_0) = -\mathbf{p}_c(s_0) & \text{in } L^1(\Omega), \\ G_b^\delta \to -\mathbf{P}_c(S_b) & \text{in } L^2(\Omega^T), \end{cases} \quad \text{as } \delta \to 0^+ \tag{2.6}$$

and

$$\begin{cases} \|G_b^\delta\|_{L^\infty(0,T;L^1(\Omega))}, \|\partial_t G_b^\delta\|_{L^1(\Omega^T)}, \|G_b^\delta\|_{L^2(0,T;H^1(\Omega))} \text{ are bounded independently of } \delta \\ G_0^\delta(x) - G_b^\delta(x, 0) \in \mathscr{H} \\ \delta < \inf_{(x,t)\in\Omega^T}\{\mathbf{P}_c^{-1}(-G_0^\delta), \mathbf{P}_c^{-1}(-G_b^\delta)\}, \quad \sup_{(x,t)\in\Omega^T}\{\mathbf{P}_c^{-1}(-G_0^\delta), \mathbf{P}_c^{-1}(-G_b^\delta)\} < 1-\delta \end{cases} \tag{2.7}$$

Let $I := (0, T]$. If $M \in \mathbf{N}$, $h := T/M$, $t_m := mh$, and $I_m := (t_{m-1}, t_m]$. For a Banach space X, let

$$I_h(X) := \{f \in L^\infty(0, T; X) : f \text{ is piecewise constant in time on each subinterval } I_m \subset I\} \tag{2.8}$$

If $f \in I_h(X)$, $f|_{I_m} = f(t_m)$ for $m \leqslant M$. We approximate for $t \in I_m$, $\alpha = w, o$

$$U_{\rm b}^{\delta,h}(x,t) := \frac{1}{h}\int_{I_m} \mathbf{p}_{\rm c}^{-1}(-G_{\rm b}^{\delta})(\tau)\,\mathrm{d}\tau,\; P_{\rm b}^{h}(x,t) := \frac{1}{h}\int_{I_m} P_{\rm b}(\tau)\,\mathrm{d}\tau,\; J_{\alpha}^{h}(x,t) := \frac{1}{h}\int_{I_m} J_{\alpha}(\tau)\,\mathrm{d}\tau \tag{2.9}$$

By **A5**, one can show that, for $\alpha = w, o$,

$$\begin{cases} U_{\rm b}^{\delta,h} \to \mathbf{p}_{\rm c}^{-1}(-G_{\rm b}^{\delta}) \\ P_{\rm b}^{h} \to P_{\rm b} \\ J_{\alpha}^{h} \to J_{\alpha} \end{cases} \quad \text{in } L^2(0,T;H^1(\Omega)) \quad \text{as } h \to 0^+ \tag{2.10}$$

Next, we write down a discretized scheme for (1.1)–(1.7) with fixed δ. Assume $\{\mathbf{e}_{1,i}\}_{i=1}^{\infty}$, $\{\mathbf{e}_{2,i}\}_{i=1}^{\infty}$ be bases of $\mathscr{H}$ and $\mathscr{W}_0$, respectively; and, for each i, $\mathbf{e}_{2,i}$ satisfies

$$\begin{cases} -\Delta_y \mathbf{e}_{2,i} = c_i \mathbf{e}_{2,i} \\ \mathbf{e}_{2,i}|_{\partial\mathscr{Q}} = 0 \end{cases} \tag{2.11}$$

for some constant c_i. Let $\mathscr{H}^{\ell}, \mathscr{W}_0^{\ell}$ denote the linear span of $\{\mathbf{e}_{1,i}\}_{i=1}^{\ell}$, $\{\mathbf{e}_{2,i}\}_{i=1}^{\ell}$ respectively. $\mathscr{Z}^{\ell} := \mathscr{H}^{\ell} \times \mathscr{H}^{\ell} \times \mathscr{W}_0^{\ell}$. Because of $(2.7)_2$, we can find $U_0^{\delta,\ell}$ such that $U_0^{\delta,\ell} - \mathbf{p}_{\rm c}^{-1}(-G_{\rm b}^{\delta}(0))$ is the L^2 projection of $\mathbf{p}_{\rm c}^{-1}(-G_0^{\delta}) - \mathbf{p}_{\rm c}^{-1}(-G_{\rm b}^{\delta}(0))$ on $\mathscr{H}^{\ell}$. Let $s_0^{\delta,\ell} := \mathscr{L}U_0^{\delta,\ell}$. The discretized scheme is to find $(S^{\delta,\ell}, U^{\delta,\ell}, P^{\delta,\ell}, s^{\delta,\ell})$ such that

$$(U^{\delta,\ell} - U_{\rm b}^{\delta,h}, P^{\delta,\ell} - P_{\rm b}^{h}, s^{\delta,\ell} - \mathscr{L}U^{\delta,\ell}) \in I_h(\mathscr{Z}^{\ell}), \tag{2.12}$$

$$S^{\delta,\ell}(0) = \widehat{\mathbf{P}}_{\rm c}^{\delta,-1}(U^{\delta,\ell}(0)), \quad U^{\delta,\ell}(0) = U_0^{\delta,\ell}, \quad s^{\delta,\ell}(0) = s_0^{\delta,\ell} \tag{2.13}$$

and if $(S^{\delta,\ell}, U^{\delta,\ell}, s^{\delta,\ell})(t_{m-1})$ is known, then $(U^{\delta,\ell} - U_{\rm b}^{\delta,h}, P^{\delta,\ell} - P_{\rm b}^{h}, s^{\delta,\ell} - \mathscr{L}U^{\delta,\ell})(t_m)$ is a zero of the mapping $\mathscr{E}^{\delta,\ell,h}: \mathfrak{R}^{3\ell} \to \mathfrak{R}^{3\ell}$ defined by

$$\begin{aligned} &\mathscr{E}^{\delta,\ell,h}(\xi_{1,1},..,\xi_{1,\ell},\xi_{2,1},..,\xi_{2,\ell},\xi_{3,1},..,\xi_{3,\ell}) \\ &\quad = (\bar{\xi}_{1,1},..,\bar{\xi}_{1,\ell},\bar{\xi}_{2,1},..,\bar{\xi}_{2,\ell},\bar{\xi}_{3,1},..,\bar{\xi}_{3,\ell}) \end{aligned} \tag{2.14}$$

where

$$(U^{\delta,\ell} - U_{\rm b}^{\delta,h}, P^{\delta,\ell} - P_{\rm b}^{h}, s^{\delta,\ell} - \mathscr{L}U^{\delta,\ell})(t_m) = \sum_{i=1}^{\ell}(\xi_{1,i}\mathbf{e}_{1,i}, \xi_{2,i}\mathbf{e}_{1,i}, \xi_{3,i}\mathbf{e}_{2,i}) \in \mathscr{Z}^{\ell} \tag{2.15}$$

$$S^{\delta,\ell}(t_m) = \widehat{\mathbf{P}}_{\rm c}^{\delta,-1}(U^{\delta,\ell}(t_m)) \tag{2.16}$$

$$\begin{aligned} \bar{\xi}_{1,i} = &\int_{\Omega} \Phi\partial^{-h} S^{\delta,\ell}(t_m)\mathbf{e}_{1,i} + \int_{\Omega} \mathbf{K}\tilde{\Lambda}_w^{\delta}(S^{\delta,\ell}(t_m))\nabla(P^{\delta,\ell}(t_m) - J_w^h)\nabla\mathbf{e}_{1,i} \\ &- \int_{\Omega} \mathbf{K}\frac{\Lambda_w^{\delta}\Lambda_o^{\delta}}{\Lambda^{\delta}}(S^{\delta,\ell}(t_m))\frac{\mathrm{d}\mathbf{p}_{\rm c}^{\delta}}{\mathrm{d}s}(U^{\delta,\ell}(t_m))\nabla U^{\delta,\ell}(t_m)\nabla\mathbf{e}_{1,i} \\ &+ \int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|}\partial^{-h} s^{\delta,\ell}(t_m)\mathscr{L}\mathbf{e}_{1,i} \end{aligned} \tag{2.17}$$

$$\bar{\xi}_{2,i}=\bar{\beta}(\delta)\int_{\Omega}\mathbf{K}(\tilde{\Lambda}^{\delta}(S^{\delta,\ell}(t_m))\nabla P^{\delta,\ell}(t_m)-\tilde{\Lambda}^{\delta}_{w}(S^{\delta,\ell}(t_m))\nabla J^{h}_{w}-\tilde{\Lambda}^{\delta}_{o}(S^{\delta,\ell}(t_m))\nabla J^{h}_{o})\nabla\mathbf{e}_{1,i} \tag{2.18}$$

$$\xi_{3,i}=\int_{\mathscr{Q}}\frac{\phi}{|\mathscr{B}|}\partial^{-h}s^{\delta,\ell}(t_m)\mathbf{e}_{2,i}-\int_{\mathscr{Q}}\frac{\mathbf{k}}{|\mathscr{B}|}\frac{\lambda^{\delta}_{w}\lambda^{\delta}_{o}}{\lambda^{\delta}}(s^{\delta,\ell}(t_m))\frac{\mathrm{d}\mathbf{p}^{\delta}_{\mathrm{c}}}{\mathrm{d}s}(s^{\delta,\ell}(t_m))\nabla_y s^{\delta,\ell}(t_m)\nabla_y\mathbf{e}_{2,i} \tag{2.19}$$

where $\bar{\beta}(\delta)$ in (2.18) is a constant satisfying

$$\bar{\beta}(\delta)>1+\sup_{z\in(0,1)}\frac{2|\Lambda^{\delta}|^{2}(z)}{\Lambda^{\delta}_{w}\Lambda^{\delta}_{o}(z)|(\mathrm{d}\mathbf{p}^{\delta}_{\mathrm{c}}/\mathrm{d}s)(\widehat{\mathbf{P}}^{\delta}_{\mathrm{c}}(z))|}$$

and Λ^{δ}, $\Lambda^{\delta}_{\alpha}$, $\tilde{\Lambda}^{\delta}$, $\tilde{\Lambda}^{\delta}_{\alpha}$ are defined in (2.3)–(2.4). (2.16)–(2.19) is obtained from (1.1)–(1.3) by using backward Euler method to approximate time derivatives of (1.1) and (1.3), and then by using regularized functions, integration by parts, and boundary conditions to obtain the variational formulation of the time-discretized equations. Here we introduce a new variable $U^{\delta,\ell}$, which is related to $S^{\delta,\ell}$ by (2.16) and is equal to $s^{\delta,\ell}$ on the boundary of matrix blocks.

For fixed δ, we will show in section 3 that a zero of (2.14)–(2.19) exists. Furthermore, we show a subsequence of the solutions of (2.12)–(2.19) converges to a solution of the following problem:

Theorem 2.1. *Under* **A1-6** *and* (2.3)–(2.7), *for each* δ, *there exist* S^{δ}, U^{δ}, P^{δ}, s^{δ} *such that for all* $(\zeta_1,\zeta_2,\eta)\in L^2(0,T;\mathscr{Z})$,

$$\Phi\partial_t S^{\delta}+\int_{\mathscr{B}}\frac{\phi}{|\mathscr{B}|}\partial_t s^{\delta}\,\mathrm{d}y\in L^2(0,T;\mathscr{H}^*),\quad \phi\partial_t s^{\delta}\in L^2(0,T;\mathscr{W}_0^*) \tag{2.20}$$

$$U^{\delta}=\widehat{\mathbf{P}}_{\mathrm{c}}(S^{\delta}),\quad (U^{\delta}-\mathbf{p}_c^{-1}(-G^{\delta}_{\mathrm{b}}),P^{\delta}-P_{\mathrm{b}},\ s^{\delta}-\mathscr{L}U^{\delta})\in L^2(0,T;\mathscr{Z}) \tag{2.21}$$

$$\delta\leqslant S^{\delta}\leqslant 1-\delta,\quad \widehat{\mathbf{P}}_{\mathrm{c}}(\delta)\leqslant s^{\delta}\leqslant\widehat{\mathbf{P}}_{\mathrm{c}}(1-\delta) \tag{2.22}$$

$$\int_{\Omega^{\mathrm{T}}}\Phi\partial_t S^{\delta}\zeta_1+\int_{\Omega^{\mathrm{T}}}\mathbf{K}\left(\tilde{\Lambda}^{\delta}_{w}(S^{\delta})\nabla(P^{\delta}-J_w)-\frac{\Lambda_w\Lambda_o}{\Lambda}(S^{\delta})\nabla\mathbf{P}_{\mathrm{c}}(S^{\delta})\right)\nabla\zeta_1$$
$$=-\int_{\mathscr{Q}^{\mathrm{T}}}\frac{\phi}{|\mathscr{B}|}\partial_t s^{\delta}\mathscr{L}\zeta_1 \tag{2.23}$$

$$\int_{\Omega^{\mathrm{T}}}\mathbf{K}\tilde{\Lambda}^{\delta}(S^{\delta})\nabla P^{\delta}\nabla\zeta_2-\int_{\Omega^{\mathrm{T}}}\mathbf{K}(\tilde{\Lambda}^{\delta}_{w}(S^{\delta})\nabla J_w+\tilde{\Lambda}^{\delta}_{o}(S^{\delta})\nabla J_o)\nabla\zeta_2=0 \tag{2.24}$$

$$\int_{\mathscr{Q}^{\mathrm{T}}}\frac{\phi}{|\mathscr{B}|}\partial_t s^{\delta}\eta-\int_{\mathscr{Q}^{\mathrm{T}}}\frac{\mathbf{k}}{|\mathscr{B}|}\frac{\lambda_w\lambda_o}{\lambda}(s^{\delta})\nabla_y\mathbf{p}_{\mathrm{c}}(s^{\delta})\nabla_y\eta=0 \tag{2.25}$$

$$U^{\delta}(x,0)=\mathbf{p}_{\mathrm{c}}^{-1}(-G^{\delta}_{0}),\quad s^{\delta}(x,y,0)=\mathscr{L}\mathbf{p}_{\mathrm{c}}^{-1}(-G^{\delta}_{0}) \tag{2.26}$$

