Construction Schemes for Fault-Tolerant Hamiltonian Graphs

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In this paper, we present three construction schemes for fault-tolerant Hamiltonian graphs. We show that applying these construction schemes on fault-tolerant Hamiltonian graphs generates graphs preserving the original Hamiltonicity property. We apply these construction schemes to generate some known families of optimal 1-Hamiltonian graphs in the literature and the Hamiltonicity properties of these graphs are the direct consequence of the construction schemes. In addition, we can use these construction schemes to propose new family of optimal 1-Hamiltonian graphs.

Keywords: Hamiltonian graphs; token rings; fault tolerance; 1-Hamiltonian graphs; 3-joins; cycle extensions

1. INTRODUCTION

The topology of an interconnection network for parallel and distributed systems can always be represented by a graph. Let $G = (V, E)$ be an undirected graph. A cycle in G that traverses every vertex exactly once is called a *Hamiltonian cycle*. A graph G is called a *Hamiltonian graph* or said to be *Hamiltonian* if it contains a Hamiltonian cycle. In this paper, we study Hamiltonian graphs which correspond to the token ring topology.

Fault tolerance is an important issue in the design of an interconnection network. When faults occur in a network, it corresponds to removing edges and/or vertices from the graph. Let $V' \subseteq V$ and $E' \subseteq E$. We use $G - V'$ to denote the subgraph of G induced by $V - V'$, and $G - E'$, the subgraph obtained by removing E' from G. Faults can be in the combination of vertices and edges. Let $F \subseteq V \cup E$. We use $G - F$ to denote the subgraph induced by $V - F$ and deleting the edges in F from the induced subgraph. If $G - V'$ is Hamiltonian for any $V' \subseteq V$ and $|V'| = k$, then G is called a *k-vertex-Hamiltonian* graph. If $G - E'$ is Hamiltonian for any $E' \subseteq E$ and $|E'| = k$, then G is called a *k-edge-Hamiltonian* graph. If $G - F$ is Hamiltonian for any $F \subseteq V \cup E$ and $|F| = k$, then G is called a *k-Hamiltonian* graph. In this paper, we are, in particular, interested in $k = 1$ and an arbitrary

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vertex or edge fault, that is, a 1-Hamiltonian graph. For convenience, we write $G - f$ instead of $G - \{f\}$, where $\{f\} \subset V \cup E$.

A k-Hamiltonian graph is said to be *optimal* if it contains the least number of edges among all k-Hamiltonian graphs having the same number of vertices. Mukhopadhyaya and Sinha [4], Harary and Hayes [1,2], Wang et al. [5], Hung et al. [3], and Wang et al. [6] proposed different families of optimal 1-Hamiltonian graphs. These optimal 1-Hamiltonian graphs are all trivalent except the ones proposed in [1,2,4] with an odd number of vertices which have exactly one vertex of degree 4 and the remaining vertices of degree 3.

In this paper, we propose two operations on graphs called 3-*join* and *cycle extension*. In addition, a variation of 3-join, called (3,4)-*join*, is also presented. Using these operations on particular graphs, the resultant graphs have nice properties. In particular, applying these operations on trivalent 1-Hamiltonian graphs can yield other trivalent 1-Hamiltonian graphs. Furthermore, by recursively applying these operations on specific simple primitive graphs, we can obtain the optimal 1-Hamiltonian graphs proposed in [1–6]. Our constructed graphs can be easily shown to be optimal 1-Hamiltonian using the properties of these operations.

This paper is organized as follows: In Section 2, we introduce the 3-join operation and its variation (3,4)-join and their properties. In addition, we show that optimal 1-Hamiltonian graphs in [1–5] can be constructed by recursively applying 3-joins and (3,4)-joins on specific primitive graphs. In Section 3, the cycle extension operation is presented. We also study its properties and show that the graphs proposed in [6] can be obtained by recursively applying cycle extensions. In addition, we apply this operation on the Petersen graph (which is 1-vertex-Hamiltonian) to generate a new family of 1-Hamiltonian graphs. Final remarks appear in Section 4.

2. 3-JOIN

In this section, we introduce two operations, called 3-*join* and (3,4)-*join*, performed on two different graphs. A 3-join is an operation applied on two vertices of degree 3 in two graphs, while a (3,4)-join is applied on a vertex of degree 3 in one graph and a vertex of degree 4 in another graph. We will show that performing these two operations on two 1-Hamiltonian graphs produces new families of 1-Hamiltonian graphs. Furthermore, those families of 1-Hamiltonian graphs proposed in [1–5] can be easily generated by applying the two operations on specific graphs and K_4. In the following discussion, we use 3-join and (3,4)-join to mean the graphs and the operations interchangeably.

