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A direct theory for the perturbed unstable nonlinear Schrödinger equation

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A direct perturbation theory for the unstable nonlinear Schrödinger equation with perturbations is developed. The linearized operator is derived and the squared Jost functions are shown to be its eigenfunctions. Then the equation of linearized operator is transformed into an equivalent 4×4 matrix form with first order derivative in t and the eigenfunctions into a four-component row. Adjoint functions and the inner product are defined. Orthogonality relations of these functions are derived and the expansion of the unity in terms of the four-component eigenfunctions is implied. The effect of damping is discussed as an example. © *2000 American Institute of Physics.* [S0022-2488(00)00405-9]

I. INTRODUCTION

The unstable nonlinear Schrödinger (UNLS, for short) equation was introduced in plasma physics[1,2] to describe the nonlinear modulation of a high frequency mode in electron beam plasma such as a system where an electron beam is injected under high frequency electric field. This equation may be considered as a prototype amplitude equation for the soliton phenomena in an unstable system. It also describes the nonlinear modulation of waves in Rayleigh-Taylor problem.[3] The UNLS equation can be expressed as

$$iu_x+u_{tt}+2|u|^2u=0, \tag{1}$$

where t and x are time and space coordinate.

Interchange of x and t in (1) leads to the conventional stable nonlinear Schrödinger (SNLS, for short) equation. Since the SNLS equation has been proved to be a completely integrable system,[4,5] it has been solved by using the inverse scattering transform (IST). Soliton solutions for the UNLS equation can be obtained by simply interchanging x and t from the soliton solutions for the SNLS equation. The UNLS equation has also been generally solved, by a similar IST, including the contribution of continuous spectrum of the spectral parameter,[2,6] which is necessary in developing a perturbation theory for the UNLS equation with perturbations. To have some insight into the physical significance of the UNLS equation and to have an effective method to study practical problems, it is necessary to consider the UNLS equation with perturbations,[7]

$$iv_x+v_{tt}+2|v|^2v=\epsilon r[v], \tag{2}$$

where ϵ is a small positive parameter and $r[v]$ is a functional of v. Since (2) has a second order derivative in t, the initial conditions must include one about $v_t(x,0)$ in addition to the one about $v(x,0)$. We choose

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$$v(x,0)=u(x,0), \quad v_t(x,0)=u_t(x,0). \tag{3}$$

However, (1) and (2) are second order partial differential equations in time. The initial value problem under the condition (3), which is very different from that for the SNLS equation,[7] has never appeared in the literature. The purpose of this work is to find the perturbed solution of (2) under the initial condition (3). This work is arranged as follows:

(1) The linearized equation for (2) is derived, and the squared Jost functions are shown to be solutions of this linearized equation by means of the Lax equations.
(2) A 4×4-matrix form of the linearized equation which has only a first order derivative in t is introduced to replace its original 2×2-matrix form with a second derivative in t;
(3) The two-component squared Jost functions are transformed into four-component ones. The four-component adjoint functions and the inner product are introduced. The orthogonality relations are then derived.
(4) The expansion of the unity in terms of the four-component squared Jost functions is implied.
(5) The secularity conditions are found and the adiabatic solution can be determined with them.
(6) Finally, the effect of damping is discussed as an example.

A brief review of the inverse scattering transform for (1) is given in the Appendix.[2,6]

II. THE LINEARIZED EQUATION

Suppose[8–11]

$$v=u^a+\epsilon q, \tag{4}$$

where u^a is the so-called adiabatic solution which has the same functional form as that of the exact soliton solution but the parameters involved may depend on t of the order of ϵ, which will be discussed in detail later. Here ϵq is the remaining term up to the order of ϵ. Substitution of (4) into (2) yields

$$iq_x+q_{tt}+4|u|^2q+2u^2\overline{q}=R[u], \tag{5}$$

$$R[u]=r[u]-s[u], \quad s[u]=\frac{1}{\epsilon}\{iu_x+u_{tt}+2|u|^2u\}. \tag{6}$$

Equation (5) is an equation up to the order of ϵ, u in the left hand side and in $r[u]$ is the exact solution, and u in $s[u]$ is the adiabatic solution. Here the bars denote complex conjugates.

Equation (5) and its complex conjugate can be combined as

$$\begin{pmatrix} i\partial_x+\partial_t^2+4|u|^2 & -2u^2 \\ -2\overline{u}^2 & -i\partial_x+\partial_t^2+4|u|^2 \end{pmatrix}\begin{pmatrix} q \\ -\overline{q} \end{pmatrix}=\begin{pmatrix} R \\ -\overline{R} \end{pmatrix}. \tag{7}$$

The initial condition (3) turns to

$$q(x,t=0)=0, \quad q_t(x,t=0)=0. \tag{8}$$

To find the perturbed solution of (2) under the initial condition (3) is equivalent to solving (7) under the initial condition (8).