2.3. *Continuous problem*

By **A8**, we may define

$$\begin{cases} \mathscr{R}:[0,1]\to\Re, & \mathscr{R}(z):=-\int_{1/2}^{z}(\Lambda_w\Lambda_o/\Lambda)(\mathrm{d}\mathbf{P}_c/\mathrm{d}S)(\xi)\,\mathrm{d}\xi \\ \mathscr{D}:[s_{\min},s_{\max}]\to\Re, & \mathscr{D}(z):=-\int_{\widehat{\mathbf{P}}_c(1/2)}^{z}(\lambda_w\lambda_o/\lambda)(\mathrm{d}\mathbf{p}_c/\mathrm{d}s)(\xi)\,\mathrm{d}\xi \end{cases} \tag{2.27}$$

In this section, we claim that a subsequence of solutions of the regularized system (2.20)–(2.26) converges to a weak solution of (1.1)–(1.7) as $\delta\to 0^+$. In fact, the limit of the subsequence of (2.20)–(2.26) is a solution of the following problem:

Theorem 2.2. *Under* **A1-8**, *there exist S, U, P, s such that for $(\zeta_1,\zeta_2,\eta)\in L^2(0,T;\mathscr{Z})$,*

$$\Phi\partial_t S+\int_{\mathscr{B}}\frac{\phi}{|\mathscr{B}|}\partial_t s\,\mathrm{d}y\in L^2(0,T;\mathscr{H}^*),\quad \phi\partial_t s\in L^2(0,T;\mathscr{W}_0^*) \tag{2.28}$$

$$U=\widehat{\mathbf{P}}_c(S),\quad (\mathscr{R}(S)-\mathscr{R}(S_b),P-P_b,\mathscr{D}(s)-\mathscr{L}\mathscr{D}(U))\in L^2(0,T;\mathscr{Z}) \tag{2.29}$$

$$0\leqslant S\leqslant 1,\quad s_{\min}\leqslant s\leqslant s_{\max} \tag{2.30}$$

$$\int_{\Omega^T}\Phi\partial_t S\,\zeta_1+\int_{\Omega^T}\mathbf{K}(\Lambda_w(S)\nabla(P-J_w)+\nabla\mathscr{R}(S))\nabla\zeta_1=-\int_{\mathscr{Q}^T}\frac{\phi}{|\mathscr{B}|}\partial_t s\;\mathscr{L}\zeta_1 \tag{2.31}$$

$$\int_{\Omega^T}\mathbf{K}\Lambda(S)\nabla P\nabla\zeta_2-\int_{\Omega^T}\mathbf{K}(\Lambda_w(S)\nabla J_w+\Lambda_o(S)\nabla J_o)\nabla\zeta_2=0 \tag{2.32}$$

$$\int_{\mathscr{Q}^T}\frac{\phi}{|\mathscr{B}|}\partial_t s\,\eta+\int_{\mathscr{Q}^T}\frac{\mathbf{k}}{|\mathscr{B}|}\nabla_y\mathscr{D}(s)\nabla_y\eta=0 \tag{2.33}$$

Moreover, for $\zeta\in L^2(0,T;\mathscr{H})\cap H^1(\Omega^T)$, $\eta\in L^2(0,T;\mathscr{W}_0)\cap H^1(0,T;L^2(\mathscr{Q}))$, $\zeta(T)=\eta(T)=0$

$$\int_{\Omega^T}\Phi\partial_t S\zeta+\int_{\mathscr{Q}^T}\frac{\phi}{|\mathscr{B}|}\partial_t s(\mathscr{L}\zeta+\eta)=-\int_{\Omega^T}\Phi(S-S_0)\partial_t\zeta-\int_{\mathscr{Q}^T}\frac{\phi}{|\mathscr{B}|}(s-s_0)\partial_t(\mathscr{L}\zeta+\eta) \tag{2.34}$$

Proof of this result will be given in section 4.

2.4. *Uniqueness*

Next, we consider the uniqueness of (2.28)–(2.34) for the case that $\Lambda_w\Lambda_o|(\mathrm{d}\mathbf{P}_c/\mathrm{d}S)|$ and $\lambda_w\lambda_o|(\mathrm{d}\mathbf{p}_c/\mathrm{d}s)|$ are bounded above (which includes degenerate case). Domain considered will be a nonsmooth domain.

Definition 2.1. *Boundary $\partial\Omega=\Upsilon_1\cup\Upsilon_2$ of the bounded domain Ω belongs to class $\mathbf{H}_*^m$, $m\geqslant 1$, if* (1) *in the vicinity of each boundary point $x\notin\bar{\Upsilon}_1\cap\bar{\Upsilon}_2$ there exists a homeomorphic transformation $x'(x)=(x_1'(x),x_2'(x),x_3'(x))\in C^m$, $|\mathrm{d}x'/\mathrm{d}x|\geqslant M>0$ ($\mathrm{d}x'/\mathrm{d}x$ is the Jacobian of the transformation) such that $x\in\Upsilon_i$, $x_3'(\Upsilon_i)=0$, $x_3'(\Omega)>0$, $i=1,2$, i.e., Υ_i can be locally straightened,* (2) *in the vicinity of each point $x\in\bar{\Upsilon}_1\cap\bar{\Upsilon}_2$ there exists a transformation $x'=x'(x)$ with the same properties mapping it at the neighbour of the edge(vertex) of a cube in variable x'.*

Besides **A1–8** in section 2.1, let us also assume, for $\alpha = w, o$,

A9. $\mathscr{R}(z) = \mathscr{D}(\widehat{\mathbf{P}}_c(z))$ for $z \in [0, 1]$,

A10. $\Lambda_w \Lambda_o |(\mathrm{d}\mathbf{P}_c/\mathrm{d}S)|(z) \in L^\infty((0, 1])$, $\lambda_w \lambda_o |(\mathrm{d}\mathbf{p}_c/\mathrm{d}s)|(z) \in L^\infty((s_{\min}, s_{\max}])$, $J_\alpha \in L^\infty(0, T; W^{1,\infty}(\Omega))$

A11. $|\Lambda_\alpha(z_1) - \Lambda_\alpha(z_2)| \leqslant \varepsilon_5 \sqrt{(\mathscr{R}(z_1) - \mathscr{R}(z_2))(z_1 - z_2)}$ for $z_1, z_2 \in [0, 1]$,

A12. $\partial\Omega \in \mathbf{H}^3_*$, $\mathscr{B}$ is smooth.

Theorem 2.3. *Under* **A1–12**, *solution of* (2.28)–(2.34) *satisfying* $P \in L^\infty(0, T; W^{1,\infty}(\Omega))$ *is unique.*

Proof of this result will be given in section 5.

3. Convergence of discretized problem

Throughout this section, δ is fixed. We shall show that a solution of (2.12)–(2.19) exists, and a subsequence of solutions of (2.12)–(2.19) converges to a solution of the regularized system (2.20)–(2.26). These are results claimed in section 2.2. Define a non-negative function $\hat{\Theta} : \mathfrak{R} \to \mathfrak{R}^+_0$ by

$$\hat{\Theta}(z) := \int_0^z (\widehat{\mathbf{P}}_c^{\delta,-1}(z) - \widehat{\mathbf{P}}_c^{\delta,-1}(\xi))\,\mathrm{d}\xi.$$

Since $\widehat{\mathbf{P}}_c^{\delta,-1}$ is a strictly increasing function (see (2.5)), as **Remark 1.2** [2], we have

$$\begin{cases} \hat{\Theta}(z) - \hat{\Theta}(z_0) \leqslant (\widehat{\mathbf{P}}_c^{\delta,-1}(z) - \widehat{\mathbf{P}}_c^{\delta,-1}(z_0))z & \text{for any } z,\ z_0 \in \mathfrak{R} \\ |\widehat{\mathbf{P}}_c^{\delta,-1}(z)| \leqslant \varepsilon\hat{\Theta}(z) + \sup_{|\xi| \leqslant 1/\varepsilon} |\widehat{\mathbf{P}}_c^{\delta,-1}(\xi)| & \text{for any positive constant } \varepsilon \end{cases} \tag{3.1}$$

Lemma 3.1. *Under* **A1-6** *and* (2.3)–(2.7), (2.12)–(2.19) *is solvable for all* δ, ℓ, h, *and solution* $U^{\delta,\ell}, P^{\delta,\ell}, s^{\delta,\ell}$ *satisfies*

$$\sup_{0 \leqslant t \leqslant T} (\|\hat{\Theta}(U^{\delta,\ell})\|_{L^1(\Omega)} + \|s^{\delta,\ell}\|^2_{L^2(\mathscr{Q})}) + (\|\nabla U^{\delta,\ell}\|^2_{L^2(\Omega^T)} + \|\nabla P^{\delta,\ell}\|^2_{L^2(\Omega^T)} + \|\nabla_y s^{\delta,\ell}\|^2_{L^2(\mathscr{Q}^T)}) \leqslant c_0 \tag{3.2}$$

where c_0 *is a constant independent of* ℓ, h.

Proof. The solvability of (2.12)–(2.19) is derived by induction. $(S^{\delta,\ell}, U^{\delta,\ell}, s^{\delta,\ell})$ (0) is known by (2.13). Assume $(S^{\delta,\ell}, U^{\delta,\ell}, s^{\delta,\ell})$ (t_{m-1}) is known. By assumptions, $\mathscr{E}^{\delta,\ell,h}$ of (2.14) is continuous. (2.10) and (2.15)–(2.19) imply

$$\begin{aligned} &\mathscr{E}^{\delta,\ell,h}(\xi_{1,1},..,\xi_{1,\ell},\xi_{2,1},\ldots,\xi_{3,\ell})(\xi_{1,1},..,\xi_{1,\ell},\xi_{2,1},\ldots,\xi_{3,\ell}) \\ &\quad \geqslant \int_\Omega \Phi\partial^{-h} S^{\delta,\ell}(U^{\delta,\ell} - U_b^{\delta,h})(t_m) \\ &\qquad + c_1\left(\int_{\mathscr{Q}} \frac{|s^{\delta,\ell}|^2}{h} + \int_\Omega |\nabla U^{\delta,\ell}|^2 + \int_\Omega |\nabla P^{\delta,\ell}|^2 + \int_{\mathscr{Q}} |\nabla_y s^{\delta,\ell}|^2\right)(t_m) - c_2 \end{aligned} \tag{3.3}$$

where c_1, c_2 are positive constants. By $(3.1)_1$,

$$\frac{\Phi}{h}(\hat{\Theta}(U^{\delta,\ell}(t_m)) - \hat{\Theta}(U^{\delta,\ell}(t_{m-1}))) \leqslant \Phi\partial^{-h}S^{\delta,\ell}(U^{\delta,\ell} - U_{\mathrm{b}}^{\delta,h})(t_m) + \Phi\partial^{-h}S^{\delta,\ell}U_{\mathrm{b}}^{\delta,h}(t_m) \tag{3.4}$$

(3.3)–(3.4) and $(3.1)_2$ imply

$$\begin{aligned}&\mathscr{E}^{\delta,\ell,h}(\xi_{1,1},..,\xi_{1,\ell},\xi_{2,1},\ldots,\xi_{3,\ell})(\xi_{1,1},..,\xi_{1,\ell},\xi_{2,1},\ldots,\xi_{3,\ell})\\ &\quad\geqslant c_3\left(\int_\Omega \frac{\hat{\Theta}(U^{\delta,\ell})}{h} + \int_{\mathscr{Q}} \frac{|s^{\delta,\ell}|^2}{h} + \int_\Omega |\nabla U^{\delta,\ell}|^2 + \int_\Omega |\nabla P^{\delta,\ell}|^2 + \int_{\mathscr{Q}} |\nabla_y s^{\delta,\ell}|^2\right)(t_m) - c_4\end{aligned} \tag{3.5}$$

Right-hand side of (3.5) is strictly positive if norm of $(\xi_{1,1},\ldots,\xi_{3,\ell})$ is large enough. So $\mathscr{E}^{\delta,\ell,h}$ of (2.14) has a zero [16] for $t = t_m$. By induction, we see (2.12)–(2.19) is solvable.