2.1. *Definitions and Properties*

Let G_1 and G_2 be two graphs. We assume that $V(G_1) \cap V(G_2) = \emptyset$ throughout the section. Let x be a vertex of a graph. We use $N(x)$ to denote an **ordered set** which consists of all of the neighbors of x, that is, $N(x)$ is an ordering of all of the neighbors of x. Henceforth, we use $N(x)$ as an ordered set. The 3-*join*, as illustrated in Figure 1, is defined as follows:

Definition 1. *Let x be a vertex of degree 3 in G_1 and y be a vertex of degree 3 in G_2. Let $N(x) = \{x_1, x_2, x_3\}$ and $N(y) = \{y_1, y_2, y_3\}$. The 3-join of G_1 and G_2 at x and y is a graph K given by*

$$\begin{aligned} V(K) &= (V(G_1) - \{x\}) \cup (V(G_2) - \{y\}) \qquad \text{and} \\ E(K) &= (E(G_1) - \{(x, x_i) | 1 \le i \le 3\}) \\ &\cup\ (E(G_2) - \{(y, y_i) | 1 \le i \le 3\}) \\ &\cup\ \{(x_i, y_i) | 1 \le i \le 3\}. \end{aligned}$$

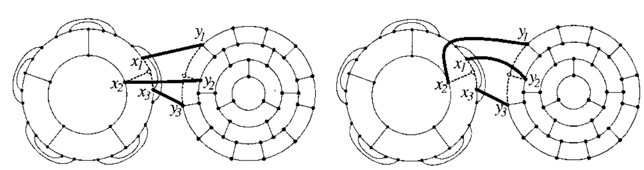
Note that different $N(x)$ and $N(y)$ generate different 3-joins of G_1 and G_2 at x and y. For example, as illustrated in Figure 1, given $N(y) = \{y_1, y_2, y_3\}$, the 3-join of G_1 and G_2 at x and y with $N(x) = \{x_1, x_2, x_3\}$ is different from the one with $N(x) = \{x_2, x_1, x_3\}$. On the other hand, each 3-join of G_1 and G_2 at x and y is uniquely determined by $N(x)$ and $N(y)$.

Throughout this subsection, we use x to represent a vertex of degree 3 in graph G_1, and y, a vertex of degree 3 in graph G_2. A graph K is said to be a 3-*join of* G_1 *and* G_2 if K is a 3-join of G_1 and G_2 at x and y with some $N(x)$ and some $N(y)$. Clearly, 3-joins of two trivalent graphs G_1 and G_2 are still trivalent. In this paper, we write a path P from x to y as $\langle x, x_1, x_2, \ldots, x_k, y\rangle$ or $\langle x \to P \to y\rangle$.

Theorem 1. *Let G_1 and G_2 be two graphs and K be a 3-join of G_1 and G_2. If both G_1 and G_2 are 1-Hamiltonian graphs, then K is a 1-Hamiltonian graph.*

Proof. Let K be a 3-join of G_1 and G_2 at x and y with $N(x) = \{x_1, x_2, x_3\}$ and $N(y) = \{y_1, y_2, y_3\}$. Let f be any fault, vertex or edge, of K.

Consider $f \ne (x_i, y_i)$ for all $1 \le i \le 3$. Without loss of generality, we may assume that $f \in (V(G_1) - \{x\}) \cup (E(G_1) - \{(x, x_i) | 1 \le i \le 3\})$. Since G_1 is 1-Hamiltonian, it follows that there is a Hamiltonian cycle H_1 in $G_1 - f$ given by $\langle x, x_i \to P \to x_j, x\rangle$, where $i, j \in \{1, 2, 3\}$ with $i \ne j$ and P denotes a path in G_1. Let k be the unique



(a) (b)

FIG. 1. Examples of 3-joins of two trivalent graphs.

element in $\{1, 2, 3\} - \{i, j\}$. Since G_2 is 1-Hamiltonian, there is a Hamiltonian cycle H_2 in $G_2 - (y, y_k)$ which can be written as $\langle y, y_i \to Q \to y_j, y\rangle$. Hence, $\langle x_i \to P \to x_j, y_j \to Q \to y_i, x_i\rangle$ forms a Hamiltonian cycle of $K - f$.

Consider $f = (x_i, y_i)$ for some i. Without loss of generality, we may assume that $f = (x_1, y_1)$. Let H_1 be a Hamiltonian cycle of $G_1 - (x, x_1)$ which can be written as $\langle x, x_2 \to P \to x_3, x\rangle$. Let H_2 be a Hamiltonian cycle of $G_2 - (y, y_1)$ which can be written as $\langle y, y_2 \to Q \to y_3, y\rangle$. Then, $\langle x_2 \to P \to x_3, y_3 \to Q \to y_2, x_2\rangle$ forms a Hamiltonian cycle of $K - f$. Hence, the theorem is proved. ▪

Using a similar proof technique in Theorem 1, we can show the following corollaries:

Corollary 1. *Let G_1 and G_2 be two graphs and K be a 3-join of G_1 and G_2. If both G_1 and G_2 are 1-edge-Hamiltonian, then K is also 1-edge-Hamiltonian.*

We consider a special 3-join on G_1 and G_2, where G_2 is K_4, as described below:

Definition 2. *Let G be a graph with a vertex v of degree 3. A 3-vertex expansion on v is a graph obtained from G by performing a 3-join at v and any vertex of K_4.*

In other words, a 3-vertex expansion on v of degree 3 is a graph obtained from G by replacing the vertex v with a K_3 and connecting the three vertices of K_3 to the three neighbors of v, one by one. Note that 3-vertex expansions on v are unique up to isomorphism.