In order to solve (7), we need to find a complete set of solutions for its homogeneous version, i.e., (7) with a vanishing right hand side. From (A2) and (A3), the Lax equations of (1), we obtain[8–11]

$$\begin{pmatrix} i\partial_x+\partial_t^2+4|u|^2 & -2u^2 \\ -2\overline{u}^2 & -i\partial_x+\partial_t^2+4|u|^2 \end{pmatrix} W=\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9}$$

Here

$$W=\begin{pmatrix} w_1^2 \\ w_2^2 \end{pmatrix}, \tag{10}$$

in which w_1 and w_2 are components of a solution of the Lax equations, w, which can be chosen as those Jost functions, $h(t,\lambda)^{-1}\psi(x,\lambda)$, $h(t,\lambda)\tilde{\psi}(x,\lambda)$, $h(t,\lambda)\phi(x,\lambda)$, or $h(t,\lambda)^{-1}\tilde{\phi}(x,\lambda)$ (see the Appendix). That is, like the case of the SNLS equation,[8] solutions of the homogeneous version of (7) can be constructed with those so-called squared Jost functions W. We denote $W=w\circ w$.

III. A TRICK TO TREAT THE SECOND ORDER DERIVATIVE IN *T*

If one can find a complete set of the squared Jost functions, solutions of (7) can be expanded in the complete set. However, owing to the fact that (7) and (9) have second derivatives in t, like the case of sine-Gordon equation,[10] it is more convenient to transform them into an equation having only a first derivative in t. Thus, equivalently, we rewrite (7) as

$$\begin{pmatrix} -i\partial_t & 0 & 1 & 0 \\ 0 & -i\partial_t & 0 & 1 \\ i\partial_x+4|u|^2 & -2u^2 & -i\partial_t & 0 \\ -2\overline{u}^2 & -i\partial_x+4|u|^2 & 0 & -i\partial_t \end{pmatrix}\begin{pmatrix} q \\ -\overline{q} \\ iq_t \\ -i\overline{q_t} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ R \\ -\overline{R} \end{pmatrix}. \tag{11}$$

Similarly, (9) is transformed to

$$\begin{pmatrix} -i\partial_t & 0 & 1 & 0 \\ 0 & -i\partial_t & 0 & 1 \\ i\partial_x+4|u|^2 & -2u^2 & -i\partial_t & 0 \\ -2\overline{u}^2 & -i\partial_x+4|u|^2 & 0 & -i\partial_t \end{pmatrix}\begin{pmatrix} W_1 \\ W_2 \\ iW_{1t} \\ iW_{2t} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{12}$$

Introducing $\Psi(x,\lambda)=\psi(x,\lambda)\circ\psi(x,\lambda)$, $\tilde{\Psi}(x,\lambda)=\tilde{\psi}(x,\lambda)\circ\tilde{\psi}(x,\lambda)$, $\Phi(x,\lambda)=\phi(x,\lambda)\circ\phi(x,\lambda)$ and $\tilde{\Phi}(x,\lambda)=\tilde{\phi}(x,\lambda)\circ\tilde{\phi}(x,\lambda)$, taking $W=h(t,\lambda)^{-2}\Psi(x,\lambda)$, for example, (12) becomes

$$\begin{pmatrix} -i\partial_t & 0 & 1 & 0 \\ 0 & -i\partial_t & 0 & 1 \\ i\partial_x+4|u|^2 & -2u^2 & -i\partial_t & 0 \\ -2\overline{u}^2 & -i\partial_x+4|u|^2 & 0 & -i\partial_t \end{pmatrix}\begin{pmatrix} \Psi(x,\lambda)_1 \\ \Psi(x,\lambda)_2 \\ \Psi(x,\lambda)_3 \\ \Psi(x,\lambda)_4 \end{pmatrix}=2\lambda\begin{pmatrix} \Psi(x,\lambda)_1 \\ \Psi(x,\lambda)_2 \\ \Psi(x,\lambda)_3 \\ \Psi(x,\lambda)_4 \end{pmatrix}, \tag{13}$$

where

$$\Psi(x,\lambda)=(\Psi(x,\lambda)_1\,\Psi(x,\lambda)_2\,\Psi(x,\lambda)_3\,\Psi(x,\lambda)_4)^T \tag{14}$$

is a four-component squared Jost function with the additional third and fourth components:

$$\begin{aligned} \Psi(x,\lambda)_3&=i2(i\lambda\psi(x,\lambda)_1-u\psi(x,\lambda)_2)\psi(x,\lambda)_1, \\ \Psi(x,\lambda)_4&=i2(-i\lambda\psi(x,\lambda)_2+\overline{u}\psi(x,\lambda)_1)\psi(x,\lambda)_2. \end{aligned} \tag{15}$$

Set

$$-\mathbf{L}(u)=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ i\partial_x+4|u|^2 & -2u^2 & 0 & 0 \\ -2\overline{u}^2 & -i\partial_x+4|u|^2 & 0 & 0 \end{pmatrix}. \tag{16}$$

Equation (15) becomes

$$\{-i\partial_t-\mathbf{L}(u)\}\mathbf{\Psi}(x,\lambda)=2\lambda\mathbf{\Psi}(x,\lambda). \tag{17}$$

It is obvious that at λ_n, one of the zeros of $a(\lambda)$,

$$\{-i\partial_t-\mathbf{L}(u)\}\mathbf{\Psi}(x,\lambda_n)=2\lambda_n\mathbf{\Psi}(x,\lambda_n), \tag{18}$$

and

$$\{-i\partial_t-\mathbf{L}(u)\}\dot{\mathbf{\Psi}}(x,\lambda_n)=2\lambda_n\dot{\mathbf{\Psi}}(x,\lambda_n)+2\mathbf{\Psi}(x,\lambda_n), \tag{19}$$

where $\dot{\mathbf{\Psi}}(x,\lambda_n)=(d/d\lambda)\,\mathbf{\Psi}(x,\lambda)|_{\lambda=\lambda_n}$.