Assume $(U^{\delta,\ell} - U_{\mathrm{b}}^{\delta,h}, P^{\delta,\ell} - P_b^h, s^{\delta,\ell} - \mathscr{L}U^{\delta,\ell}) = \sum_{i=1}^{\ell}(\xi_{1,i}\mathbf{e}_{1,i}, \xi_{2,i}\mathbf{e}_{1,i}, \xi_{3,i}\mathbf{e}_{2,i})$ is a zero of (2.14). Then

$$\mathscr{E}^{\delta,\ell,h}(\xi_{1,1},..,\xi_{1,\ell},\xi_{2,1},\ldots,\xi_{3,\ell})(\xi_{1,1},..,\xi_{1,\ell},\xi_{2,1},\ldots,\xi_{3,\ell}) = 0. \tag{3.6}$$

Integrating (3.6) over $[0, t_m]$, we obtain by (2.10),

$$\begin{aligned}&\int_0^{t_m}\int_\Omega \Phi\partial^{-h}S^{\delta,\ell}(U^{\delta,\ell} - U_{\mathrm{b}}^{\delta,h}) + \int_0^{t_m}\int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|}\partial^{-h}s^{\delta,\ell}(s^{\delta,\ell} - \mathscr{L}U_{\mathrm{b}}^{\delta,h})\\ &\quad + c_5\left(\int_0^{t_m}\int_\Omega |\nabla U^{\delta,\ell}|^2 + \int_0^{t_m}\int_\Omega |\nabla P^{\delta,\ell}|^2 + \int_0^{t_m}\int_{\mathscr{Q}} |\nabla_y s^{\delta,\ell}|^2\right) \leqslant c_6\end{aligned} \tag{3.7}$$

where c_5, c_6 are constants independent of ℓ, h. By $(3.1)_1$

$$\frac{\Phi}{h}(\hat{\Theta}(U^{\delta,\ell}(t)) - \hat{\Theta}(U^{\delta,\ell}(t-h))) \leqslant \Phi\partial^{-h}S^{\delta,\ell}(U^{\delta,\ell} - U_{\mathrm{b}}^{\delta,h})(t) + \Phi\partial^{-h}S^{\delta,\ell}U_{\mathrm{b}}^{\delta,h}(t) \tag{3.8}$$

Integrating (3.8) over $\Omega \times [0, t_m]$, we obtain

$$\begin{aligned}&\frac{1}{h}\int_{t_m-h}^{t_m}\int_\Omega \Phi\hat{\Theta}(U^{\delta,\ell}) \leqslant \int_0^{t_m}\int_\Omega \Phi\partial^{-h}S^{\delta,\ell}(U^{\delta,\ell} - U_{\mathrm{b}}^{\delta,h}) + \int_\Omega \Phi\hat{\Theta}(U^{\delta,\ell}(0))\\ &\quad - \int_0^{t_m-h}\int_\Omega \Phi(S^{\delta,\ell} - S^{\delta,\ell}(0))\partial^h U_{\mathrm{b}}^{\delta,h} + \frac{1}{h}\int_{t_m-h}^{t_m}\int_\Omega \Phi(S^{\delta,\ell} - S^{\delta,\ell}(0))U_{\mathrm{b}}^{\delta,h}\end{aligned} \tag{3.9}$$

where $S^{\delta,\ell}(t) = S^{\delta,\ell}(0)$ for $-h < t < 0$. A similar idea as (3.9), we have

$$\begin{aligned}&\frac{1}{h}\int_{t_m-h}^{t_m}\int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|}\frac{|s^{\delta,\ell}|^2}{2} \leqslant \int_0^{t_m}\int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|}\partial^{-h}s^{\delta,\ell}(s^{\delta,\ell} - \mathscr{L}U_{\mathrm{b}}^{\delta,h}) + \int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|}\frac{|s^{\delta,\ell}(0)|^2}{2}\\ &\quad - \int_0^{t_m-h}\int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|}(s^{\delta,\ell} - s^{\delta,\ell}(0))\mathscr{L}\partial^h U_{\mathrm{b}}^{\delta,h} + \frac{1}{h}\int_{t_m-h}^{t_m}\int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|}(s^{\delta,\ell} - s^{\delta,\ell}(0))\mathscr{L}U_{\mathrm{b}}^{\delta,h}\end{aligned} \tag{3.10}$$

where $s^{\delta,\ell}(t) = s^{\delta,\ell}(0)$ for $-h < t < 0$. Since $\mathbf{p}_c^{-1}(-G_b^\delta)$ is smooth, $\|\partial^h U_b^{\delta,h}\|_{L^1(0,T;L^\infty(\Omega))\cap L^2(\Omega^T)}$ and $\|U_b^{\delta,h}\|_{L^\infty(\Omega^T)}$ are bounded by a constant independent of h. (3.7), (3.9)–(3.10), $(3.1)_2$, and discrete Gronwall's inequality imply (3.2). □

From now on, $(S^{\delta,\ell}, \mathrm{U}^{\delta,\ell}, P^{\delta,\ell}, s^{\delta,\ell})$ will be a solution of (2.12)–(2.19). By (2.17)–(2.19), one can see

$$0 = \int_\Omega \Phi\partial^{-h} S^{\delta,\ell}(t_m)\zeta_1 + \int_\Omega \mathbf{K}\tilde{\Lambda}_w^\delta(S^{\delta,\ell}(t_m))\nabla(P^{\delta,\ell}(t_m) - J_w^h)\nabla\zeta_1$$
$$- \int_\Omega \mathbf{K}\frac{\Lambda_w^\delta\Lambda_o^\delta}{\Lambda^\delta}(S^{\delta,\ell}(t_m))\frac{\mathrm{d}\mathbf{p}_c^\delta}{\mathrm{d}s}(U^{\delta,\ell}(t_m))\nabla U^{\delta,\ell}(t_m)\nabla\zeta_1 + \int_{\mathscr{Q}}\frac{\phi}{|\mathscr{B}|}\partial^{-h}s^{\delta,\ell}(t_m)\mathscr{L}\zeta_1 \quad (3.11)$$

$$0 = \int_\Omega \mathbf{K}\tilde{\Lambda}^\delta(S^{\delta,\ell}(t_m))\nabla P^{\delta,\ell}(t_m)\nabla\zeta_2 - \int_\Omega \mathbf{K}(\tilde{\Lambda}_w^\delta(S^{\delta,\ell}(t_m))\nabla J_w^h + \tilde{\Lambda}_o^\delta(S^{\delta,\ell}(t_m))\nabla J_o^h)\nabla\zeta_2 \quad (3.12)$$

$$0 = \int_{\mathscr{Q}}\frac{\phi}{|\mathscr{B}|}\partial^{-h}s^{\delta,\ell}(t_m)\eta - \int_{\mathscr{Q}}\frac{\mathbf{k}}{|\mathscr{B}|}\frac{\lambda_w^\delta\lambda_o^\delta}{\lambda^\delta}(s^{\delta,\ell}(t_m))\frac{\mathrm{d}\mathbf{p}_c^\delta}{\mathrm{d}s}(s^{\delta,\ell}(t_m))\nabla_y s^{\delta,\ell}(t_m)\nabla_y\eta \quad (3.13)$$

for $(\zeta_1, \zeta_2, \eta) \in \mathscr{X}^\ell$.

Lemma 3.2. *For any small* $\varepsilon(>0)$, *solutions of* (2.12)–(2.19) *satisfy*

$$\int_\varepsilon^T\int_\Omega \Phi(S^{\delta,\ell}(x,t) - S^{\delta,\ell}(x,t-\varepsilon))(U^{\delta,\ell}(x,t) - U^{\delta,\ell}(x,t-\varepsilon)) \leqslant c\varepsilon$$

where c is a constant independent of ℓ, h, ε.

Proof. For fixed k, we add (3.11), (3.13) for $m = j+1, \ldots, j+k$, test the resulting equations by

$$\zeta_{1,j} := \zeta_1 = h^2k\partial^{-kh}(U^{\delta,\ell} - U_b^{\delta,h})(t_{j+k}),\ \eta_j := \eta = h^2k\partial^{-kh}(s^{\delta,\ell} - \mathscr{L}U^{\delta,\ell})(t_{j+k})$$

then sum equations for $j = 1, \ldots, M-k$ to obtain (note $t_m = mh$)

$$\sum_{j=1}^{M-k}\left\{\int_\Omega \Phi(S^{\delta,\ell}(t_{j+k}) - S^{\delta,\ell}(t_j))(U^{\delta,\ell}(t_{j+k}) - U^{\delta,\ell}(t_j)) + \int_{\mathscr{Q}}\frac{\phi}{|\mathscr{B}|}|s^{\delta,\ell}(t_{j+k}) - s^{\delta,\ell}(t_j)|^2\right\}$$
$$= \sum_{j=1}^{M-k}\left\{\int_\Omega \Phi(kh)^2\partial^{-kh}S^{\delta,\ell}(t_{j+k})\partial^{-kh}U_b^{\delta,h}(t_{j+k})\right.$$
$$\left. + \int_{\mathscr{Q}}\frac{\phi}{|\mathscr{B}|}(kh)^2\partial^{-kh}s^{\delta,\ell}(t_{j+k})\partial^{-kh}\mathscr{L}U_b^{\delta,h}(t_{j+k})\right\}$$

$$+\sum_{j=1}^{M-k}\sum_{m=j+1}^{j+k}\left\{-\int_\Omega \mathbf{K}\left(\tilde{\Lambda}_w^\delta(S^{\delta,\ell})\nabla(P^{\delta,\ell}-J_w^h)\right.\right.$$
$$\left.-\frac{\Lambda_w^\delta\Lambda_o^\delta}{\Lambda^\delta}(S^{\delta,\ell})\frac{\mathrm{d}\mathbf{p}_c^\delta}{\mathrm{d}s}(U^{\delta,\ell})\nabla U^{\delta,\ell}\right)(t_m)\nabla\zeta_{1,j}$$
$$\left.+\int_{\mathscr{Q}}\frac{\mathbf{k}}{|\mathscr{B}|}\frac{\lambda_w^\delta\lambda_o^\delta}{\lambda^\delta}\frac{\mathrm{d}\mathbf{p}_c^\delta}{\mathrm{d}s}(s^{\delta,\ell})\nabla_y s^{\delta,\ell}(t_m)\nabla_y\eta_j\right\} \tag{3.14}$$

By Lemma 3.1. and rearranging the indices j, m, the right-hand side of (3.14) is bounded by ck. So

$$\int_{kh}^{T}\int_\Omega \Phi(S^{\delta,\ell}(t)-S^{\delta,\ell}(t-kh))(U^{\delta,\ell}(t)-U^{\delta,\ell}(t-kh))\leqslant ckh \tag{3.15}$$

Since $U^{\delta,\ell}$ is a step function in time, we see that this estimate is also satisfied if we replace kh by any positive constant ε. Therefore we complete the proof of this lemma. □

If $d\in\mathbf{N}$, $\sigma:=T/d$, $I_i^\sigma:=[(i-1)\sigma, i\sigma)$, define $\mathscr{A}^\sigma: L^1([0,T))\to L^1(0,T))$ by

$$\mathscr{A}^\sigma(\zeta)(t):=\frac{1}{\sigma}\int_{I_i^\sigma}\zeta(\tau)\,\mathrm{d}\tau \quad \text{for } t\in I_i^\sigma \tag{3.16}$$

Lemma 3.3. *As $\sigma\to 0^+$, $\mathscr{A}^\sigma(U^{\delta,\ell})$ converges to $U^{\delta,\ell}$ strongly in $L^2(\Omega^T)$ and uniformly in ℓ, h.*

Proof. Define, for each ℓ, h,

$$\mathscr{G}^{\ell,h}(\varepsilon,\mathscr{M}):=\left\{t\in(\varepsilon,T): \|U^{\delta,\ell}\|_{H^1(\Omega)}(t)+\|U^{\delta,\ell}\|_{H^1(\Omega)}(t-\varepsilon)\right.$$
$$\left.+\frac{1}{\varepsilon}\int_\Omega \Phi\varepsilon^2\partial^{-\varepsilon}S^{\delta,\ell}(x,t)\partial^{-\varepsilon}U^{\delta,\ell}(x,t)\,\mathrm{d}x>\mathscr{M}\right\} \tag{3.17}$$

By Lemmas 3.1 and 3.2 and (3.17), $\int_{\mathscr{G}^{\ell,h}(\varepsilon,\mathscr{M})}\mathscr{M}\,\mathrm{d}t\leqslant c$, where c is independent of ℓ, h. So

$$|\mathscr{G}^{\ell,h}(\varepsilon,\mathscr{M})|\leqslant c/\mathscr{M}, \quad \text{for all } \ell, h \tag{3.18}$$

Next, we claim: If $t\in(\varepsilon,T)-\mathscr{G}^{\ell,h}(\varepsilon,\mathscr{M})$, then

$$\|U^{\delta,\ell}(\cdot,t)-U^{\delta,\ell}(\cdot,t-\varepsilon)\|_{L^2(\Omega)}\leqslant\beta_{\mathscr{M}}(\varepsilon) \tag{3.19}$$

with a continuous function $\beta_{\mathscr{M}}$ (independent of ℓ, h) satisfying $\beta_{\mathscr{M}}(0)=0$.

Proof of claim. If not, there is a positive constant c_1 such that, as $\varepsilon\to 0^+$, for each ε there are $t_\varepsilon, \ell_\varepsilon, U^{\delta,\ell_\varepsilon}(t_\varepsilon), U^{\delta,\ell_\varepsilon}(t_\varepsilon-\varepsilon)$ satisfying

$$\begin{cases}\|U^{\delta,\ell_\varepsilon}\|_{H^1(\Omega)}(t_\varepsilon)+\|U^{\delta,\ell_\varepsilon}\|_{H^1(\Omega)}(t_\varepsilon-\varepsilon)\leqslant\mathscr{M}\\ \int_\Omega \Phi\varepsilon^2\partial^{-\varepsilon}S^{\delta,\ell_\varepsilon}(x,t_\varepsilon)\partial^{-\varepsilon}U^{\delta,\ell_\varepsilon}(x,t_\varepsilon)\,\mathrm{d}x\leqslant\mathscr{M}\varepsilon\\ \|U^{\delta,\ell_\varepsilon}(t_\varepsilon)-U^{\delta,\ell_\varepsilon}(t_\varepsilon-\varepsilon)\|_{L^2(\Omega)}\geqslant c_1>0\end{cases} \tag{3.20}$$

By $(3.20)_1$, there is a subsequence of $\{U^{\delta,\ell_\varepsilon}(t_\varepsilon), U^{\delta,\ell_\varepsilon}(t_\varepsilon-\varepsilon)\}$ (same index for subsequence) such that the subsequence converges to $\{f_1, f_2\}$ weakly in $H^1(\Omega)$ [14]. It implies $\{U^{\delta,\ell_\varepsilon}(t_\varepsilon), U^{\delta,\ell_\varepsilon}(t_\varepsilon-\varepsilon)\}$ converges to $\{f_1, f_2\}$ strongly in $L^2(\Omega)$ and pointwise almost everywhere. Furthermore, by $(3.20)_3$,

$$\|f_1 - f_2\|_{L^2(\Omega)} \geqslant c_1 > 0 \tag{3.21}$$

By (2.5), $(3.20)_2$, and Lebesque dominant theorem [17],

$$\int_\Omega \Phi(\widehat{\mathbf{P}}_c^{\delta,-1}(f_1) - \widehat{\mathbf{P}}_c^{\delta,-1}(f_2))(f_1 - f_2)\,\mathrm{d}x$$

$$= \lim_{\varepsilon\to 0^+} \int_\Omega \Phi\varepsilon^2 \partial^{-\varepsilon} S^{\delta,\ell_\varepsilon}(x, t_\varepsilon)\partial^{-\varepsilon} U^{\delta,\ell_\varepsilon}(x, t_\varepsilon)\,\mathrm{d}x = 0 \tag{3.22}$$

(3.22) implies $f_1 = f_2$ almost everywhere, which contradicts to (3.21). So the claim is true.