Corollary 2. *Let G be a trivalent 1-edge-Hamiltonian graph and $V'(G) = \{v \in V(G) | G - v$ is not Hamiltonian$\}$. Let G^* denote the graph obtained from G by performing a sequence of 3-vertex expansions on every vertex $v \in V'(G)$. Then, G^* is 1-Hamiltonian.*

One may ask whether the converse of Theorem 1 holds. To answer this question, we consider the graph M shown in Figure 2(a) which is constructed by a sequence of 3-vertex expansions on $K_{3,3}$. To be specific, let $\{a_i, b_i | 1 \le i \le 3\}$ be the vertex set of $K_{3,3}$ and $\{(a_i, b_j) | 1 \le i, j \le 3\}$ be the edge set of $K_{3,3}$. The construction of M is given as follows:

1. Perform 3-vertex expansions on vertices a_1, a_2, and a_3, that is, replace each a_i with a K_3 given by vertex set $\{a_{i,m} | 1 \le m \le 3\}$.
2. Perform 3-vertex expansions on vertices b_1 and b_3, that is, replace each b_i for $i = 1, 3$ with a K_3 given by vertex set $\{b_{i,m} | 1 \le m \le 3\}$.

Since $K_{3,3}$ and K_4 are 1-edge-Hamiltonian, it follows from Corollary 1 that M is also 1-edge-Hamiltonian. Note that $M - b_2$ is not Hamiltonian. It follows that M is not 1-Hamiltonian. Furthermore, it can be easily verified that $V'(M) = \{b_2\}$. By performing a 3-vertex expansion on b_2 of M, we obtain a new graph M^* as shown in Figure 2(b). In other words, M^* is a 3-join of M and K_4. Note that M^* is 1-Hamiltonian following from Corollary 2, whereas M is not. This provides a counterexample for the converse of Theorem 1. Moreover, note that a 3-join operation on a 1-Hamiltonian graph and a non-1-Hamiltonian graph may generate a 1-Hamiltonian graph.

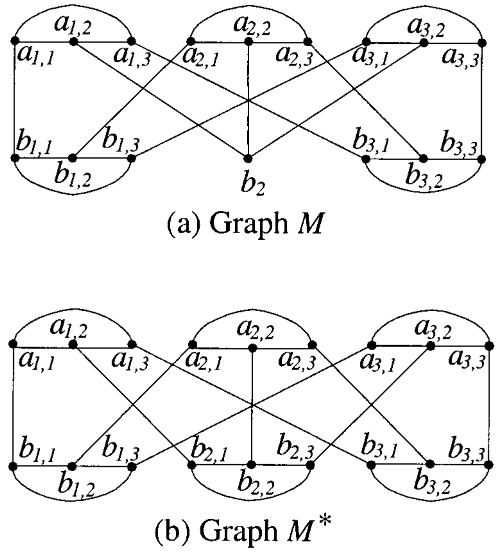


FIG. 2. A counterexample for the converse of Theorem 1.

2.2. *Families of 1-Hamiltonian 3-Join Graphs*

In this subsection, we show that known trivalent 1-Hamiltonian graphs proposed in [3–5] and those with even vertices proposed in [1,2] can be generated by a sequence of 3-joins on some specific graphs and K_4.

Harary and Hayes [1,2] proposed a family of 1-Hamiltonian graphs, denoted by $H(k)$ for $k \ge 4$, where

$$V(H(k)) = \{0, 1, 2, \ldots, k-1\}, \text{ and}$$

$$E(H(k)) = \begin{cases} \{(i, i+1) | 0 \le i \le k-2\} \\ \quad \cup\{(0, k/2), (0, k-1)\} \\ \quad \cup\{(i, k-i) | 1 \le i \le k/2 - 1\} \\ \qquad\qquad\qquad \text{for } k \text{ even}, \\ \{(0,1), (0,2), (0, k-1), (0, k-2)\} \\ \quad \cup\{(i, i+2) | 1 \le i \le k-3\} \\ \quad \cup\{(i, i+1) | 1 \le i \le k-2 \text{ and } i \text{ odd}\} \\ \qquad\qquad\qquad \text{for } k \text{ odd}. \end{cases}$$

Examples of $H(4), H(5), H(8)$, and $H(9)$ are shown in Figure 3.

Note that $H(4)$ is, indeed, a complete graph K_4 which is the smallest 1-Hamiltonian graph. Furthermore, $H(k)$ for all $k \ge 4$ and even are trivalent. The following theorem can be easily verified:

Theorem 2. *Let $k \ge 6$ be an even integer. Given that $H(4) = K_4, H(k)$ can be obtained by a 3-vertex expansion at the vertex 0 [or $(k-2)/2$] in $H(k-2)$ and then relabeling vertices. Furthermore, $H(k)$ is 1-Hamiltonian.*

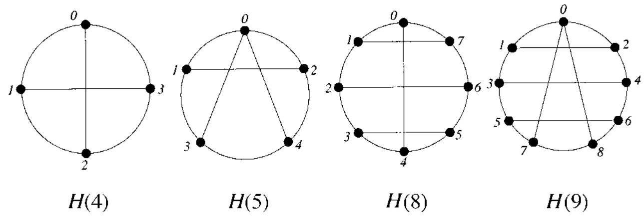


FIG. 3. The graphs $H(k)$.

$H(k)$ for k odd will be discussed in Section 2.3.