Similarly, we have equations for other four-component squared Jost functions, $\mathbf{\Phi}(x,\lambda)$, $\tilde{\mathbf{\Psi}}(x,\lambda)$ and $\tilde{\mathbf{\Phi}}(x,\lambda)$, similar to (17)–(19).

Introducing

$$\mathbf{q}=(q\;-\overline{q}\;iq_t\;-i\overline{q_t})^T, \quad \mathbf{R}=(0\;\;0\;\;R\quad-\overline{R})^T, \tag{20}$$

(11) can be rewritten as

$$\{-i\partial_t-\mathbf{L}(u)\}\mathbf{q}=\mathbf{R}. \tag{21}$$

IV. ADJOINT FUNCTIONS AND INNER PRODUCTS

We now introduce adjoint functions and inner products. The essential point is that the inner product of a squared Jost function with its adjoint function is proportional to the $\delta(\lambda-\lambda')$ function in the continuous spectrum.[8–11] Definition of the inner product is given by

$$\langle\mathbf{\Psi}(\lambda')|\mathbf{\Psi}(\lambda)\rangle=\int_{-\infty}^{\infty}dx\,\mathbf{\Psi}(x,\lambda')^A\mathbf{\Psi}(x,\lambda). \tag{22}$$

We choose the adjoint function to be

$$\mathbf{\Psi}(x,\lambda)^A=(-\mathbf{\Phi}(x,\lambda)_4\;-\mathbf{\Phi}(x,\lambda)_3\;\mathbf{\Phi}(x,\lambda)_2\;\mathbf{\Phi}(x,\lambda)_1), \tag{23}$$

where $\mathbf{\Phi}_3$ and $\mathbf{\Phi}_4$ are as in (15), replacing components of ψ with those of ϕ.

From the Lax equation (A2) we obtain

$$\frac{d}{dx}\mathrm{W}[\varphi(x,\lambda'),\psi(x,\lambda)]=-i2(\lambda'^2-\lambda^2)(\varphi(x,\lambda')_1\psi(x,\lambda)_2+\varphi(x,\lambda')_2\psi(x,\lambda)_1)+2(\lambda'-\lambda)$$
$$\times(u\varphi(x,\lambda')_2\psi(x,\lambda)_2+\overline{u}\varphi(x,\lambda')_1\psi(x,\lambda)_1). \tag{24}$$

where $\mathrm{W}[\cdots]$ is the Wronskian determinant.[2,5] From (14) and (23) we have

$$\mathbf{\Psi}(x,\lambda')^A\mathbf{\Psi}(x,\lambda)=[2(\lambda'+\lambda)(\varphi_1(x,\lambda')\psi_2(x,\lambda)+\varphi_2(x,\lambda')\psi_1(x,\lambda))+i2u\varphi_2(x,\lambda')\psi_2(x,\lambda)$$
$$+i2\overline{u}\varphi_1(x,\lambda')\psi_1(x,\lambda)][\varphi_1(x,\lambda')\psi_2(x,\lambda)-\varphi_2(x,\lambda')\psi_1(x,\lambda)]. \tag{25}$$

Hence we find

$$\frac{d}{dx}\{W[\varphi(x,\lambda'),\psi(x,\lambda)]\}^2=-i2(\lambda'-\lambda)\boldsymbol{\Psi}(x,\lambda')^A\boldsymbol{\Psi}(x,\lambda). \tag{26}$$

Therefore, the inner product is

$$\langle\boldsymbol{\Psi}(\lambda')|\boldsymbol{\Psi}(\lambda)\rangle=\lim_{L\to\infty}\frac{1}{-i2(\lambda'-\lambda)}\{W[\varphi(x,\lambda'),\psi(x,\lambda)]\}^2\Big|_{x=-L}^{x=L}, \tag{27}$$

where λ and λ' should be considered as those approaching the real or the imaginary axis from the first or the third quadrant. The limit is considered as the Cauchy principal value

$$\lim_{L\to\infty}P\,\frac{1}{-i2(\lambda'-\lambda)}e^{-i4(\lambda'^2-\lambda^2)L}=\pi\lambda\,\delta(\lambda^2-\lambda'^2). \tag{28}$$

Hence the values of (27) at the upper and at the lower limits can be found. We thus find

$$\langle\boldsymbol{\Psi}(\lambda')|\boldsymbol{\Psi}(\lambda)\rangle=\pi a(\lambda)^2 2\lambda\,\delta(\lambda^2-\lambda'^2). \tag{29}$$

It is obvious that

$$\langle\boldsymbol{\Psi}(\lambda_m)|\boldsymbol{\Psi}(\lambda_n)\rangle=0. \tag{30}$$

Applying the operator $d^2/d\lambda^2$ to (26), and setting $\lambda=\lambda'=\lambda_n$, we obtain

$$\frac{d^2}{d\lambda^2}\frac{d}{dx}\{W[\varphi(x,\lambda'),\psi(x,\lambda)]\}^2_{\lambda=\lambda'=\lambda_n}\Big|=i4\boldsymbol{\Psi}(x,\lambda_n)^A\dot{\boldsymbol{\Psi}}(x,\lambda_n). \tag{31}$$