(3.18)–(3.19) imply

$$\int_\varepsilon^T \|U^{\delta,\ell}(\cdot, t) - U^{\delta,\ell}(\cdot, t-\varepsilon)\|^2_{L^2(\Omega)}\,\mathrm{d}t \to 0 \tag{3.23}$$

uniformly in ℓ, h as $\varepsilon \to 0^+$. By (3.16), (3.23),

$$\begin{aligned}
\int_0^T \|U^{\delta,\ell} - \mathscr{A}^\sigma(U^{\delta,\ell})\|^2_{L^2(\Omega)}\,\mathrm{d}t &= \sum_{i=1}^d \int_{I_i^\sigma} \left\| \frac{1}{\sigma}\int_{I_i^\sigma} (U^{\delta,\ell}(\cdot, t) - U^{\delta,\ell}(\cdot, \tau))\,\mathrm{d}\tau \right\|^2_{L^2(\Omega)} \mathrm{d}t \\
&\leqslant \sum_{i=1}^d \int_{I_i^\sigma} \frac{1}{\sigma} \int_{t-i\sigma}^{t-(i-1)\sigma} \|U^{\delta,\ell}(\cdot, t) - U^{\delta,\ell}(\cdot, t-\varepsilon)\|^2_{L^2(\Omega)}\,\mathrm{d}\varepsilon\,\mathrm{d}t \\
&\leqslant \frac{2}{\sigma}\int_0^\sigma \int_\varepsilon^T \|U^{\delta,\ell}(\cdot, t) - U^{\delta,\ell}(\cdot, t-\varepsilon)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,\mathrm{d}\varepsilon \to 0
\end{aligned}$$

uniformly in ℓ, h as $\sigma \to 0^+$. So we complete the proof of this lemma. □

Lemma 3.4. *There are subsequences of* $U^{\delta,\ell}, S^{\delta,\ell}$ *converging to* $U^\delta, S^\delta (= \widehat{\mathbf{P}}_c^{\delta,-1}(U^\delta))$ *pointwise and in* $L^2(\Omega^T)$ *strongly.*

Proof. By Lemma 3.1, $\|U^{\delta,\ell}\|_{L^2(0,T;H^1(\Omega))} \leqslant c$, which is independent of ℓ, h. So for all σ,

$$\|\mathscr{A}^\sigma(U^{\delta,\ell})\|_{L^2(0,T;H^1(\Omega))} \leqslant \text{constant (independent of } \ell, h). \tag{3.24}$$

By Lemma 3.3, (3.24), and diagonal process, we can find a subsequence of $U^{\delta,\ell}$ converging to U^δ in $L^2(\Omega^T)$ strongly and pointwise. By (2.5) and convergence of $U^{\delta,\ell}$ in $L^2(\Omega^T)$, we can also find a convergent subsequence for $S^{\delta,\ell}$. □

Define $\mathscr{D}^\delta: \mathfrak{R} \to \mathfrak{R}$ by

$$\mathscr{D}^\delta(z) := \int_{\widehat{\mathbf{P}}_c(1/2)}^z \frac{\lambda_w^\delta \lambda_o^\delta}{\lambda^\delta} \left|\frac{\mathrm{d}\mathbf{p}_c^\delta}{\mathrm{d}s}\right| (\xi)\,\mathrm{d}\xi \tag{3.25}$$

Lemma 3.5. *There exist subsequences of $S^{\delta,\ell}$, $U^{\delta,\ell}$, $P^{\delta,\ell}$, $s^{\delta,\ell}$ (same indices for subsequences) such that, as $\ell \to \infty$, $h \to 0^+$,*

$$\begin{cases} U^{\delta,\ell}, S^{\delta,\ell} \to U^\delta, S^\delta & \text{in } L^2(\Omega^{\mathrm{T}}) \text{ strongly},\\ P^{\delta,\ell}, S^{\delta,\ell}, U^{\delta,\ell} \to P^\delta, S^\delta, U^\delta & \text{in } L^2(0,T;H^1(\Omega)) \text{ weakly},\\ s^{\delta,\ell} \to s^\delta & \text{in } L^2(0,T;\mathscr{W}) \text{ weakly},\\ \Phi\partial^{-h}S^{\delta,\ell} + \int_{\mathscr{B}}(\phi/|\mathscr{B}|)\,\partial^{-h}s^{\delta,\ell} \to \Phi\partial_t S^\delta & \\ \qquad + \int_{\mathscr{B}}(\phi/|\mathscr{B}|)\,\partial_t s^\delta & \text{in } L^2(0,T;\mathscr{H}^*) \text{ weakly},\\ \partial^{-h}s^{\delta,\ell} \to \partial_t s^\delta & \text{in } L^2(0,T;\mathscr{W}_0^*) \text{ weakly},\\ s^{\delta,\ell}(T) \to s^\delta(T) & \text{in } L^2(\mathscr{Q}) \text{ weakly},\\ s^{\delta,\ell}(0) \to \mathscr{L}\mathbf{p}_c^{-1}(-G_0^\delta) = s^\delta(0) & \text{in } L^2(\mathscr{Q}) \text{ strongly},\\ \mathscr{D}^\delta(s^{\delta,\ell}) \to \mathscr{D}^\delta(s^\delta) & \text{in } L^2(0,T;\mathscr{W}) \text{ weakly} \end{cases} \tag{3.26}$$

Proof. By (2.13) and Lemmas 3.1, 3.4, there exist $S^\delta, U^\delta, \mathrm{P}^\delta, s^\delta, \tilde{\mathscr{D}}, \tilde{s}$ such that, as $\ell \to \infty, h \to 0^+$,

$$\begin{cases} U^{\delta,\ell}, S^{\delta,\ell} \to U^\delta, S^\delta & \text{in } L^2(\Omega^{\mathrm{T}}) \text{ strongly},\\ P^{\delta,\ell}, S^{\delta,\ell}, U^{\delta,\ell} \to P^\delta, S^\delta, U^\delta & \text{in } L^2(0,T;H^1(\Omega)) \text{ weakly},\\ s^{\delta,\ell}, \mathscr{D}^\delta(s^{\delta,\ell}) \to s^\delta, \tilde{\mathscr{D}} & \text{in } L^2(0,T;\mathscr{W}) \text{ weakly},\\ s^{\delta,\ell}(T) \to \tilde{s} & \text{in } L^2(\mathscr{Q}) \text{ weakly},\\ s^{\delta,\ell}(0) \to \mathscr{L}\mathbf{p}_c^{-1}(-G_0^\delta) & \text{in } L^2(\mathscr{Q}) \text{ strongly} \end{cases} \tag{3.27}$$

Let $\mathscr{H}^\ell$, $\mathscr{W}_0^\ell$ be finite-dimensional subspaces of $\mathscr{H}$, $\mathscr{W}_0$, respectively. If $\mathscr{P}_{1,0}$ (average function as (2.9)) is defined to be $\mathscr{P}_{1,0}(f)\,(x,y,t) := 1/h\int_{I_m} f(x,y,\tau)\,\mathrm{d}\tau$, $t \in I_m$, and if $\mathscr{P}_{1,1}: I_h(\mathscr{H}) \to I_h(\mathscr{H}^\ell)$ is the projection (see (2.8) for I_h), then define a map $\mathscr{P}_1: L^2(0,T;\mathscr{H}) \to I_h(\mathscr{H}^\ell)$ by $\mathscr{P}_1 = \mathscr{P}_{1,1}\circ\mathscr{P}_{1,0}$. Similarly, if $\mathscr{P}_{2,1}: I_h(\mathscr{W}_0) \to I_h(\mathscr{W}_0^\ell)$ is the projection, then define $\mathscr{P}_2: L^2(0,T;\mathscr{W}_0) \to I_h(\mathscr{W}_0^\ell)$ by $\mathscr{P}_2 = \mathscr{P}_{2,1}\circ\mathscr{P}_{1,0}$. By (3.11), (3.13), and Lemma 3.1,

$$\begin{aligned} &\int_{\Omega^{\mathrm{T}}} \Phi\partial^{-h}S^{\delta,\ell}\mathscr{P}_1(\zeta_1) + \int_{\mathscr{Q}^{\mathrm{T}}} \frac{\phi}{|\mathscr{B}|}\,\partial^{-h}s^{\delta,\ell}(\mathscr{L}\mathscr{P}_1(\zeta_1) + \mathscr{P}_2(\eta)) \\ &\quad \leqslant c(\|\zeta_1\|_{L^2(0,T;\mathscr{H})} + \|\eta\|_{L^2(0,T;\mathscr{W}_0)}) \end{aligned} \tag{3.28}$$

for $\zeta_1 \in L^2(0,T;\mathscr{H})$, $\eta \in L^2(0,T;\mathscr{W}_0)$. (3.27)–(3.28) imply $(3.26)_4$ and $(3.26)_5$.

For each $i \geqslant 1$, and $f \in C^1[0,T]$, we obtain from (3.13)

$$\begin{aligned} &-\int_0^{T-h}\int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|}\,s^{\delta,\ell}\partial^h\mathscr{P}_{1,0}(f)(t)\mathbf{e}_{2,i} + \int_{\mathscr{Q}^{\mathrm{T}}} \frac{\mathbf{k}}{|\mathscr{B}|}\,\nabla_y\mathscr{D}^\delta(s^{\delta,\ell})\mathscr{P}_{1,0}(f)(t)\nabla_y\mathbf{e}_{2,i} \\ &\quad = -\frac{1}{h}\int_{T-h}^{T}\int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|}\,s^{\delta,\ell}(T)\mathscr{P}_{1,0}(f)(t)\mathbf{e}_{2,i} + \int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|}\,s^{\delta,\ell}(0)f(0)\mathbf{e}_{2,i} \end{aligned} \tag{3.29}$$

Let $\ell \to \infty$, $h \to 0^+$, (2.13) and (3.27) imply

$$-\int_{\mathscr{Q}^T} \frac{\phi}{|\mathscr{B}|} s^\delta \partial_t f(t) \mathbf{e}_{2,i} + \int_{\mathscr{Q}^T} \frac{\mathbf{k}}{|\mathscr{B}|} \nabla_y \tilde{\mathscr{D}} f(t) \nabla_y \mathbf{e}_{2,i}$$

$$= -\int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|} \tilde{s} f(T) \mathbf{e}_{2,i} + \int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|} \mathscr{L} \mathbf{p}_c^{-1}(-G_0^\delta) f(0) \mathbf{e}_{2,i} \tag{3.30}$$

Applying Green's theorem for (3.30) in the t variable yields, by $(3.26)_5$,

$$\int_{\mathscr{Q}^T} \frac{\phi}{|\mathscr{B}|} \partial_t s^\delta f(t) \mathbf{e}_{2,i} + \int_{\mathscr{Q}^T} \frac{\mathbf{k}}{|\mathscr{B}|} \nabla_y \tilde{\mathscr{D}} f(t) \nabla_y \mathbf{e}_{2,i}$$

$$= -\int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|} (\tilde{s} - s^\delta(T)) f(T) \mathbf{e}_{2,i} + \int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|} (\mathscr{L} \mathbf{p}_c^{-1}(-G_0^\delta) - s^\delta(0)) f(0) \mathbf{e}_{2,i} \tag{3.31}$$

Since $\{\mathbf{e}_{2,i}\}_{i=1}^\infty$ is a basis of $\mathscr{W}_0$, (3.31) implies $\tilde{s} = s^\delta(T)$, $s^\delta(0) = \mathscr{L} \mathbf{p}_c^{-1}(-G_0^\delta)$ (that is, $(3.26)_{6,7}$), and for $\eta \in L^2(0, T; \mathscr{W}_0)$,

$$\int_{\mathscr{Q}^T} \frac{\phi}{|\mathscr{B}|} \partial_t s^\delta \eta + \int_{\mathscr{Q}^T} \frac{\mathbf{k}}{|\mathscr{B}|} \nabla_y \tilde{\mathscr{D}} \nabla_y \eta = 0 \tag{3.32}$$

Finally we show $\mathscr{D}^\delta(s^\delta) = \tilde{\mathscr{D}}$. Since $\mathscr{D}^\delta(z)$ is an increasing function, for all $\ell \in \mathbf{N}$, $f \in L^2(\mathscr{Q}^T)$

$$0 \leqslant \int_{\mathscr{Q}^T} \frac{\phi}{|\mathscr{B}|} (\mathscr{D}^\delta(s^{\delta,\ell}) - \mathscr{D}^\delta(f))(s^{\delta,\ell} - f) \tag{3.33}$$

By **A2** and (2.11), one can find $\varphi^{\delta,\ell} \in \mathscr{W}_0^\ell$, φ^δ, $\rho^{\delta,\ell}$, $\rho^\delta \in \mathscr{W}_0$ satisfying, for all $x \in \Omega$,

$$\begin{cases} -\nabla_y \left(\frac{\mathbf{k}}{|\mathscr{B}|} \nabla_y \varphi^{\delta,\ell} \right) = \frac{\phi}{|\mathscr{B}|} (s^{\delta,\ell} - \mathscr{L} U^{\delta,\ell}), \\ \varphi^{\delta,\ell}|_{\partial\mathscr{B}} = 0, \end{cases} \begin{cases} -\nabla_y \left(\frac{\mathbf{k}}{|\mathscr{B}|} \nabla_y \varphi^\delta \right) = \frac{\phi}{|\mathscr{B}|} (s^\delta - \mathscr{L} U^\delta), \\ \varphi^\delta|_{\partial\mathscr{B}} = 0 \end{cases} \tag{3.34}$$

$$\begin{cases} \nabla_y \left(\frac{\mathbf{k}}{|\mathscr{B}|} \nabla_y \rho^{\delta,\ell} \right) = \frac{\phi}{|\mathscr{B}|} \mathscr{L} U^{\delta,\ell}, \\ \rho^{\delta,\ell}|_{\partial\mathscr{B}} = 0, \end{cases} \begin{cases} \nabla_y \left(\frac{\mathbf{k}}{|\mathscr{B}|} \nabla_y \rho^\delta \right) = \frac{\phi}{|\mathscr{B}|} \mathscr{L} U^\delta, \\ \rho^\delta|_{\partial\mathscr{B}} = 0 \end{cases} \tag{3.35}$$