Mukhopadhyaya and Sinha [4] proposed a family of 1-Hamiltonian graphs, denoted by $M(k)$. Let $k \geq 4$ be an integer and t be a nonnegative integer. To define $M(k)$, we first introduce $B(i,t)$, as illustrated in Figure 4, which is defined as follows:

$$V(B(i,t)) = \{x^r_{i,j} \mid 1 \leq j \leq t\} \cup \{x^l_{i,j} \mid 1 \leq j \leq t\} \cup \{y_i, z_i\},$$

and

$$\begin{aligned} E(B(i,t)) = {} & \{(x^l_{i,j}, x^l_{i,j-1}) \mid 1 < j \leq t\} \\ & \cup \{(x^r_{i,j}, x^r_{i,j+1}) \mid 1 \leq j \leq t-1\} \\ & \cup \{(x^l_{i,j}, x^r_{i,j}) \mid 1 \leq j \leq t\} \\ & \cup \{(x^l_{i,1}, y_i), (y_i, z_i), (y_i, x^r_{i,1})\}. \end{aligned}$$

It can be easily verified that $B(i,t)$, for $t \geq 2$, is isomorphic to the graph obtained from performing a 3-vertex expansion on the vertex y_i of $B(i,t-1)$. The graph $M(k)$ for k even is constructed as follows. (For convenience, when $t = 0$, we write $y_i = x^l_{i,0} = x^r_{i,0}$.)

1. If $k = 6t+4$, then $M(k)$ is constructed from three $B(i,t)$ for all $0 \leq i \leq 2$ by identifying z_1, z_2, and z_3 into a single vertex z and adding the edges $(x^l_{0,t}, x^r_{1,t}), (x^l_{1,t}, x^r_{2,t})$, and $(x^l_{2,t}, x^r_{0,t})$.
2. If $k = 6t+6$, then $M(k)$ is constructed from $B(0,t+1), B(1,t)$, and $B(2,t)$ by identifying z_1, z_2, and z_3 into a single vertex z and adding the edges $(x^l_{0,t+1}, x^r_{1,t}), (x^l_{1,t}, x^r_{2,t})$, and $(x^l_{2,t}, x^r_{0,t+1})$.
3. If $k = 6t+8$, then $M(k)$ is constructed from $B(0,t+1), B(1,t+1)$, and $B(2,t)$ by identifying z_1, z_2, and z_3 into a single vertex z and adding the edges $(x^l_{0,t+1}, x^r_{1,t+1}), (x^l_{1,t+1}, x^r_{2,t})$, and $(x^l_{2,t}, x^r_{0,t+1})$.

Note that $M(4)$ is, indeed, a complete graph K_4. The graph $M(k)$ for $k \geq 5$ and odd is constructed as follows:

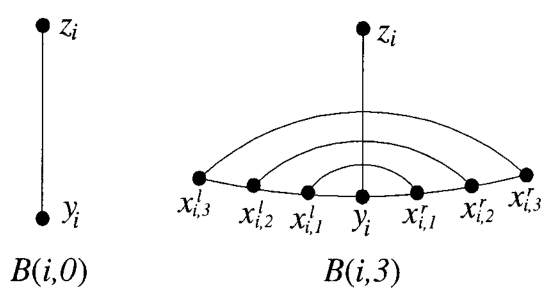


FIG. 4. The graphs $B(i,t)$.

1. If $k = 8t+5$, $M(k)$ is constructed from four $B(i,t)$ for all $0 \leq i \leq 3$ by identifying all z_i into a single vertex z and adding the edges $(x^l_{0,t}, x^r_{1,t}), (x^l_{1,t}, x^r_{2,t}), (x^l_{2,t}, x^r_{3,t})$, and $(x^l_{3,t}, x^r_{0,t})$.
2. If $k = 8t+7$, $M(k)$ is constructed from $B(0,t+1)$ and $B(i,t)$ for $i = 1,2,3$ by identifying all z_i into a single vertex z and adding the edges $(x^l_{0,t+1}, x^r_{1,t}), (x^l_{1,t}, x^r_{2,t}), (x^l_{2,t}, x^r_{3,t})$, and $(x^l_{3,t}, x^r_{0,t+1})$.
3. If $k = 8t+9$, $M(k)$ is constructed from $B(i,t+1)$ for $i = 0,1$ and $B(j,t)$ for $j = 2,3$ by identifying all z_i into a single vertex z and adding the edges $(x^l_{0,t+1}, x^r_{1,t+1}), (x^l_{1,t+1}, x^r_{2,t}), (x^l_{2,t}, x^r_{3,t})$, and $(x^l_{3,t}, x^r_{0,t+1})$.
4. If $k = 8t+11$, $M(k)$ is constructed from $B(i,t+1)$ for $i = 0,1,2$ and $B(3,t)$ by identifying all z_i into a single vertex z and adding the edges $(x^l_{0,t+1}, x^r_{1,t+1}), (x^l_{1,t+1}, x^r_{2,t+1}), (x^l_{2,t+1}, x^r_{3,t})$, and $(x^l_{3,t}, x^r_{0,t+1})$.