Integration leads to

$$\langle\boldsymbol{\Psi}(\lambda_m)|\dot{\boldsymbol{\Psi}}(\lambda_n)\rangle=i\,\tfrac{1}{2}\dot{a}(\lambda_n)^2\delta_{mn}. \tag{32}$$

Applying the operator $\{d^3/d\lambda^3+3d/d\lambda'\,d^2/d\lambda^2\}$ to (27), setting $\lambda=\lambda'=\lambda_n$, upon integration we have

$$\left\{\frac{d^3}{d\lambda^3}+3\frac{d}{d\lambda'}\frac{d^2}{d\lambda^2}\right\}\{W[\varphi(x,\lambda'),\psi(x,\lambda)]\}^2\big|_{\lambda=\lambda'=\lambda_n}\big|_{-L}^{L}=i12\langle\dot{\boldsymbol{\Psi}}(\lambda_n)|\dot{\boldsymbol{\Psi}}(\lambda_n)\rangle. \tag{33}$$

Finally we obtain

$$\langle\dot{\boldsymbol{\Psi}}(\lambda_m)|\dot{\boldsymbol{\Psi}}(\lambda_n)\rangle=i\,\tfrac{1}{2}\dot{a}(\lambda_n)\ddot{a}(\lambda_n)\delta_{mn}. \tag{34}$$

Having defined $\tilde{\boldsymbol{\Psi}}(x,\lambda)$'s adjoint function in a similar way, we also have

$$\langle\tilde{\boldsymbol{\Psi}}(\lambda')|\tilde{\boldsymbol{\Psi}}(\lambda)\rangle=-\pi\tilde{a}(\lambda)^2 2\lambda\,\delta(\lambda^2-\lambda'^2), \tag{35}$$

$$\langle\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\lambda}_m)|\tilde{\boldsymbol{\Psi}}(\overline{\lambda}_n)\rangle=\langle\tilde{\boldsymbol{\Psi}}(\overline{\lambda}_m)|\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\lambda}_n)\rangle=i\,\tfrac{1}{2}\dot{\tilde{a}}(\overline{\lambda}_n)^2\delta_{mn}, \tag{36}$$

and

$$\langle\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\lambda}_m)|\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\lambda}_n)\rangle=i\,\tfrac{1}{2}\dot{\tilde{a}}(\overline{\lambda}_n)\ddot{\tilde{a}}(\overline{\lambda}_n)\delta_{mn}. \tag{37}$$

Now we have the desired orthogonality relations.

V. THE EXPANSION OF THE UNITY

If the above squared Jost functions form a complete set, like the case of the SNLS equation,[8] a state $\mathbf{q}(x)$ can be expanded in terms of them:

$$\mathbf{q}(x)=\frac{1}{\pi}\int_{\Gamma}d\lambda\{f(\lambda)\boldsymbol{\Psi}(x,\lambda)+\tilde{f}(\lambda)\tilde{\boldsymbol{\Psi}}(x,\lambda)\}+\sum_{n}\{f_n\boldsymbol{\Psi}(x,\lambda_n)+g_n\dot{\boldsymbol{\Psi}}(x,\lambda_n)\}+\sum_{n}\{\tilde{f}_n\tilde{\boldsymbol{\Psi}}(\overline{\lambda}_n)+\tilde{g}_n\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\lambda}_n)\}. \tag{38}$$

By using the orthogonality relations we obtain

$$f(\lambda)=\frac{1}{a(\lambda)^2}\langle\boldsymbol{\Psi}(\lambda)|\mathbf{q}\rangle,\quad g_n=-i\frac{2}{\dot{a}(\lambda_n)^2}\langle\boldsymbol{\Psi}(\lambda_n)|\mathbf{q}\rangle \tag{39}$$

and

$$f_n=-i\frac{2}{\dot{a}(\lambda_n)^2}\langle\dot{\boldsymbol{\Psi}}(\lambda_n)|\mathbf{q}\rangle+i\frac{2\ddot{a}(\lambda_n)}{\dot{a}(\lambda_n)^3}\langle\boldsymbol{\Psi}(\lambda_n)|\mathbf{q}\rangle, \tag{40}$$

and similarly

$$\tilde{f}(\lambda)=-\frac{1}{\tilde{a}(\lambda)^2}\langle\tilde{\boldsymbol{\Psi}}(\lambda)|\mathbf{q}\rangle,\quad \tilde{g}_n=-i\frac{2}{\dot{\tilde{a}}(\overline{\lambda}_n)^2}\langle\tilde{\boldsymbol{\Psi}}(\overline{\lambda}_n)|\mathbf{q}\rangle \tag{41}$$

and

$$\tilde{f}_n=-i\frac{2}{\dot{\tilde{a}}(\overline{\lambda}_n)^2}\langle\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\lambda}_n)|\mathbf{q}\rangle+i\frac{2\ddot{\tilde{a}}(\overline{\lambda}_n)}{\dot{\tilde{a}}(\overline{\lambda}_n)^3}\langle\tilde{\boldsymbol{\Psi}}(\overline{\lambda}_n)|\mathbf{q}\rangle. \tag{42}$$