Note, by (3.34),

$$\int_{\mathscr{Q}^T} \frac{\phi}{|\mathscr{B}|} \mathscr{D}^\delta(s^{\delta,\ell}) s^{\delta,\ell} = \int_{\mathscr{Q}^T} \frac{\phi}{|\mathscr{B}|} \mathscr{D}^\delta(\mathscr{L} U^{\delta,\ell}) s^{\delta,\ell} + \int_{\mathscr{Q}^T} (\mathscr{D}^\delta(s^{\delta,\ell}) - \mathscr{D}^\delta(\mathscr{L} U^{\delta,\ell})) \frac{\phi}{|\mathscr{B}|} \mathscr{L} U^{\delta,\ell}$$

$$-\int_{\mathscr{Q}^T} (\mathscr{D}^\delta(s^{\delta,\ell}) - \mathscr{D}^\delta(\mathscr{L} U^{\delta,\ell})) \nabla_y \left(\frac{\mathbf{k}}{|\mathscr{B}|} \nabla_y \varphi^{\delta,\ell} \right) \tag{3.36}$$

By Green's theorem, (2.15), (3.13), (3.34)–(3.35),

$$
\begin{aligned}
&-\int_{\mathcal{Q}^{\mathrm{T}}} (\mathcal{D}^{\delta}(s^{\delta,\ell}) - \mathcal{D}^{\delta}(\mathcal{L}U^{\delta,\ell}))\nabla_y\left(\frac{\mathbf{k}}{|\mathcal{B}|}\nabla_y\varphi^{\delta,\ell}\right)\\
&=\int_{\mathcal{Q}^{\mathrm{T}}} \frac{\mathbf{k}}{|\mathcal{B}|}\nabla_y\mathcal{D}^{\delta}(\mathrm{s}^{\delta,\ell})\nabla_y\varphi^{\delta,\ell}\\
&=-\int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi}{|\mathcal{B}|}\partial^{-h}s^{\delta,\ell}\varphi^{\delta,\ell}\\
&=-\int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi}{|\mathcal{B}|}\partial^{-h}s^{\delta,\ell}(\varphi^{\delta,\ell}-\rho^{\delta,\ell}) - \int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi}{|\mathcal{B}|}\partial^{-h}s^{\delta,\ell}\rho^{\delta,\ell}\\
&\leqslant -\frac{1}{2}\int_{\mathcal{Q}} \frac{\mathbf{k}}{|\mathcal{B}|}|\nabla_y(\varphi^{\delta,\ell}-\rho^{\delta,\ell})|^2\Big|_0^T - \int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi}{|\mathcal{B}|}\partial^{-h}s^{\delta,\ell}\rho^{\delta,\ell}
\end{aligned} \tag{3.37}
$$

$(3.26)_{1,5,6}$, (3.34)–(3.35), Hölder inequality, and Green's theorem imply

$$
\int_{\mathcal{Q}} \frac{\mathbf{k}}{|\mathcal{B}|}|\nabla_y(\varphi^{\delta}-\rho^{\delta})|^2(T) \leqslant \liminf_{\ell\to\infty}\int_{\mathcal{Q}} \frac{\mathbf{k}}{|\mathcal{B}|}|\nabla_y(\varphi^{\delta,\ell}-\rho^{\delta,\ell})|^2(T) \tag{3.38}
$$

$$
\int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi}{|\mathcal{B}|}\partial_t s^{\delta}\rho^{\delta} = \lim_{\ell\to\infty}\int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi}{|\mathcal{B}|}\partial^{-h}s^{\delta,\ell}\rho^{\delta,\ell} \tag{3.39}
$$

Taking limit supremum both sides of (3.33) and employing (3.27), (3.36)–(3.39), we obtain

$$
\begin{aligned}
0 \leqslant &\int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi}{|\mathcal{B}|}\mathcal{D}^{\delta}(\mathcal{L}U^{\delta})s^{\delta} + \int_{\mathcal{Q}^{\mathrm{T}}} (\tilde{\mathcal{D}} - \mathcal{D}^{\delta}(\mathcal{L}U^{\delta}))\frac{\phi}{|\mathcal{B}|}\mathcal{L}U^{\delta} - \int_{\mathcal{Q}} \frac{\mathbf{k}}{2|\mathcal{B}|}|\nabla_y(\varphi^{\delta}-\rho^{\delta})|^2\Big|_0^{\mathrm{T}}\\
&-\int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi}{|\mathcal{B}|}\partial_t s^{\delta}\rho^{\delta} - \int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi\mathcal{D}^{\delta}(f)(s^{\delta}-f)+\phi\tilde{\mathcal{D}}f}{|\mathcal{B}|}
\end{aligned} \tag{3.40}
$$

Set $\eta = \varphi^{\delta}$ in (3.32), by $(3.26)_5$,

$$
0 = \int_{\mathcal{Q}} \frac{\mathbf{k}}{2|\mathcal{B}|}|\nabla_y(\varphi^{\delta}-\rho^{\delta})|^2\Big|_0^{\mathrm{T}} + \int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi}{|\mathcal{B}|}\partial_t s^{\delta}\rho^{\delta} + \int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi}{|\mathcal{B}|}(\tilde{\mathcal{D}} - \mathcal{D}^{\delta}(\mathcal{L}U^{\delta}))(s^{\delta}-\mathcal{L}U^{\delta}) \tag{3.41}
$$

(3.40)–(3.41) imply, for all $f\in L^2(\mathcal{Q}^{\mathrm{T}})$,

$$
0 \leqslant \int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi}{|\mathcal{B}|}(\tilde{\mathcal{D}} - \mathcal{D}^{\delta}(f))(s^{\delta}-f)
$$

If $f = s^{\delta} - \tilde{\beta}\tilde{\rho}$, where $\tilde{\beta} > 0$ and $\tilde{\rho}\in L^2(\mathcal{Q}^{\mathrm{T}})$, then

$$
0 \leqslant \int_{\mathcal{Q}^{\mathrm{T}}} \frac{\phi}{|\mathcal{B}|}(\tilde{\mathcal{D}} - \mathcal{D}^{\delta}(s^{\delta}-\tilde{\beta}\tilde{\rho}))\tilde{\rho}
$$

Let $\tilde{\beta} \to 0^+$, then we have, for all $\tilde{\rho} \in L^2(\mathscr{Q}^{\mathrm{T}})$,

$$0 \leqslant \int_{\mathscr{Q}^{\mathrm{T}}} \frac{\phi}{|\mathscr{B}|}(\tilde{\mathscr{D}} - \mathscr{D}^\delta(s^\delta))\tilde{\rho},$$

which implies $\tilde{\mathscr{D}} = \mathscr{D}^\delta(s^\delta)$. So we complete the proof of this lemma. □

Lemma 3.6. $\delta \leqslant S^\delta \leqslant 1 - \delta$ *and* $\widehat{\mathbf{P}}_{\mathrm{c}}(\delta) \leqslant s^\delta \leqslant \widehat{\mathbf{P}}_{\mathrm{c}}(1-\delta)$.

Proof. (2.10), (2.15)–(2.19), and Lemma 3.5 imply that for all $\zeta_1, \zeta_2 \in L^2(0, T; \mathscr{H})$, $\eta \in L^2(0, T; \mathscr{W}_0)$,

$$U^\delta = \widehat{\mathbf{P}}_{\mathrm{c}}^\delta(S^\delta), (U^\delta - \mathbf{p}_{\mathrm{c}}^{-1}(-G_{\mathrm{b}}^\delta), P^\delta - P_{\mathrm{b}}, s^\delta - \mathscr{L}U^\delta) \in L^2(0, T; \mathscr{Z}) \tag{3.42}$$

$$\int_{\Omega^{\mathrm{T}}} \Phi \partial_t S^\delta \zeta_1 + \int_{\Omega^{\mathrm{T}}} \mathbf{K}\left(\tilde{\Lambda}_w^\delta \nabla(P^\delta - J_w) - \frac{\Lambda_w^\delta \Lambda_o^\delta}{\Lambda^\delta}\frac{\mathrm{d}\mathbf{p}_{\mathrm{c}}^\delta}{\mathrm{d}s}(U^\delta)\nabla U^\delta\right)\nabla\zeta_1 = -\int_{\mathscr{Q}^{\mathrm{T}}} \frac{\phi}{|\mathscr{B}|}\partial_t s^\delta \mathscr{L}\zeta_1 \tag{3.43}$$

$$\int_{\Omega^{\mathrm{T}}} \mathbf{K}\tilde{\Lambda}^\delta \nabla P^\delta \nabla \zeta_2 - \int_{\Omega^{\mathrm{T}}} \mathbf{K}(\tilde{\Lambda}_w^\delta \nabla J_w + \tilde{\Lambda}_o^\delta \nabla J_o)\nabla\zeta_2 = 0 \tag{3.44}$$

$$\int_{\mathscr{Q}^{\mathrm{T}}} \frac{\phi}{|\mathscr{B}|}\partial_t s^\delta \eta - \int_{\mathscr{Q}^{\mathrm{T}}} \frac{\mathbf{k}}{|\mathscr{B}|}\frac{\lambda_w^\delta \lambda_o^\delta}{\lambda^\delta}\frac{\mathrm{d}\mathbf{p}_{\mathrm{c}}^\delta}{\mathrm{d}s}(s^\delta)\nabla_y s^\delta \nabla_y \eta = 0 \tag{3.45}$$

$$U^\delta(x, 0) = \mathbf{p}_{\mathrm{c}}^{-1}(-G_0^\delta), \quad s^\delta(x, y, 0) = \mathscr{L}\mathbf{p}_{\mathrm{c}}^{-1}(-G_0^\delta) \tag{3.46}$$

where $\tilde{\Lambda}^\delta, \tilde{\Lambda}_\alpha^\delta, \Lambda^\delta, \Lambda_\alpha^\delta$ are functions of S^δ and $\lambda^\delta, \lambda_\alpha^\delta$ are functions of s^δ for $\alpha = w, o$.

By (2.5), $(2.7)_3$, and (3.42), $\max\{U^\delta - \widehat{\mathbf{P}}_{\mathrm{c}}^\delta(1-\delta), 0\} \in L^2(0, T; \mathscr{H})$. Let $\zeta_1 = \zeta_2 = \max\{U^\delta - \widehat{\mathbf{P}}_{\mathrm{c}}^\delta(1-\delta), 0\}$ in (3.43)–(3.44) and let $\eta = \max\{s^\delta - \widehat{\mathbf{P}}_{\mathrm{c}}^\delta(1-\delta), 0\} - \mathscr{L}\zeta_1$ in (3.45), we see that, by (2.4), $(2.7)_3$, and (3.46),

$$\int_{\Omega^{\mathrm{T}}} \Phi \partial_t (S^\delta - (1-\delta))\zeta_1$$

$$+ c_1\left(\int_{\Omega^{\mathrm{T}}} |\nabla U^\delta|^2 \mathscr{X}_{\zeta_1} + \int_{\mathscr{Q}} |s^\delta - \widehat{\mathbf{P}}_{\mathrm{c}}^\delta(1-\delta)|^2 \mathscr{X}_{\eta + \mathscr{L}\zeta_1}(T) + \int_{\mathscr{Q}^{\mathrm{T}}} |\nabla_y s^\delta|^2 \mathscr{X}_\eta\right) \leqslant 0 \tag{3.47}$$

where c_1 is a positive number and $\mathscr{X}_{\zeta_1}, \mathscr{X}_{\eta+\mathscr{L}\zeta_1}(T), \mathscr{X}_\eta$ are characteristic functions defined in (2.2). By $(2.5)_4$, (3.47) implies $S^\delta \leqslant 1 - \delta$, $s^\delta \leqslant \widehat{\mathbf{P}}_{\mathrm{c}}^\delta(1-\delta) = \widehat{\mathbf{P}}_{\mathrm{c}}(1-\delta)$. Similarly, let $\zeta_1 = \max\{-U^\delta + \widehat{\mathbf{P}}_{\mathrm{c}}^\delta(\delta), 0\}$ in (3.43) and let $\eta = \max\{-s^\delta + \widehat{\mathbf{P}}_{\mathrm{c}}^\delta(\delta), 0\} - \mathscr{L}\zeta_1$ in (3.45), we have $S^\delta \geqslant \delta$, $s^\delta \geqslant \widehat{\mathbf{P}}_{\mathrm{c}}^\delta(\delta) = \widehat{\mathbf{P}}_{\mathrm{c}}(\delta)$. □

Theorem 2.1 is a direct result of (2.3)–(2.5) and Lemmas 3.5, 3.6.