The graphs $M(4), M(5), M(18)$, and $M(25)$ are shown in Figure 5. The following theorem can be easily verified:

Theorem 3. *Let k and t be nonnegative integers. Let $M(4)$ and $M(5)$ be given as in Figure 5.*

(*i*) *$M(k)$ for $k \geq 6$ and even can be obtained by a 3-vertex expansion of $M(k-2)$ at the vertex y_0 if $k = 6t+6$, at y_1 if $k = 6t+8$, and at y_2 if $y = 6t+10$.*
(*ii*) *$M(k)$ for $k \geq 7$ and odd can be obtained by a 3-vertex expansion of $M(k-2)$ at the vertex y_0 if $k = 8t+7$, at y_1 if $k = 8t+9$, at y_2 if $k = 8t+11$, and at y_3 if $y = 8t+13$.*
(*iii*) *$M(k)$ is 1-Hamiltonian for all $k \geq 4$.*

Wang et al. [5] presented a new family of 1-Hamiltonian graphs $W(k)$, as illustrated in Figure 6, which is constructed from $B(i,k)$ for $0 \leq i \leq 2k$ by joining all z_i with a cycle $\langle z_0, z_1, \ldots, z_{2k}, z_0 \rangle$ and adding edges $(x^l_{2k,k}, x^r_{0,k})$ and $(x^l_{i,k}, x^r_{i+1,k})$ for all $0 \leq i \leq 2k-1$. Let $D(k)$ denote the graph obtained from $B(i,0)$ for $0 \leq i \leq 2k$ by joining all z_i and y_i with cycles $\langle z_0, z_1, \ldots, z_{2k}, z_0 \rangle$ and $\langle y_0, y_1, \ldots, y_{2k}, y_0 \rangle$, respectively. It can be easily verified that $D(k)$ is 1-Hamiltonian. The following theorem can also be easily verified:

Theorem 4. *Let $k \geq 1$ be an integer.*

(*i*) *$W(k)$ can be generated by performing k 3-vertex expansions at y_i in $D(k)$ for $0 \leq i \leq 2k$.*
(*ii*) *$W(k)$ is 1-Hamiltonian.*

Hung et al. [3] proposed a family of 1-Hamiltonian graphs, called a *Christmas tree*, denoted by $CT(k)$. In their construction, they, indeed, employed 3-vertex ex-

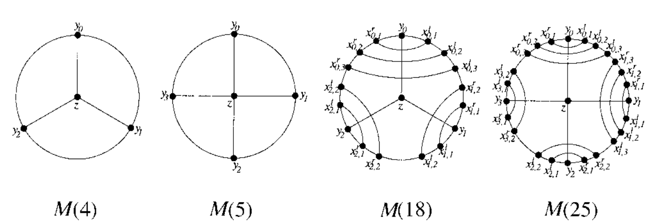


FIG. 5. The graphs $M(k)$.

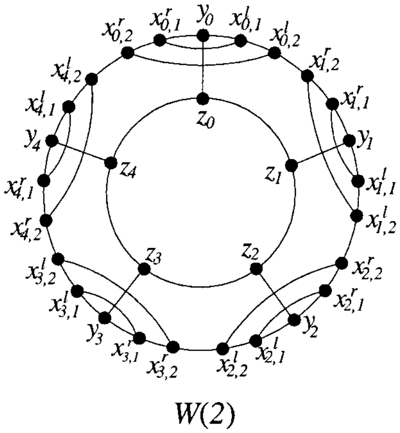


FIG. 6. The graph $W(2)$.

pansions. Let r be an arbitrary vertex in K_4. The graphs $CT(k)$ are defined as follows: $CT(1) = K_4$; for $k \geq 2, CT(k)$ is recursively constructed by performing a 3-vertex expansion in $CT(k-1)$ on every vertex which is at the distance of $k-1$ from the vertex r. We illustrate an example of $CT(3)$ in Figure 7. By this construction, $CT(k)$ is obviously 1-Hamiltonian.

2.3. (3,4)-Join

To construct the graphs $H(k)$ proposed in [1,2] for k odd, we introduce a variation of 3-join, called (3,4)-*join*, which is performed at a vertex of degree 3 in one graph and a vertex of degree 4 in another graph.

Definition 3. *Let x be a vertex of degree 3 in G_1 and y be a vertex of degree 4 in G_2. Let $N(x) = \{x_1, x_2, x_3\}$ and $N(y) = \{y_1, y_2, y_3, y_4\}$. The (3,4)-join of G_1 and G_2 is defined as the graph K such that*

$$\begin{aligned} V(K) &= (V(G_1) - \{x\}) \cup (V(G_2) - \{y\}) \qquad \textit{and} \\ E(K) &= (E(G_1) - \{(x, x_i) | 1 \leq i \leq 3\}) \\ &\cup\ (E(G_2) - \{(y, y_i) | 1 \leq i \leq 4\}) \\ &\cup\ \{(x_i, y_i) | 1 \leq i \leq 3\} \cup \{(x_3, y_4)\}. \end{aligned}$$

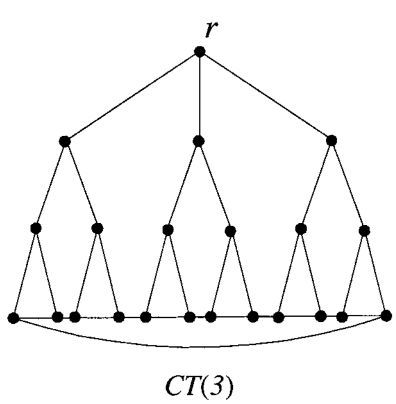


FIG. 7. The Christmas tree $CT(3)$.

We call a graph K a (3,4)-*join* of G_1 and G_2 if K is a (3,4)-join at some vertex x of degree 3 in G_1 and some vertex y of degree 4 in G_2 with some specific $N(x)$ and $N(y)$. Similar to the 3-join operation, each (3,4)-join of G_1 and G_2 is uniquely defined by $N(x)$ and $N(y)$.