Substituting them into (38), we obtain

$$\begin{aligned}\delta(x-y)=&\frac{1}{\pi}\int_{\Gamma}d\lambda\frac{1}{a(\lambda)^2}\boldsymbol{\Psi}(x,\lambda)\boldsymbol{\Psi}(y,\lambda)^A+\sum_n i\frac{2\ddot{a}(\lambda_n)}{\dot{a}(\lambda_n)^3}\boldsymbol{\Psi}(x,\lambda_n)\boldsymbol{\Psi}(y,\lambda_n)^A\\&-\sum_n i\frac{2}{\dot{a}(\lambda_n)^2}\{\dot{\boldsymbol{\Psi}}(x,\lambda_n)\boldsymbol{\Psi}(y,\lambda_n)^A+\boldsymbol{\Psi}(x,\lambda_n)\dot{\boldsymbol{\Psi}}(y,\lambda_n)^A\}\\&-\frac{1}{\pi}\int_{\Gamma}d\lambda\frac{1}{\tilde{a}(\lambda)^2}\tilde{\boldsymbol{\Psi}}(x,\lambda)\tilde{\boldsymbol{\Psi}}(y,\lambda)^A+\sum_n i\frac{2\ddot{\tilde{a}}(\overline{\lambda}_n)}{\dot{\tilde{a}}(\overline{\lambda}_n)}\tilde{\boldsymbol{\Psi}}(x,\overline{\lambda}_n)\tilde{\boldsymbol{\Psi}}(y,\overline{\lambda}_n)^A\\&-\sum_n i\frac{2}{\dot{\tilde{a}}(\overline{\lambda}_n)}\{\dot{\tilde{\boldsymbol{\Psi}}}(x,\overline{\lambda}_n)\tilde{\boldsymbol{\Psi}}(y,\overline{\lambda}_n)^A+\tilde{\boldsymbol{\Psi}}(x,\overline{\lambda}_n)\dot{\tilde{\boldsymbol{\Psi}}}(y,\overline{\lambda}_n)^A\}.\end{aligned} \tag{43}$$

This is the expansion of the unity in terms of the squared Jost functions.

VI. SECULARITY CONDITIONS

Suppose $\mathbf{q}$ in (21) can be expanded in the form of (38) (the coefficients may be dependent on t). Substituting it into (21) and performing the inner product with $\boldsymbol{\Psi}(x,\lambda)^A$, $\boldsymbol{\Psi}(x,\lambda_m)^A$ and $\dot{\boldsymbol{\Psi}}(x,\lambda_m)^A$ from the left, respectively, by using the orthogonality relations, we obtain

$$\{-if_t(\lambda)+2\lambda f(\lambda)\}a(\lambda)^2=\langle\boldsymbol{\Psi}(\lambda)|\mathbf{R}\rangle, \tag{44}$$

$$\{-ig_{nt}+2\lambda_n g_n\}i\,\tfrac{1}{2}\dot{a}(\lambda_n)^2=\langle\boldsymbol{\Psi}(\lambda_n)|\mathbf{R}\rangle, \tag{45}$$

and

$$\{-if_{nt}+2\lambda_n f_n+2g_n\}i\,\tfrac{1}{2}\dot{a}(\lambda_n)^2+\{-ig_{nt}+2\lambda_n g_n\}\tfrac{1}{2}\dot{a}(\lambda_n)\ddot{a}(\lambda_n)=\langle\dot{\boldsymbol{\Psi}}(\lambda_n)|\mathbf{R}\rangle. \tag{46}$$

Similarly, we also have

$$-\{-i\tilde{f}_t(\lambda)-2\lambda\tilde{f}(x,\lambda)\}\tilde{a}(\lambda)^2=\langle\tilde{\boldsymbol{\Psi}}(\lambda)|\mathbf{R}\rangle, \tag{47}$$

$$\{-i\tilde{g}_{nt}-2\overline{\lambda_n}\tilde{g}_n\}i\,\tfrac{1}{2}\dot{\tilde{a}}(\overline{\lambda_n})^2=\langle\tilde{\boldsymbol{\Psi}}(\overline{\lambda_n})|\mathbf{R}\rangle, \tag{48}$$

and

$$\{-i\tilde{f}_{nt}-2\overline{\lambda_n}\tilde{f}_n-2\tilde{g}_n\}i\,\tfrac{1}{2}\dot{\tilde{a}}(\overline{\lambda_n})^2+\{-i\tilde{g}_{nt}-2\overline{\lambda_n}\tilde{g}_n\}i\,\tfrac{1}{2}\dot{\tilde{a}}(\overline{\lambda_n})\ddot{\tilde{a}}(\overline{\lambda_n})=\langle\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\lambda_n})|\mathbf{R}\rangle. \tag{49}$$