4. Convergence of continuous problem

In this section, we are going to show that a subsequence of solutions of the regularized systems (2.20)–(2.26) converges to a weak solution of (1.1)–(1.7) as $\delta \to 0^+$. Idea of proof for above claim is almost same as that in section 3. By A4, let us give some notations:

$$\begin{cases} \Gamma:(-\infty,0]\to(0,1] & \Gamma(z):=\mathbf{P}_c^{-1}(-z), \\ \gamma:(-\infty,0]\to(s_{\min},s_{\max}] & \gamma(z):=\mathbf{p}_c^{-1}(-z), \\ \Theta:(-\infty,0]\to\Re_0^+ & \Theta(z):=\int_0^z(\Gamma(z)-\Gamma(\xi))\,d\xi, \\ \theta:(-\infty,0]\to\Re_0^+ & \theta(z):=\int_0^z(\gamma(z)-\gamma(\xi))\,d\xi, \\ \hat{\mathscr{D}}:(s_{\min},s_{\max}]\to\Re & \hat{\mathscr{D}}(z):=\int_{\hat{\mathbf{P}}_c(1/2)}^z\sqrt{\lambda_w\lambda_o}|d\mathbf{p}_c/ds|(\xi)\,d\xi, \\ G^\delta:=-\mathbf{P}_c(S^\delta),\ g^\delta:=-\mathbf{p}_c(s^\delta), & V^\delta:=\hat{\mathscr{D}}(U^\delta),\ v^\delta:=\hat{\mathscr{D}}(s^\delta) \end{cases} \tag{4.1}$$

It is easy to see that $G^\delta - G_b^\delta \in L^2(0,T;\mathscr{H})$ by (2.21)–(2.22) of Theorem 2.1 and

$$\begin{cases} \Theta(z)-\Theta(z_0)\leqslant(\Gamma(z)-\Gamma(z_0))z \\ \theta(z)-\theta(z_0)\leqslant(\gamma(z)-\gamma(z_0))z \end{cases} \quad \text{for any } z, z_0\in(-\infty,0] \tag{4.2}$$

Define

$$\begin{cases} P_w^\delta:=P^\delta-\frac{1}{2}\left(\mathbf{P}_c(S^\delta)+\int_0^{\mathbf{P}_c(S^\delta)}\left(\frac{\Lambda_o}{\Lambda}-\frac{\Lambda_w}{\Lambda}\right)(\mathbf{P}_c^{-1}(\xi))\,d\xi\right) \\ P_{w,b}^\delta:=\mathbf{P}_b-\frac{1}{2}\left(-G_b^\delta+\int_0^{-G_b^\delta}\left(\frac{\Lambda_o}{\Lambda}-\frac{\Lambda_w}{\Lambda}\right)(\mathbf{P}_c^{-1}(\xi))\,d\xi\right) \\ P_o^\delta:=\mathbf{P}_c(S^\delta)+P_w^\delta \\ P_{o,b}^\delta:=-G_b^\delta+P_{w,b}^\delta \end{cases} \tag{4.3}$$

(2.23)–(2.24) and (4.3) imply, for $\zeta_w, \zeta_o \in L^2(0,T;\mathscr{H})$,

$$P_w^\delta - P_{w,b}^\delta,\ P_o^\delta - P_{o,b}^\delta \in L^2(0,T;\mathscr{H}) \tag{4.4}$$

$$\int_{\Omega^T}\Phi\partial_t S^\delta\zeta_w+\int_{\Omega^T}\mathbf{K}(\Lambda_w\nabla P_w^\delta-\tilde{\Lambda}_w^\delta\nabla J_w+(\tilde{\Lambda}_w^\delta-\Lambda_w)\nabla P^\delta)\nabla\zeta_w=-\int_{\mathscr{Q}^T}\frac{\phi}{|\mathscr{B}|}\partial_t s^\delta\mathscr{L}\zeta_w \tag{4.5}$$

$$-\int_{\Omega^T}\Phi\partial_t S^\delta\zeta_o+\int_{\Omega^T}\mathbf{K}(\Lambda_o\nabla P_o^\delta-\tilde{\Lambda}_o^\delta\nabla J_o+(\tilde{\Lambda}_o^\delta-\Lambda_o)\nabla P^\delta)\nabla\zeta_o=\int_{\mathscr{Q}^T}\frac{\phi}{|\mathscr{B}|}\partial_t s^\delta\mathscr{L}\zeta_o \tag{4.6}$$

where $\tilde{\Lambda}_\alpha^\delta, \Lambda_\alpha, \alpha = w, o$ are functions of S^δ.

Lemma 4.1. *Under* **A1–7**, *solutions of* (2.20)–(2.26) *satisfy* (*by notation of* (4.1))

$$\sup_{0\leqslant t\leqslant T}\left(\int_\Omega\Theta(G^\delta)+\int_{\mathscr{Q}}\frac{\theta(g^\delta)}{|\mathscr{B}|}\right)+\|\nabla P^\delta\|^2_{L^2(\Omega^T)}+\int_{\Omega^T}\Lambda_o\Lambda_w(S^\delta)|\nabla\mathbf{P}_c(S^\delta)|^2$$
$$+\int_{\mathscr{Q}^T}\lambda_o\lambda_w(s^\delta)|\nabla_y\mathbf{p}_c(s^\delta)|^2+\|V^\delta\|^2_{L^2(0,T;H^1(\Omega))}+\|v^\delta\|^2_{L^2(0,T;\mathscr{W})}\leqslant c_0$$

where c_0 is a constant independent of δ.

Proof. Set $\zeta_2 = P^\delta - P_{\rm b}$ in (2.24), we obtain

$$\|\nabla P^\delta\|^2_{L^2(\Omega^\tau)} \leqslant c_0 \tag{4.7}$$

By $(4.2)_1$, for all $t, \varepsilon > 0$ and $x \in \Omega$,

$$\Theta(G^\delta(t)) - \Theta(G^\delta(t-\varepsilon)) \leqslant (S^\delta(t) - S^\delta(t-\varepsilon))\, G^\delta(t) \tag{4.8}$$

where $G^\delta(t) = G^\delta(0)$ for $-\varepsilon < t < 0$. We multiply (4.8) by Φ/ε and integrate over Ω^τ to obtain

$$\frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau}\int_\Omega \Phi\Theta(G^\delta) \leqslant \int_{\Omega^\tau}\Phi\partial^{-\varepsilon}S^\delta(G^\delta - G^\delta_{\rm b}) + \int_\Omega \Phi\Theta(G^\delta(0))$$
$$-\int_0^{\tau-\varepsilon}\int_\Omega \Phi(S^\delta - S^\delta(0))\partial^\varepsilon G^\delta_{\rm b} + \frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau}\int_\Omega \Phi(S^\delta - S^\delta(0))\, G^\delta_{\rm b} \tag{4.9}$$

Similar to (4.9) by $(4.2)_2$, we obtain

$$\frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau}\int_{\mathcal{Q}}\frac{\phi}{|\mathcal{B}|}\,\theta(g^\delta) \leqslant \int_{\mathcal{Q}^\tau}\frac{\phi}{|\mathcal{B}|}\,\partial^{-\varepsilon}s^\delta(g^\delta - \mathcal{L}G^\delta_{\rm b}) + \int_{\mathcal{Q}}\frac{\phi}{|\mathcal{B}|}\,\theta(g^\delta(0))$$
$$-\int_0^{\tau-\varepsilon}\int_{\mathcal{Q}}\frac{\phi}{|\mathcal{B}|}(s^\delta - s^\delta(0))\mathcal{L}\partial^\varepsilon G^\delta_b + \frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau}\int_{\mathcal{Q}}\frac{\phi}{|\mathcal{B}|}(s^\delta - s^\delta(0))\mathcal{L}G^\delta_{\rm b}. \tag{4.10}$$

Summing (4.9) and (4.10), letting $\varepsilon \to 0^+$, by (2.22) and (4.2), we have, for almost all $\tau \in (0, T)$,

$$\int_\Omega \Phi\Theta(G^\delta)(\tau) + \int_{\mathcal{Q}}\frac{\phi}{|\mathcal{B}|}\,\theta(g^\delta)(\tau) \leqslant \int_{\Omega^\tau}\Phi\partial_t S^\delta(G^\delta - G^\delta_{\rm b}) + \int_{\mathcal{Q}^\tau}\frac{\phi}{|\mathcal{B}|}\,\partial_t s^\delta(g^\delta - \mathcal{L}G^\delta_{\rm b})$$
$$+ c(\|G^\delta_0\|_{L^1(\Omega)}, \|G^\delta_{\rm b}\|_{L^\infty(0,T;L^1(\Omega))}, \|\partial_t G^\delta_{\rm b}\|_{L^1(\Omega^T)}). \tag{4.11}$$

Setting $\zeta_w = P^\delta_w - P^\delta_{w,{\rm b}}$ in (4.5), $\zeta_o = P^\delta_o - P^\delta_{o,{\rm b}}$ in (4.6), $\eta = g^\delta - \mathcal{L}G^\delta$ in (2.25), and summing the three equations, we obtain, by (4.7),

$$\int_{\Omega^\tau}\Phi\partial_t S^\delta(G^\delta - G^\delta_{\rm b}) + \sum_{\alpha = w,o}\int_{\Omega^\tau}\Lambda_\alpha(S^\delta)|\nabla P^\delta_\alpha|^2 + \int_{\mathcal{Q}^\tau}\frac{\phi}{|\mathcal{B}|}\,\partial_t s^\delta(g^\delta - \mathcal{L}G^\delta_{\rm b})$$
$$+ \int_{\mathcal{Q}^\tau}\lambda_w\lambda_o(s^\delta)|\nabla_y g^\delta|^2 \leqslant c(\|\mathbf{K}\|_{L^\infty(\Omega)}, \|\nabla P_b\|_{L^2(\Omega^T)}, \|\nabla G^\delta_{\rm b}\|_{L^2(\Omega^T)}, \tau) \tag{4.12}$$

By (2.6)–(2.7), **A5,6**, and (4.11)–(4.12), we obtain

$$\sup_{0\leqslant t\leqslant T}\left(\int_\Omega \Theta(G^\delta) + \int_{\mathcal{Q}}\frac{\theta(g^\delta)}{|\mathcal{B}|}\right) + \sum_{\alpha = w,o}\int_{\Omega^T}\Lambda_\alpha(S^\delta)|\nabla P^\delta_\alpha|^2 + \int_{\mathcal{Q}^T}\lambda_o\lambda_w(s^\delta)|\nabla_y g^\delta|^2 \leqslant c_0 \tag{4.13}$$

where c_0 is a constant independent of δ. By $(4.3)_3$ and (4.13),

$$\int_{\Omega^T}\Lambda_w\Lambda_o(S^\delta)|\nabla\mathbf{P}_{\rm c}(S^\delta)|^2 = \int_{\Omega^T}\Lambda_w\Lambda_o(S^\delta)|\nabla P^\delta_o - \nabla P^\delta_w|^2 \leqslant c_0 \tag{4.14}$$

By **A7**, (2.7), (2.21), and (4.13)–(4.14), one can easily see that $\|V^\delta\|_{L^2(0,T;H^1(\Omega))}$ and $\|v^\delta\|_{L^2(0,T;\mathscr{W})}$ are bounded independent of δ. Therefore we complete the proof of this lemma. □

Lemma 4.2. *Under* **A1–7**, *for any* $f \in C_0^\infty(\Omega)$ *and sufficiently small* ε, *solutions of* (2.20)–(2.26) *satisfy*

$$\int_\varepsilon^T \int_\Omega \Phi f(x)(S^\delta(x,\tau) - S^\delta(x,\tau-\varepsilon))(V^\delta(x,\tau) - V^\delta(x,\tau-\varepsilon)) \leqslant c_1 \varepsilon \|f\|_{W^{1,\infty}(\Omega)}$$

where c_1 *is independent of* δ, ε *and* V^δ *is defined in* $(4.1)_6$.

Proof. Proof is similar to that in Lemma 3.2. Let $f \in C_0^\infty(\Omega)$. In (2.23) and (2.25), we set

$$\zeta_1(x,t) := f(x) \int_{\max(t,\varepsilon)}^{\min(t+\varepsilon,T)} \varepsilon \partial^{-\varepsilon} V^\delta(x,\tau)\,\mathrm{d}\tau$$

$$\eta(x,y,t) := f(x) \int_{\max(t,\varepsilon)}^{\min(t+\varepsilon,T)} \varepsilon \partial^{-\varepsilon}(v^\delta - \mathscr{L}V^\delta)(x,y,\tau)\,\mathrm{d}\tau$$

where V^δ and v^δ are defined in $(4.1)_6$. By Theorem 2.1, $\zeta_1 \in L^2(0,T;\mathscr{H})$ and $\eta \in L^2(0,T;\mathscr{W}_0)$. By Fubini's theorem,

$$\begin{aligned}
&\int_\varepsilon^T \int_\Omega \Phi f(x)\varepsilon^2 \partial^{-\varepsilon} S^\delta(x,\tau) \partial^{-\varepsilon} V^\delta(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \int_\varepsilon^T \int_{\mathscr{Q}} \frac{\phi}{|\mathscr{B}|} f(x)\varepsilon^2 \partial^{-\varepsilon} s^\delta(x,y,\tau) \partial^{-\varepsilon} v^\delta(x,y,\tau)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_{\Omega^{\mathrm{T}}} \Phi \partial_t S^\delta(x,t)\zeta_1\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathscr{Q}^{\mathrm{T}}} \frac{\phi}{|\mathscr{B}|} \partial_t s^\delta(x,y,t)(\eta + \mathscr{L}\zeta_1)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{\Omega^{\mathrm{T}}} \mathbf{K}\left(\tilde{\Lambda}_w^\delta(S^\delta)\nabla(P^\delta - J_w) + \nabla\mathscr{R}(S^\delta)\right)\nabla\zeta_1 - \int_{\mathscr{Q}^{\mathrm{T}}} \frac{\mathbf{k}}{|\mathscr{B}|}\nabla_y \mathscr{D}(s^\delta)\nabla_y \eta \qquad (4.15)
\end{aligned}$$

By Fubini's theorem and Lemma 4.1, the right-hand side of (4.15) is bounded by $c_1\varepsilon\|f\|_{W^{1,\infty}(\Omega)}$, where c_1 is independent of δ, ε. So we complete the proof of this lemma. □

By Lemma 4.2 and by performing similar argument as Lemmas 3.3 and 3.4, one can obtain the following result:

Corollary 4.1 *There is a subsequence of* V^δ *converging to* V *pointwise and in* $L^2(\Omega^{\mathrm{T}})$ *strongly.*