Since we are interested in 1-Hamiltonian graphs, we restrict our discussion on (3,4)-join to particular graphs, where (i) G_1 and G_2 are 1-Hamiltonian, (ii) G_1 has one vertex x of degree 3, and (iii) G_2 has one vertex y of degree 4. Note that not all $N(x)$ and $N(y)$ can induce 1-Hamiltonian graphs by using (3,4)-join on G_1 and G_2. For example, Figure 8(a) and (b) illustrates two (3,4)-joins of $G_1 = K_4$ and $G_2 = H(5)$ with different $N(x)$ and $N(y)$. Given $N(x) = \{x_1, x_2, x_3\}$ in $G_1, N(y)$ of G_2 in Figure 8(a) and (b) are given by $\{y_1, y_2, y_3, y_4\}$ and $\{y_1, y_3, y_2, y_4\}$, respectively. It can be observed that the graph in Figure 8(a) is 1-Hamiltonian, but the one in Figure 8(b) is not 1-Hamiltonian since it is not Hamiltonian when deleting the vertex x_3 from the graph. The ordering of $N(x)$ and $N(y)$ plays an important role in determining the 1-Hamiltonicity of the resulting (3,4)-join.

It is interesting to find sufficient conditions of $N(x), N(y), G_1$, and G_2 for the resulting (3,4)-join to have a Hamiltonian property. For example, let $N(x) = \{x_1, x_2, x_3\}$ and $N(y) = \{y_1, y_2, y_3, y_4\}$. If G_1 and G_2 have a Hamiltonian cycle containing the edges $(x, x_i), (x, x_j)$ and $(y, y_i), (y, y_j)$, respectively, with $i, j \in \{1, 2, 3\}$ and $i \neq j$, then the resulting (3,4)-join is also Hamiltonian. In the following theorem, we give a sufficient condition of $N(x), N(y), G_1$, and G_2 for the resulting (3,4)-join to be 1-Hamiltonian.

Theorem 5. *Let G_1 and G_2 be two 1-Hamiltonian graphs, where G_1 has a vertex x of degree 3 with $N(x) = \{x_1, x_2, x_3\}$ and G_2 has a vertex y of degree 4 with $N(y) = \{y_1, y_2, y_3, y_4\}$. Let $f_1 \in V(G_1) \cup E(G_1)$ and $f_2 \in V(G_2) \cup E(G_2)$ with $f_1 \neq x$ and $f_2 \neq y$. Suppose that $G_1, G_2, N(x)$, and $N(y)$ satisfy the following conditions:*

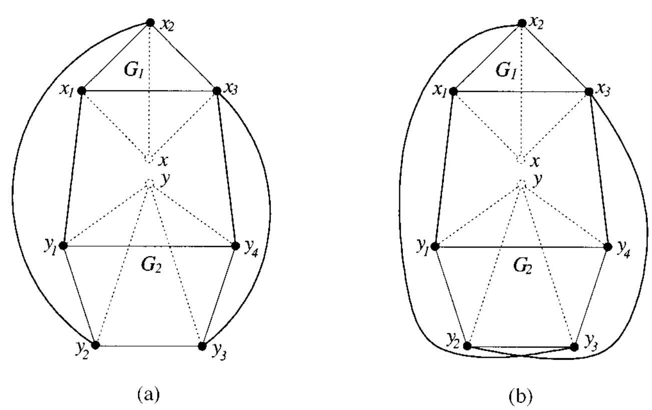


FIG. 8. Examples of (3,4)-joins of graphs K_4 and $H(5)$, where the graph in (a) is 1-Hamiltonian and the one in (b) is not.

($A1$) *For every Hamiltonian cycle in $G_1 - f_1$ without containing an edge (x, x_i) with $i \in \{1,2\}$, there is a Hamiltonian cycle in $G_2 - (y, y_i)$ containing the edge (y, y_{3-i});*
($A2$) *For every Hamiltonian cycle in $G_1 - f_1$ without containing the edge (x, x_3), there is a Hamiltonian cycle in G_2 containing the edges (y, y_1) and (y, y_2);*
($A3$) *For every Hamiltonian cycle in $G_2 - f_2$ containing the edges (y, y_3) and (y, y_4) with $f_2 \notin \{y_1, y_2\}$, there is a Hamiltonian path in $G_2 - f_2$ from y_1 to y_2 containing (y, y_3) and (y, y_4);*
($A4$) *There is a Hamiltonian cycle of $G_2 - y_i$ containing the edge (y, y_{3-i}) for $i \in \{1,2\}$.*

Let K be the (3,4)-join of G_1 and G_2 at x and y with the given $N(x)$ and $N(y)$. Then, K is a 1-Hamiltonian graph.

Proof. Let f be any fault, vertex or edge of K. We distinguish the following cases of f:

CASE 1. $f \in (V(G_1)-\{x\}) \cup (E(G_1)-\{(x,x_i)|1 \le i \le 3\})$.

Since G_1 is 1-Hamiltonian and x is of degree 3, there is a Hamiltonian cycle H_1 in $G_1 - f$ without containing an edge (x, x_i), where $i \in \{1,2,3\}$.