We can see that $g_n(t)$, $f_n(t)$, $\tilde{g}_n(t)$ and $\tilde{f}_n(t)$, the expansion coefficients of the discrete spectrum, may tend to infinity as t grows, unless the right hand sides of those relevant equations above vanish. In order to eliminate such leading secularities, modulations of those parameters characterizing soliton solutions must be so selected that the full source $\mathbf{R}[u]$ is orthogonal to the entire discrete subspace. Explicitly, we demand[8–11]

$$\langle\boldsymbol{\Psi}(\lambda_n)|\mathbf{R}\rangle=\mathbf{0},\quad \langle\dot{\boldsymbol{\Psi}}(\lambda_{\mathbf{n}})|\mathbf{R}\rangle=\mathbf{0}, \tag{50}$$

and

$$\langle\tilde{\boldsymbol{\Psi}}(\overline{\lambda_n})|\mathbf{R}\rangle=\mathbf{0},\quad \langle\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\lambda_{\mathbf{n}}})|\mathbf{R}\rangle=\mathbf{0}. \tag{51}$$

It is easy to show that (51) are just complex conjugates of (50) and are not independent of them. The so-called secularity conditions (50) become

$$\int_{-\infty}^{\infty}dx\{\boldsymbol{\Phi}_2(x,\lambda_n)R[u]-\boldsymbol{\Phi}_1(x,\lambda_n)\overline{R[u]}\}=0, \tag{52}$$

and

$$\int_{-\infty}^{\infty}dx\{\dot{\boldsymbol{\Phi}}_2(x,\lambda_n)R[u]-\dot{\boldsymbol{\Phi}}_1(x,\lambda_n)\overline{R[u]}\}=0. \tag{53}$$

They give $4N$ real conditions for the N-soliton case. In the N-soliton case, we have just $4N$ parameters. By means of these secularity conditions we can determine the time dependence of the parameters up to the order of ϵ in the adiabatic solution. After determining the adiabatic solution, from (44) we can determine $f(\lambda)$ as a function of t. Finally, we can find q.

VII. A SINGLE SOLITON CASE

The secularity conditions (52) and (53) can be rewritten as

$$\mathcal{S}_1=\mathcal{R}_1 \tag{54}$$

and

$$\mathcal{S}_2=\mathcal{R}_2, \tag{55}$$

with

$$\mathcal{S}_1=\int_{-\infty}^{\infty}dx\{\mathbf{\Phi}_2(X,\lambda_1)e^{i2\delta}s[u]-\mathbf{\Phi}_1(X,\lambda_1)e^{-i2\delta}\overline{s[u]}\},\tag{56}$$

$$\mathcal{S}_2=\int_{-\infty}^{\infty}dx\{\dot{\mathbf{\Phi}}_2(X,\lambda_1)e^{i2\delta}s[u]-\dot{\mathbf{\Phi}}_1(X,\lambda_1)e^{-i2\delta}\overline{s[u]}\},\tag{57}$$

and $\mathcal{R}_1$ and $\mathcal{R}_2$ are obtained simply by replacing $s[u]$ with $r[u]$ from (56) and (57), respectively.

For the single soliton solution,

$$u=2\nu\,\mathrm{sech}\,Xe^{-i\varphi},\tag{58}$$

where the parameter $\lambda_1=\mu+i\nu$. We assume λ_1 lies within the first quadrant without loss of generality, hence $\mu>0$ and $\nu>0$:

$$X=2\nu[-t+4\mu(x-x_1)],\quad \varphi=-2\mu t+4(\mu^2-\nu^2)x+\varphi_0,\tag{59}$$

where x_1 and φ_0 are real constants.

For the adiabatic solution, μ,ν,x_1,φ_0 may be dependent on t of the order of ϵ. We write

$$X=8\mu\nu z,\;\; z=x-\hat{x},\;\; \frac{d}{dt}\hat{x}=\frac{1}{4\mu},\tag{60}$$

and

$$\varphi=4(\mu^2-\nu^2)z+2\delta,\;\; \frac{d}{dt}2\delta=-\frac{\mu^2+\nu^2}{\mu}.\tag{61}$$

Simple algebra yields

$$\begin{aligned}s[u_1]=&16\nu\nu_\tau\,\mathrm{sech}\,X\mathrm{th}Xe^{-i\varphi}-8\nu^2[8(\nu\mu)_\tau z-8(\nu\mu)\hat{x}_\tau][\mathrm{sech}\,X-2\,\mathrm{sech}^3X]e^{-i\varphi}\\&+8\nu\mu[4(\mu^2-\nu^2)_\tau z-4(\mu^2-\nu^2)\hat{x}_\tau+2\delta_\tau]\,\mathrm{sech}\,Xe^{-i\varphi}+i8(\nu\mu)_\tau\mathrm{sech}\,Xe^{-i\varphi}\\&-i8\nu^2[4(\mu^2-\nu^2)_\tau z-4(\mu^2-\nu^2)\hat{x}_\tau+2\delta_\tau]\,\mathrm{sech}\,X\mathrm{th}Xe^{-i\varphi}\\&-i8\nu\mu[8(\nu\mu)_\tau z-8(\nu\mu)\hat{x}_\tau]\,\mathrm{sech}\,X\mathrm{th}Xe^{-i\varphi}.\end{aligned}\tag{62}$$