Lemma 4.3. *Under* **A1–8**, *there exist subsequences of solutions* $S^\delta, U^\delta, P^\delta, s^\delta$ *of* (2.20)–(2.26) (*same indices for subsequences*) *such that, as* $\delta \to 0^+$,

$$\begin{cases} P^\delta \to P & \text{in } L^2(0,T;H^1(\Omega)) \text{ weakly},\\ S^\delta, U^\delta, \mathscr{D}(U^\delta) \to S, U, \mathscr{D}(U) & \text{in } L^r(\Omega^{\mathrm{T}}) \text{ strongly}, \ 1 \leqslant r < \infty,\\ \mathscr{R}(S^\delta), \mathscr{D}(U^\delta) \to \mathscr{R}(S), \mathscr{D}(U) & \text{in } L^2(0,T;H^1(\Omega)) \text{ weakly},\\ s^\delta \to s & \text{in } L^2(\mathscr{Q}^T) \text{ weakly},\\ \Phi\partial_t S^\delta + \int_{\mathscr{B}}(\phi/|\mathscr{B}|)\,\partial_t s^\delta \to \Phi\partial_t S + \int_{\mathscr{B}}(\phi/|\mathscr{B}|)\,\partial_t s & \text{in } L^2(0,T;\mathscr{H}^*) \text{ weakly},\\ \partial_t s^\delta \to \partial_t s & \text{in } L^2(0,T;\mathscr{W}_0^*) \text{ weakly},\\ s^\delta(T) \to s(T) & \text{in } L^2(\mathscr{Q}) \text{ weakly},\\ \mathscr{D}(s^\delta) \to \mathscr{D}(s) & \text{in } L^2(0,T;\mathscr{W}) \text{ weakly} \end{cases}$$

Proof. By Theorem 2.1, Lemma 4.1, Corollary 4.1, (2.6)–(2.7), (2.21), **A7, 8**, and (4.1), we easily obtain subsequences of S^δ, U^δ, P^δ, s^δ such that, as $\delta \to 0^+$,

$$\begin{cases} P^\delta \to P & \text{in } L^2(0,T;H^1(\Omega)) \text{ weakly},\\ S^\delta, U^\delta, \mathscr{D}(U^\delta) \to S, U, \mathscr{D}(U) & \text{in } L^r(\Omega^{\mathrm{T}}) \text{ strongly}, \ 1 \leqslant r < \infty,\\ \mathscr{R}(S^\delta), \mathscr{D}(U^\delta) \to \mathscr{R}(S), \mathscr{D}(U) & \text{in } L^2(0,T;H^1(\Omega)) \text{ weakly},\\ s^\delta \to s & \text{in } L^2(\mathscr{Q}^{\mathrm{T}}) \text{ weakly},\\ s^\delta(T) \to s^* & \text{in } L^2(\mathscr{Q}) \text{ weakly},\\ \mathscr{D}(s^\delta) \to \mathscr{D}^* & \text{in } L^2(0,T;\mathscr{W}) \text{ weakly} \end{cases}$$

To show the following results:

$$\begin{cases} \Phi\partial_t S^\delta + \int_{\mathscr{B}}(\phi/|\mathscr{B}|)\,\partial_t s^\delta \to \Phi\partial_t S + \int_{\mathscr{B}}(\phi/|\mathscr{B}|)\,\partial_t s & \text{in } L^2(0,T;\ \mathscr{H}^*) \text{ weakly},\\ \partial_t s^\delta \to \partial_t s & \text{in } L^2(0,T;\ \mathscr{W}_0^*) \text{ weakly},\\ s^* = s(T),\\ \mathscr{D}^* = \mathscr{D}(s), \end{cases}$$

one can follow the argument in Lemma 3.5. One remark concerning the proof for $\mathscr{D}^* = \mathscr{D}(s)$ is that: By **A8**, one can extend the increasing function $\mathscr{D}$ to $\mathfrak{R}$ continuously and linearly with slope 1. Then, instead of (3.33), we consider

$$0 \leqslant \int_{\mathscr{Q}^{\mathrm{T}}} \frac{\phi}{|\mathscr{B}|}(\mathscr{D}(s^\delta) - \mathscr{D}(f))(s^\delta - f) \quad \text{for } f \in L^2(\mathscr{Q}^{\mathrm{T}})$$

because of the boundedness in s^δ (see Lemma 3.6). Rest of its proof is similar as Lemma 3.5. □

Proof of Theorem 2.2. (2.28)–(2.33) are direct results of Theorem 2.1, Lemma 4.3, and (2.6)–(2.7). For $\zeta \in L^2(0,T;\mathscr{H}) \cap H^1(\Omega^T)$, $\eta \in L^2(0,T;\mathscr{W}_0) \cap H^1(0,T;L^2(\mathscr{Q}))$, $\zeta(T) = \eta(T) = 0$, by Theorem 2.1 and integration by parts,

$$\int_{\Omega^T} \Phi \partial_t S^\delta \zeta + \int_{\mathscr{Q}^T} \frac{\phi}{|\mathscr{B}|} \partial_t s^\delta (\mathscr{L}\zeta + \eta)$$

$$= \int_{\Omega^T} \Phi(\mathbf{P}_c^{-1}(-G_0^\delta) - S^\delta)\partial_t \zeta + \int_{\mathscr{Q}^T} \frac{\phi}{|\mathscr{B}|} (\mathbf{p}_c^{-1}(-G_0^\delta) - s^\delta)\partial_t(\mathscr{L}\zeta + \eta)$$

By (2.6), Lemma 4.3, and Lebesque dominant theorem, we obtain (2.34). So we complete the proof of Theorem 2.2.

5. Uniqueness

We now consider the uniqueness of (2.28)–(2.34). Assume S_i, P_i, s_i, $i = 1, 2$ are two solutions of (2.28)–(2.34), **A1–12** hold (see sections 2.1, 2.4), and ζ_1, ζ_2, η are smooth functions satisfying

$$\zeta_1(T) = \eta(T) = 0, \quad \zeta_1|_{\Upsilon_1^T} = \zeta_2|_{\Upsilon_1^T} = \eta|_{\partial \mathscr{Q}^T} = 0, \quad \nabla\zeta_1 \cdot \vec{n}|_{\Upsilon_2^T} = \nabla\zeta_2 \cdot \vec{n}|_{\Upsilon_2^T} = 0 \tag{5.1}$$

By subtracting one solution from the other, (2.28)–(2.34), and integration by parts, we have

$$-\int_{\Omega^T} \Phi(S_1 - S_2)\partial_t\zeta_1 - \int_{\Omega^T} (\mathscr{R}(S_1) - \mathscr{R}(S_2))\nabla(\mathbf{K}\nabla\zeta_1) - \int_{\Omega^T} (P_1 - P_2)\nabla(\mathbf{K}\Lambda_w(S_1)\nabla\zeta_1)$$

$$+ \int_{\Omega^T} \mathbf{K}(\Lambda_w(S_1) - \Lambda_w(S_2))\nabla(P_2 - J_w)\nabla\zeta_1 - \int_{\mathscr{Q}^T} \frac{\phi}{|\mathscr{B}|} (s_1 - s_2)\mathscr{L}\partial_t\zeta_1 = 0 \tag{5.2}$$

$$-\int_{\Omega^T} (P_1 - P_2)\nabla(\mathbf{K}\Lambda(S_1)\nabla\zeta_2) + \sum_{\alpha\in\{w,o\}} \int_{\Omega^T} \mathbf{K}(\Lambda_\alpha(S_1) - \Lambda_\alpha(S_2))\nabla(P_2 - J_\alpha)\nabla\zeta_2 = 0 \tag{5.3}$$

$$-\int_{\mathscr{Q}^T} \frac{\phi}{|\mathscr{B}|} (s_1 - s_2)\partial_t\eta - \int_{\mathscr{Q}^T} \frac{\mathbf{k}}{|\mathscr{B}|} (\mathscr{D}(s_1) - \mathscr{D}(s_2))\Delta_y\eta$$

$$+ \int_{\Omega^T} \frac{\mathbf{k}}{|\mathscr{B}|} (\mathscr{D}(\widehat{\mathbf{P}}_c(S_1)) - \mathscr{D}(\widehat{\mathbf{P}}_c(S_2))) \int_{\mathscr{B}} \Delta_y\eta = 0 \tag{5.4}$$

By (5.2)–(5.4) and **A9**, we obtain

$$\int_{\Omega^T} (S_1 - S_2)\left(-\Phi\partial_t\zeta_1 - \mathscr{F}_1^v\nabla(\mathbf{K}\nabla\zeta_1) + \mathscr{F}_2\nabla\zeta_1 + \mathscr{F}_3\nabla\zeta_2 + \frac{\mathbf{k}}{|\mathscr{B}|}\mathscr{F}_1^v \int_{\mathscr{B}} \Delta_y\eta\right)$$

$$-\int_{\Omega^T} (P_1 - P_2)\nabla(\mathbf{K}\Lambda(S_1)\nabla\zeta_2 + \mathbf{K}\Lambda_w(S_1)\nabla\zeta_1)$$

$$+\int_{\mathscr{Q}^T} (s_1 - s_2)\left(-\frac{\phi}{|\mathscr{B}|}\partial_t(\eta + \mathscr{L}\zeta_1) - \mathscr{F}_4^v \frac{\mathbf{k}}{|\mathscr{B}|}\Delta_y\eta\right)$$

$$= -\int_{\Omega^T} (S_1 - S_2)v\left(\nabla(\mathbf{K}\nabla\zeta_1) - \frac{\mathbf{k}}{|\mathscr{B}|}\int_{\mathscr{B}} \Delta_y\eta\right) - \int_{\mathscr{Q}^T} (s_1 - s_2)v\frac{\mathbf{k}}{|\mathscr{B}|}\Delta_y\eta \tag{5.5}$$

where $\mathscr{F}_1^v = \mathscr{F}_1 + v$, $\mathscr{F}_4^v = \mathscr{F}_4 + v$, and

$$\mathscr{F}_1 := \begin{cases} (\mathscr{R}(S_1) - \mathscr{R}(S_2))/(S_1 - S_2) & \text{if } S_1 \neq S_2 \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

$$\mathscr{F}_2 := \begin{cases} (\mathbf{K}(\Lambda_w(S_1) - \Lambda_w(S_2))\nabla(P_2 - J_w))/S_1 - S_2 & \text{if } S_1 \neq S_2 \\ 0 & \text{otherwise} \end{cases} \tag{5.7}$$

$$\mathscr{F}_3 := \begin{cases} (\sum_{\alpha \in \{w,o\}} \mathbf{K}(\Lambda_\alpha(S_1) - \Lambda_\alpha(S_2))\nabla(P_2 - J_\alpha))/S_1 - S_2 & \text{if } S_1 \neq S_2, \\ 0 & \text{otherwise} \end{cases} \tag{5.8}$$

$$\mathscr{F}_4 := \begin{cases} (\mathscr{D}(s_1) - \mathscr{D}(s_2)(/(s_1 - s_2) & \text{if } s_1 \neq s_2 \\ 0 & \text{otherwise} \end{cases} \tag{5.9}$$

Define

$$\mathscr{H}_1 := \{\zeta : \zeta \in H^1(\Omega^{\mathrm{T}}) \cap L^2(0, T; H^2(\Omega)), \zeta\,(0) = 0, \zeta\,|_{\Upsilon_1^{\mathrm{T}}} = \nabla\zeta \cdot \vec{n}|_{\Upsilon_2^{\mathrm{T}}} = 0\}$$

$$\tilde{\mathscr{H}}_1 := \{\zeta : \zeta \in H^1(\Omega^{\mathrm{T}}) \cap L^\infty(0, T; H^1(\Omega)), \zeta(0) = 0, \zeta|_{\Upsilon_1^{\mathrm{T}}} = 0\}$$

$$\mathscr{H}_2 := \{\zeta : \zeta \in L^2(0, T; H^2(\Omega)), \zeta|_{\Upsilon_1^{\mathrm{T}}} = \nabla\zeta \cdot \vec{n}|_{\Upsilon_2^{\mathrm{T}}} = 0\}$$

$$\mathscr{H}_3 := \{\eta : \eta \in H^1(0, T; L^2(\mathscr{Q})) \cap L^2(0, T; L^2(\Omega, H^2(\mathscr{B}))), \eta(0) = 0, \eta|_{\partial\mathscr{Q}^{\mathrm{T}}} = 0\}$$

Next, we consider the following auxiliary problem for fixed v:

Lemma 5.1. *Assume* $\mathscr{F}_2, \mathscr{F}_3, \Lambda_w \in L^\infty(\Omega^{\mathrm{T}})$; $0 < d_1 < \Phi, \mathbf{K}, \phi, \mathbf{k}, \Lambda, \mathscr{F}_1^v, \mathscr{F}_4^v < d_2 < \infty$; $\partial\Omega \in \mathbf{H}_*^3$; *and* $\mathscr{B}$ *is smooth. For any* $(f_1, f_2, f_3) \in L^2(\Omega^{\mathrm{T}}) \times L^2(\Omega^{\mathrm{T}}) \times L^2(\mathscr{Q}^{\mathrm{T}})$, *there is a unique* $(\zeta_1, \zeta_2, \eta) \in \tilde{\mathscr{H}}_1 \times L^2(0, T; H^1(\Omega)) \times \mathscr{H}_3$ *such that*

$$\Phi\partial_t\zeta_1 - \mathscr{F}_1^v\nabla(\mathbf{K}\nabla\zeta_1) + \mathscr{F}_2\nabla\zeta_1 + \mathscr{F}_3\nabla\zeta_2 + \frac{\mathbf{k}}{|\mathscr{B}|}\mathscr{F}_1^v\int_{\mathscr{B}}\Delta_y\eta = f_1 \tag{5.10}$$

$$-\nabla(\mathbf{K}\Lambda\nabla\zeta_2 + \mathbf{K}\Lambda_w\nabla\zeta_1) = f_2 \tag{5.11}$$

$$\frac{\phi}{|\mathscr{B}|}\partial_t(\eta + \mathscr{L}\zeta_1) - \mathscr{F}_4^v\frac{\mathbf{k}}{|\mathscr{B}|}\Delta_y\eta = f_3 \tag{5.12}$$