Consider $i \in \{1,2\}$. Then, H_1 can be written as $H_1 = \langle x, x_3 \to P_1 \to x_{3-i}, x\rangle$, where P_1 is a path in G_1. By (A1), there is a Hamiltonian cycle H_2 in $G_2 - (y, y_i)$ containing the edge (y, y_{3-i}) which can be written as $H_2 = \langle y, y_{3-i} \to P_2 \to y_j, y\rangle$, where $j \in \{3,4\}$ and P_2 is a path in G_2. Then, $\langle x_{3-i}, y_{3-i} \to P_2 \to y_j, x_3 \to P_1 \to x_{3-i}\rangle$ is a Hamiltonian of $K - f$.

Consider $i = 3$. Then, H_1 can be written as $H_1 = \langle x, x_2 \to P_1 \to x_1, x\rangle$, where P_1 is a path in G_1. By (A2), there is a Hamiltonian cycle H_2 in G_2 containing the edges (y, y_1) and (y, y_2) which can be written as $H_2 = \langle y, y_1 \to P_2 \to y_2, y\rangle$, where P_2 is a path in G_2. Thus, $\langle x_1, y_1 \to P_2 \to y_2, x_2 \to P_1 \to x_1\rangle$ is a Hamiltonian cycle of $K - f$.

CASE 2. $f \in (V(G_2)-\{y\}) \cup (E(G_2)-\{(y,y_i)|1 \le i \le 4\})$.

Consider $f \notin \{y_1, y_2\}$. Let H_2 be a Hamiltonian cycle in $G_2 - f$. We first assume that H_2 contains both (y, y_3) and (y, y_4). By (A3), there is a Hamiltonian path P_2 in $G_2 - f$ from y_1 to y_2 containing (y, y_3) and (y, y_4). Since G_1 is 1-Hamiltonian, there is a Hamiltonian cycle H_1 of $G_1 - x_3$ which can be written as $\langle x, x_2 \to P_1 \to x_1, x\rangle$, where P_1 is a path of $G_1 - x_3$. Let P_3 denote the path obtained from P_2 by replacing the edges (y, y_3) and (y, y_4) with (x_3, y_3) and (x_3, y_4). Then, $\langle y_1 \to P_3 \to y_2, x_2 \to P_1 \to x_1, y_1\rangle$ is a Hamiltonian cycle of $K - f$. Assume that H_2 contains at most one of (y, y_3) and (y, y_4). Since y is of degree 4, H_2 contains at least one of (y, y_1) and (y, y_2). If H_2 contains both (y, y_1) and (y, y_2), then let H_1 be a Hamiltonian cycle of $G_1 - (x, x_3)$. If H_2 contains (y, y_i) for $i = 1$ or 2, then let H_1 be a Hamiltonian cycle of $G_1 - (x, x_{3-i})$. Using similar arguments in Case 1, we can find a Hamiltonian cycle in $K - f$.

Consider $f \in \{y_1, y_2\}$. By (A4), there is a Hamiltonian cycle in $G_2 - y_i$ containing the edge (y, y_{3-i}), where $i \in \{1,2\}$. In other words, this Hamiltonian cycle contains either (y, y_3) or (y, y_4). Similar to the above-mentioned case, we can find a Hamiltonian cycle in $K - f$.

CASE 3. $f \in \{(x_i, y_i)|1 \le i \le 3\} \cup \{(x_3, y_4)\}$.

Consider $f = (x_1, y_1)$ or (x_2, y_2). Let H_1 be a Hamiltonian cycle of $G_1 - (x, x_i)$ with $i \in \{1,2\}$ which can be written as $H_1 = \langle x, x_{3-i} \to P_1 \to x_3, x\rangle$, where P_1 is a path in G_1. By (A1), there is a Hamiltonian cycle H_2 of $G_2 - (y, y_i)$ containing the edge (y, y_{3-i}) which is given by $H_2 = \langle y, y_j \to P_2 \to y_{3-i}, y\rangle$, where $j \in \{3,4\}$ and P_2 is a path in G_2. Then, $\langle x_3, y_j \to P_2 \to y_{3-i}, x_{3-i} \to P_1 \to x_3\rangle$ is a Hamiltonian cycle of $K - f$.

Consider $f = (x_3, y_3)$ or (x_3, y_4). Let H_1 be a Hamiltonian cycle of $G_1 - (x, x_3)$ which can be written as $H_1 = \langle x, x_2 \to P_1 \to x_1, x\rangle$, where P_1 is a path in G_1. By (A2), there is a Hamiltonian cycle H_2 in G_2 containing the edges (y, y_1) and (y, y_2) which is given by $H_2 = \langle y, y_1 \to P_2 \to y_2, y\rangle$, where P_2 is a path in G_2. Then, $\langle x_1, y_1 \to P_2 \to y_2, x_2 \to P_1 \to x_1\rangle$ is a Hamiltonian cycle of $K - f$.

Hence, this theorem follows. ∎

However, it is, in general, difficult to verify whether G_1 and G_2 satisfy the conditions (A1), (A2), (A3), and (A4). Nonetheless, some specific graphs can be easily verified to satisfy these conditions. An example is shown in the proof of the following theorem which states the construction of $H(k)$ for $k \ge 5$ and odd as proposed in [1,2]:

Theorem 6. *Let $\{x, x_1, x_2, x_3\}$ be the vertex set of K_4. Let $H(5)$ be given as shown in Figure 3. Then, $H(k)$ for $k \ge 7$ and odd can be obtained by a (3,4)-join at the vertex x in K_4 and the vertex 0 in $H(k-2)$. Furthermore, $H(k)$ is 1-Hamiltonian.*

Proof. Although the ordering of $N(x)$ in K_4 can be arbitrary, we arbitrarily define $N(x) = \{x_1, x_2, x_3\}$. Let $N(0)$ in $H(5)$ be given by $N(0) = \{1,2,3,4\}$. By performing a (3,4)-join on K_4 and $H(5)$ at the vertices x and 0, we obtain a graph which is isomorphic to $H(7)$ by relabeling x_3 with 0, x_1 with 1, x_2 with 2, and i with $i+2$ for $1 \le i \le 4$.