Except unimportant factors (see Appendix) which can be dropped from both sides of (54) and (55), $\mathbf{\Phi}(x,\lambda_1)$ and $\dot{\mathbf{\Phi}}(x,\lambda_1)$ can be replaced by

$$\mathbf{\Phi}(X,\lambda_1)e^{-i2\delta\sigma_3},\quad \dot{\mathbf{\Phi}}(X,\lambda_1)e^{-i2\delta\sigma_3},\tag{63}$$

respectively, where

$$\Phi(X,\lambda_1)=\tfrac{1}{4}\,\mathrm{sech}^2\,Xe^{-i4\bar{\lambda}_1^2z\sigma_3},\tag{64}$$

and

$$\dot{\Phi}(X,\lambda_1)=-i2\lambda_1 z\,\mathrm{sech}^2\,Xe^{i4\bar{\lambda}_1^2z\sigma_3}-i\frac{1}{2\nu}\,\mathrm{sech}\,Xe^{i4(\mu^2-\nu^2)z}\begin{pmatrix}1\\0\end{pmatrix}.\tag{65}$$

We obtain

$$\mathcal{S}_1=\frac{1}{2}\mu_\tau\left(\frac{1}{\nu}-\frac{\nu}{3\mu^2}\right)+i\frac{1}{2}\nu_\tau\left(\frac{1}{\nu}+\frac{\nu}{\mu^2}\right)\tag{66}$$

and

$$\mathcal{S}_2=4\lambda_1\left\{\left(\frac{3}{2}-\frac{\nu^2}{2\mu^2}+i\frac{\mu}{2\nu}-i\frac{5\nu}{6\mu}\right)\hat{x}_\tau-\left(\frac{1}{4\mu^2}+i\frac{1}{4\mu\nu}\right)\delta_\tau\right\}-i\hat{x}_\tau\left\{\frac{8\nu}{3}-4\frac{\mu^2}{\nu}+4\nu\right\}-i\frac{2}{\nu}\delta_\tau. \tag{67}$$

VIII. EFFECT OF DAMPING

The perturbation term for damping is $-i\Gamma u_1$, and Γ can be chosen as the small parameter ϵ. That is,

$$r[u_1]=-iu_1=-i2\nu\,\mathrm{sech}\,Xe^{-i\varphi}. \tag{68}$$

We have

$$\mathcal{R}_1=-i\nu\int_{-\infty}^{\infty}dz\,\mathrm{sech}^2X=-i\frac{1}{4\mu} \tag{69}$$

and

$$\mathcal{R}_2=0. \tag{70}$$

The secularity conditions (54) and (55) become

$$\mu_\tau=0,\quad \frac{\mu^2+\nu^2}{\mu^2\nu}\nu_\tau=-\frac{1}{4\mu}, \tag{71}$$

and

$$\hat{x}_\tau=0,\quad \delta_\tau=0. \tag{72}$$

Hence, up to the order of ϵ, we have

$$\frac{d}{dt}\mu=0,\quad \frac{\mu^2+\nu^2}{\mu^2\nu}\frac{d}{dt}\nu=-\Gamma\frac{1}{4\mu}, \tag{73}$$

and

$$\frac{d}{dt}\hat{x}=-\Gamma\frac{1}{4\mu},\quad \frac{d}{dt}\delta=\frac{\mu^2+\nu^2}{2\mu}. \tag{74}$$

Equations (73) and (74) yield

$$\mu=\mu_0,\quad \log\left(\frac{\nu}{\nu_0}\right)+\frac{1}{2\mu_0^2}(\nu^2-\nu_0^2)=-\Gamma\frac{1}{4\mu}t, \tag{75}$$

and

$$\hat{x}=x_1-\Gamma\frac{1}{4\mu}t,\quad \delta=\delta_0+\frac{1}{2}\mu t+\frac{1}{2\mu}\int_0^t dt\,\nu^2. \tag{76}$$

Here μ_0, ν_0, x_1 and δ_0 are constants.

After determination of the adiabatic solution, the right hand side of (47) is given, and we can find $f(t,\lambda)$ and then $\mathbf{q}(x,t)$. Finally, we obtain $q(x,t)$.

IX. DISCUSSION

We have developed a direct perturbation theory for the perturbed UNLS equation. Because of the second order derivative in t, the perturbation theory is essentially different from that for the perturbed SNLS equation involving only the first derivative in t.

In a single soliton case, by substituting the explicit expressions of the Jost solutions into the right hand side of (43), like the case of dark solitons of SNLS,[12] we can see that it is indeed equal to $\delta(x-y)$. Hence the completeness relation (43) is shown in this case. However, for the multi-soliton case the explicit expressions of the Jost solutions are very complicated so that it is impossible to substitute them into the right hand side of (43) and to show it is equal to $\delta(x-y)$. This problem will be discussed separately.

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APPENDIX: A REVIEW OF THE INVERSE SCATTERING TRANSFORM FOR THE UNLS EQUATION

We review the inverse scattering transform[2,6] for the unperturbed equation (1) with the boundary condition

$$u\rightarrow 0, \quad \text{as} \quad |x|\rightarrow\infty. \tag{A1}$$

Two Lax equations for the UNLS equation are obtained from those for the SNLS equation[2] by interchanging their roles. Starting from the first Lax equation

$$\partial_x w(x,t,\lambda)=\begin{pmatrix} -i2\lambda^2+|u|^2 & 2\lambda u-iu_t \\ -2\lambda\overline{u}-i\overline{u_t} & i2\lambda^2-|u|^2 \end{pmatrix} w(x,t,\lambda), \tag{A2}$$

and by using the boundary conditions (A1), the analyticity of the Jost functions can be found and the equation of IST can be derived. Then, by using the second Lax equation,

$$\partial_t w(x,t,\lambda)=\begin{pmatrix} i\lambda & -u \\ \overline{u} & -i\lambda \end{pmatrix} w(x,t,\lambda), \tag{A3}$$

the t dependence of the scattering data can be determined.