Moreover,

$$\sup_{\tau \leqslant T}\left(\int_\Omega |\nabla\zeta_1|^2 + \int_{\mathscr{Q}} |\nabla_y\eta|^2\right)(\tau) + \int_{\Omega^{\mathrm{T}}} |\nabla\zeta_2|^2 + \int_{\Omega^{\mathrm{T}}} \frac{|\partial_t\zeta_1|^2}{d_2}$$

$$+ \int_{\mathscr{Q}^{\mathrm{T}}} d_1|\Delta_y\eta|^2 + \int_{\Omega^{\mathrm{T}}} d_1|\nabla(\mathbf{K}\nabla\zeta_1)|^2 + \int_{\mathscr{Q}^{\mathrm{T}}} |\partial_t\eta|^2$$

$$\leqslant c\left(d_2, \left\|\Lambda_w, \frac{|\mathscr{F}_2|^2}{\mathscr{F}_1^v}, \frac{|\mathscr{F}_3|^2}{\mathscr{F}_1^v}\right\|_{L^\infty(\Omega^{\mathrm{T}})}\right)\left(\int_{\Omega^{\mathrm{T}}} \frac{|f_1|^2}{\mathscr{F}_1^v} + \int_{\Omega^{\mathrm{T}}} |f_2|^2 + \int_{\mathscr{Q}^{\mathrm{T}}} \frac{|f_3|^2}{\mathscr{F}_4^v}\right) \tag{5.13}$$

Proof. First let us consider smooth coefficients case. That is, we assume Φ, $\mathbf{K}$, ϕ, $\mathbf{k}$, Λ_w, Λ, $\mathscr{F}_1^v$, $\mathscr{F}_2$, $\mathscr{F}_3$, $\mathscr{F}_4^v$ are all smooth, and Φ, $\mathbf{K}$, ϕ, $\mathbf{k}$, Λ, $\mathscr{F}_1^v$, $\mathscr{F}_4^v$ are bounded above and below by d_2 and d_1 as non-smooth coefficients case. For $\sigma \in [0, 1]$, define a map $\mathscr{G}^\sigma : \mathscr{H}_1 \times \mathscr{H}_2 \times \mathscr{H}_3 \to L^2(\Omega^T) \times L^2(\Omega^T) \times L^2(\mathscr{Q}^T)$ by $\mathscr{G}^\sigma(\zeta_1, \zeta_2, \eta) = (f_1, f_2, f_3)$, where

$$f_1 = \Phi\partial_t\zeta_1 - \mathscr{F}_1^v\nabla(\mathbf{K}\nabla\zeta_1) + \mathscr{F}_2\nabla\zeta_1 + \sigma\mathscr{F}_3\nabla\zeta_2 + \sigma\frac{\mathbf{k}}{|\mathscr{B}|}\mathscr{F}_1^v\int_{\mathscr{B}}\Delta_y\eta \tag{5.14}$$

$$f_2 = -\nabla(\mathbf{K}\Lambda\nabla\zeta_2 + \sigma\mathbf{K}\Lambda_w\nabla\zeta_1) \tag{5.15}$$

$$f_3 = \sigma\frac{\phi}{|\mathscr{B}|}\mathscr{L}\partial_t\zeta_1 + \frac{\phi}{|\mathscr{B}|}\partial_t\eta - \mathscr{F}_4^v\frac{\mathbf{k}}{|\mathscr{B}|}\Delta_y\eta \tag{5.16}$$

It is easy to check $\mathscr{G}^\sigma$ is a bounded linear function and $\mathscr{G}^0$ (i.e. $\sigma = 0$) is one-to-one and onto (see sections 4,5 of Chapter 5 of [4]).

Multiply (5.14) by $(1/\mathscr{F}_1^v)\partial_t\zeta_1$, (5.15) by $\beta\zeta_2$ where $\beta(>1)$ is a constant depending on $\| |\mathscr{F}_3|^2/\mathscr{F}_1^v\Phi\|_{L^\infty(\Omega^T)}$, and (5.16) by $-(\mathbf{k}/\phi)\Delta_y\eta$, then integrate (5.14)–(5.15) over Ω^τ and (5.16) over $\mathscr{Q}^\tau$, then by integration by parts along with boundary and initial conditions to obtain

$$\begin{aligned}&\int_{\Omega^\tau}\frac{\Phi}{\mathscr{F}_1^v}|\partial_t\zeta_1|^2 + \int_{\Omega}\mathbf{K}\frac{|\nabla\zeta_1|^2}{2}(\tau) + \int_{\Omega^\tau}\left(\frac{\mathscr{F}_2}{\mathscr{F}_1^v}\nabla\zeta_1 + \sigma\frac{\mathscr{F}_3}{\mathscr{F}_1^v}\nabla\zeta_2\right)\partial_t\zeta_1\\ &\quad + \sigma\int_{Q^\tau}\frac{\mathbf{k}}{|\mathscr{B}|}\Delta_y\eta\mathscr{L}\partial_t\zeta_1 = \int_{\Omega^\tau}\frac{f_1}{\mathscr{F}_1^v}\partial_t\zeta_1\end{aligned} \tag{5.17}$$

$$\beta\int_{\Omega^\tau}\mathbf{K}\Lambda|\nabla\zeta_2|^2 + \sigma\beta\int_{\Omega^\tau}\mathbf{K}\Lambda_w\nabla\zeta_1\nabla\zeta_2 = \beta\int_{\Omega^\tau}f_2\zeta_2 \tag{5.18}$$

$$-\sigma\int_{Q^\tau}\frac{\mathbf{k}}{|\mathscr{B}|}\Delta_y\eta\mathscr{L}\partial_t\zeta_1 + \int_{\mathscr{Q}}\frac{\mathbf{k}}{|\mathscr{B}|}\frac{|\nabla_y\eta|^2}{2}(\tau) + \int_{Q^\tau}\frac{\mathbf{k}^2\mathscr{F}_4^v}{\phi|\mathscr{B}|}|\Delta_y\eta|^2 = -\int_{Q^\tau}\frac{\mathbf{k}}{\phi}f_3\Delta_y\eta \tag{5.19}$$

Summing (5.17)–(5.19), we have

$$\begin{aligned}&\int_{\Omega}|\nabla\zeta_1|^2(\tau) + \int_{\mathscr{Q}}|\nabla_y\eta|^2(\tau) + \beta\int_{\Omega^\tau}|\nabla\zeta_2|^2 + \int_{\Omega^\tau}\frac{1}{\mathscr{F}_1^v}|\partial_t\zeta_1|^2 + \int_{Q^\tau}\mathscr{F}_4^v|\Delta_y\eta|^2\\ &\quad\leqslant c_1\left(\left\|\Lambda_w, \frac{|\mathscr{F}_2|^2}{\mathscr{F}_1^v}, \frac{|\mathscr{F}_3|^2}{\mathscr{F}_1^v}\right\|_{L^\infty(\Omega^T)}\right)\left(\int_{\Omega^\tau}|\nabla\zeta_1|^2 + \int_{\Omega^\tau}\frac{|f_1|^2}{\mathscr{F}_1^v} + \int_{\Omega^\tau}|f_2|^2 + \int_{Q^\tau}\frac{|f_3|^2}{\mathscr{F}_4^v}\right)\end{aligned} \tag{5.20}$$

where c_1 is a constant depending on its parameters. By Gronwall's inequality, (5.20) implies

$$\sup_{\tau \leqslant T}\left(\int_{\Omega}|\nabla\zeta_1|^2 + \int_{\mathscr{Q}}|\nabla_y \eta|^2\right)(\tau) + \beta\int_{\Omega^T}|\nabla\zeta_2|^2 + \int_{\Omega^T}\frac{1}{\mathscr{F}_1^v}|\partial_t\zeta_1|^2 + \int_{\mathscr{Q}^T}\mathscr{F}_4^v|\Delta_y\eta|^2$$
$$\leqslant c_2\left(\|\Lambda_w, \frac{|\mathscr{F}_2|^2}{\mathscr{F}_1^v}, \frac{|\mathscr{F}_3|^2}{\mathscr{F}_1^v}\|_{L^\infty(\Omega^T)}\right)\left(\int_{\Omega^T}\frac{|f_1|^2}{\mathscr{F}_1^v} + \int_{\Omega^T}|f_2|^2 + \int_{\mathscr{Q}^T}\frac{|f_3|^2}{\mathscr{F}_4^v}\right) \quad (5.21)$$

where c_2 is a constant depending on its parameter. (5.14)–(5.16) and (5.21) imply that

$$\int_{\Omega^T} d_1|\nabla(\mathbf{K}\nabla\zeta_1)|^2 + \int_{\mathscr{Q}^T}|\partial_t\eta|^2$$
$$\leqslant c_3\left(d_2, \|\Lambda_w, \frac{|\mathscr{F}_2|^2}{\mathscr{F}_1^v}, \frac{|\mathscr{F}_3|^2}{\mathscr{F}_1^v}\|_{L^\infty(\Omega^T)}\right)\left(\int_{\Omega^T}\frac{|f_1|^2}{\mathscr{F}_1^v} + \int_{\Omega^T}|f_2|^2 + \int_{\mathscr{Q}^T}\frac{|f_3|^2}{\mathscr{F}_4^v}\right) \quad (5.22)$$

$$\|\zeta_1\|_{\mathscr{H}_1} + \|\zeta_2\|_{\mathscr{H}_2} + \|\eta\|_{\mathscr{H}_3} \leqslant c_4(\|f_1\|_{L^2(\Omega^T)} + \|f_2\|_{L^2(\Omega^T)} + \|f_3\|_{L^2(Q^T)}) \quad (5.23)$$

where constant c_4 depends on smooth coefficients of (5.10)–(5.12). By (5.23) and method of continuity [14], we see $\mathscr{G}^1$ is also a one-to-one and onto map. So we show the unique solvability of (5.10)–(5.12) for smooth coefficients case. By uniform bound (5.21)–(5.22) and passing to limit, one can find a unique solution of (5.10)–(5.12) for non-smooth coefficient case. Moreover, the solution satisfies (5.13). So we complete the proof of this lemma. □

Proof of Theorem 2.3. Let $f_1 = \mathscr{R}(S_1) - \mathscr{R}(S_2), f_2 = P_1 - P_2, f_3 = \mathscr{D}(s_1) - \mathscr{D}(s_2)$ in (5.10)–(5.12), then we obtain the corresponding solution $(\zeta_1^v, \zeta_2^v, \eta^v)$ for each v by (5.6)–(5.9), $P_2 \in L^\infty(0, T; W^{1,\infty}(\Omega))$, **A10–12**, and Lemma 5.1. After substitution $t \to T - t$ for the solution $(\zeta_1^v, \zeta_2^v, \eta^v)$, then we plug it into (5.5) to obtain

$$\int_{\Omega^T}(S_1 - S_2)(\mathscr{R}(S_1) - \mathscr{R}(S_2)) + \int_{\Omega^T}|P_1 - P_2|^2 + \int_{\mathscr{Q}^T}(s_1 - s_2)(\mathscr{D}(s_1) - \mathscr{D}(s_2))$$
$$= -\int_{\Omega^T}(S_1 - S_2)v\left(\nabla(\mathbf{K}\nabla\zeta_1^v) - \frac{\mathbf{k}}{|\mathscr{B}|}\int_{\mathscr{B}}\Delta_y\eta^v\right) - \int_{\mathscr{Q}^T}(s_1 - s_2)\,v\frac{\mathbf{k}}{|\mathscr{B}|}\Delta_y\eta^v \quad (5.24)$$

By Lemma 5.1, the right-hand side of (5.24) is bounded by $c\sqrt{v}$, where c is a constant independent of v. Letting $v \to 0^+$, the right-hand side of (5.24) goes to 0, which implies the uniqueness of (2.28)–(2.34).

Acknowledgement

The author would like to thank the anonymous referee's suggestions for improving the presentation of this paper.

References

1. Adams RA. *Sobolev Spaces*. Academic Press: New York, NY, 1975.
2. Alt HW, Luckhaus S. Quasilinear elliptic-parabolic differential equations. *Math. Z.* 1983; **183**: 311–341.

3. Alt HW, DiBenedetto E. Nonsteady flow of water and oil through inhomogeneous porous media. *Annali Scu. Norm. Sup. Pisa Cl. Sci.* 1985; **12**(4): 335–392.
4. Antontsev SN, Kazhikhov AV, Monakhov VN. *Boundary Value Problems in Mechanics in Non-homogeneous Fluids*. Elsevier: Amsterdam, 1990.
5. Arbogast T, Douglas J, Paes Leme PJ. Two models for the waterflooding of naturally fractured reservoirs. In *Proc.* 10*th SPE Symp. on Reservoir Simulation*, SPE 18425. Society of Petroleum Engineers, Dallas, TX, 1989.
6. Arbogast T. The existence of weak solutions to single porosity and simple dual-porosity models of two-phase incompressible flow. *Nonlinear Anal.* 1992; **19**(11): 1009–1031.
7. Bourgeat A, Luckhaus S, Mikelic A. Convergence of the homogenization process for a double-porosity model of immiscible two-phase flow. *SIAM J. Math. Anal.* 1996; **27**(6): 1520–1543.
8. Chavent G, Jaffre J. *Mathematical Models and Finite Elements for Reservoir Simulation*. North-Holland: Amsterdam, 1986.
9. Chen Z. Analysis of large-scale averaged models for two-phase flow in fractured reservoirs. *J. Math. Anal. Appl.* 1998; **223**(1): 158–181.
10. Jim Jr. Douglas, Paes-Leme PJ. Finite difference methods for a model for immiscible displacement in naturally fractured petroleum reservoirs. *Mat.Apl. Comput*. V. 1992; **11**(1): 3–16.
11. Douglas Jr. J, Hensley JL, Arbogast T. A dual-porosity model for waterflooding in naturally fractured reservoirs. *Comput. Methods Appl. Mech. Engng.* 1991; **87**: 157–174.
12. Douglas J, Pereira F, Yeh LM. A parallelizable characteristic scheme for two phase immiscible flow in porous media I: single porosity models. *Comput. Appl. Math.* 1995; **14**: 73–96.
13. Douglas J, Pereira F, Yeh LM. A parallel method for two-phase flows in naturally fractured reservoirs. *Comput. Geosci.* 1997; **3**(4): 333–368.
14. Gilbarg D, Trudinger NS. *Elliptic Partial Differential Equations of Second Order*. (2nd edn). Springer: Berlin, 1983.
15. Kroener D, Luckhaus S. Flow of oil and water in a porous medium. *J. Differential Equations* 1984; **55**: 276–288.
16. Lions JL. Quelques méthodes de résolution des problems aux limites non linéaries. Dund: Paris, 1969.
17. Royden HL. *Real Analysis* (2nd edn). Macmillan: London, 1970.