From the Lax equation (A2) and the boundary condition (A1), the asymptotic solution in the limit of $|x|\rightarrow\infty$ of (A2) is

$$E(x,\lambda)=e^{-i2\lambda^2 x\sigma_3}. \tag{A4}$$

In comparison with the asymptotic solution for the SNLS equation, $e^{-i\lambda x\sigma_3}$, one can see that the parameter in the exponential, λ, is replaced by $2\lambda^2$ in the UNLS case. This leads to the followoing.

(1) The domain of definition of the asymptotic solution for the SNLS equation is for real λ, namely, on the real axis in the complex λ-plane. The domain of definition of the asymptotic solution for the UNLS equation is for real λ^2, namely, on the real axis in the complex λ-plane where $\lambda^2>0$, as well as on the imaginary axis where $\lambda^2<0$.

(2) Jost functions are defined by

$$(\tilde{\psi}\ \psi)(x,\lambda)\rightarrow E(x,\lambda) \quad \text{as} \quad x\rightarrow\infty, \tag{A5}$$

and

$$(\varphi\,\tilde{\varphi})(x,\lambda)\to E(x,\lambda)\quad \text{as}\quad x\to-\infty; \tag{A6}$$

the monodromy matrix is introduced as well:

$$(\varphi\tilde{\varphi})(x,\lambda)=(\tilde{\psi}\psi)(x,\lambda)\begin{pmatrix} a(\lambda) & -\tilde{b}(\lambda) \\ b(\lambda) & \tilde{a}(\lambda)\end{pmatrix}, \tag{A7}$$

similarly in both cases. In the SNLS case $\psi(x,\lambda)$, $\varphi(x,\lambda)$ and $a(\lambda)$ are analytic in the upper half plane of complex λ-plane, and $\tilde{\psi}(x,\lambda)$, $\tilde{\varphi}(x,\lambda)$ and $\tilde{a}(\lambda)$ are analytic in the lower plane. Moreover, $b(\lambda)$ and $\tilde{b}(\lambda)$ cannot be analytically continued out of the real axis. The zeros of $a(\lambda)$ lie in the upper plane. On the other hand, in the UNLS case, $\psi(x,\lambda)$, $\varphi(x,\lambda)$ and $a(\lambda)$ are analytic in the first and third quadrants, and $\tilde{\psi}(x,\lambda)$, $\tilde{\varphi}(x,\lambda)$ and $\tilde{a}(\lambda)$ are analytic in the second and fourth quadrants. Moreover, $b(\lambda)$ and $\tilde{b}(\lambda)$ cannot be analytically continued out of the real and the imaginary axes. The zeros of $a(\lambda)$ lie in the first or the third quadrants.

(3) By using the usual procedure, we can obtain the equation of inverse scattering transform of Zakharov-Shabat type,

$$\tilde{\psi}(x,\lambda)=\{E_{\cdot 2}(x,\lambda)+R(x,\lambda)+J(x,\lambda)\}e^{-i2\lambda^2 x}, \tag{A8}$$

where $E_{\cdot 2}=(0\ \ e^{i2\lambda^2 x})^T$,

$$R(x,\lambda)=i\sum_n \frac{1}{\lambda-\lambda_n}c_n\psi(x,\lambda_n)e^{i2\lambda_n^2 x}, \tag{A9}$$

$$J(x,\lambda)=\frac{1}{2\pi}\int_\Gamma d\lambda'\frac{1}{\lambda-\lambda'}r(\lambda')\psi(x,\lambda')e^{i2\lambda'^2 x}. \tag{A10}$$

Here c_n and $r(\lambda')$ are the usual symbols.[2] The path of integration is

$$\Gamma=(0,+\infty)\cup(0,-\infty)\cup(i\infty,i0)\cup(-i\infty,i0). \tag{A11}$$

(4) By using the Lax equation (6), we can obtain the t dependence of scattering data in (12), Simply, the Jost functions $\psi(x,\lambda)$, etc., which are determined by only one of the Lax equations, can be extended to those to satisfy simultaneously the two Lax equations. For example,

$$h(t,\lambda)\tilde{\psi}(x,\lambda),\quad h(t,\lambda)^{-1}\psi(x,\lambda),\quad h(t,\lambda)=e^{i\lambda t}. \tag{A12}$$

The scattering data are replaced by

$$r(\lambda)\to r(\lambda)h(t,\lambda)^{-2},\quad c_n\to c_n h(t,\lambda_n)^{-2}, \tag{A13}$$

etc.

The soliton solutions correspond to a reflectionless potential and in this case the continuous spectrum disappears. The poles of the transmission coefficient $a(\lambda)^{-1}$ lie within the first or the third quadrants. However, it has been shown[13] that the forms of the soliton solutions depend on the absolute values of the imaginary part of these poles and the values of the real parts. Thus the soliton solutions of the UNLS equation can be obtained from those of the SNLS equation by simply interchanging x and t.

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