Suppose that $H(l)$ is a (3,4)-join on K_4 and $H(l-2)$ for $l \ge 7$. Let $N(0)$ in $H(l)$ be given by $N(0) = \{1, 2, l-2, l-1\}$. Similarly, by performing a (3,4)-join on the vertex x in K_4 and the vertex 0 in $H(l)$, we obtain a graph which is isomorphic to $H(l+2)$ by relabeling x_3 with 0, x_1 with 1, x_2 with 2, and i with $i+2$ for all $1 \le i \le l-1$. Hence, $H(k)$ can be obtained from K_4 and $H(k-$

2) by performing a (3,4)-join on the vertex x and the vertex 0.

It can be verified that $H(5)$ is 1-Hamiltonian. To prove that $H(k)$ for $k \geq 7$ is 1-Hamiltonian, we need to show that (A1), (A2), (A3), and (A4) are satisfied. To this end, let $f_1 \in \{x_1, x_2, x_3\} \cup E(K_4), f_2 \in (V(H(k)) - \{0\}) \cup E(H(k))$, and $N(0) = \{1, 2, k-2, k-1\}$ in $H(k)$. To show that conditions (A1) and (A2) are satisfied, we need to construct a Hamiltonian cycle in $H(k) - (0, 1)$ containing the edge (0, 2), a Hamiltonian cycle in $H(k) - (0, 2)$ containing the edge (0, 1), and a Hamiltonian cycle in $H(k)$ containing the two edges (0, 1) and (0, 2).

The desired Hamiltonian cycle in $H(k) - (0, 1)$ can be constructed as follows:

$$\langle 1, 3, 4, 6, 5, 7, 8, \ldots, k-3, k-1, k-2, 0, 2, 1 \rangle \quad \text{if } (k-1)/2 \text{ is odd},$$
$$\langle 1, 3, 4, 6, 5, 7, 8, \ldots, k-4, k-2, k-1, 0, 2, 1 \rangle \quad \text{if } (k-1)/2 \text{ is even}.$$

Also, the desired Hamiltonian cycle in $H(k) - (0, 2)$ can be constructed as follows:

$$\langle 1, 2, 4, 3, 5, 6, \ldots, k-4, k-2, k-1, 0, 1 \rangle \quad \text{if } (k-1)/2 \text{ is odd},$$
$$\langle 1, 2, 4, 3, 5, 6, \ldots, k-3, k-1, k-2, 0, 1 \rangle \quad \text{if } (k-1)/2 \text{ is even}.$$

Furthermore, we can construct a Hamiltonian cycle in $H(k)$ as follows:

$$\langle 0, 1, 3, \ldots, k-2, k-1, k-3, \ldots, 2, 0 \rangle,$$

which contains the two edges (0, 1) and (0, 2). Therefore, (A1) and (A2) are satisfied.

Let $\mathscr{H}$ be any Hamiltonian cycle in $H(k) - f_2$ for $f_2 \in (V(H(k)) - \{0, 1, 2\}) \cup E(H(k))$ containing the edges $(0, k-2)$ and $(0, k-1)$. Since the vertices 1 and 2 are of degree 3, $\mathscr{H}$ contains $\langle 3, 1, 2, 4 \rangle$ as a subpath. Then, $\mathscr{H} - (1, 2)$ is the desired Hamiltonian path in $H(k) - f_2$ containing the edges $(0, k-2)$ and $(0, k-1)$. It follows that (A3) is satisfied.

To satisfy (A4), we construct a Hamiltonian cycle of $H(k) - 1$ containing (0, 2) as follows:

$$\langle 3, 5, 6, 8, 7, \ldots, k-2, k-1, 0, 2, 4, 3 \rangle \quad \text{if } (k-1)/2 \text{ is odd},$$
$$\langle 3, 5, 6, 8, 7, \ldots, k-3, k-1, k-2, 0, 2, 4, 3 \rangle \quad \text{if } (k-1)/2 \text{ is even},$$

and the desired Hamiltonian cycle of $H(k) - 2$ containing (0, 1) is

$$\langle 4, 6, 5, 7, 8, \ldots, k-3, k-1, k-2, 0, 1, 3, 4 \rangle \quad \text{if } (k-1)/2 \text{ is odd},$$
$$\langle 4, 6, 5, 7, 8, \ldots, k-4, k-2, k-1, 0, 1, 3, 4 \rangle \quad \text{if } (k-1)/2 \text{ is even}.$$

It follows that (A4) is satisfied.

Therefore, $H(k)$ is 1-Hamiltonian following from recursively applying Theorem 5. ∎

Theorem 6 states that $H(k)$ for $k \geq 7$ and odd proposed by Harary and Hayes [1,2] can be obtained from a (3,4)-join of K_4 and $H(k-2)$.

3. CYCLE EXTENSION

Let G be a graph and $C = \langle x_0, x_1, \ldots, x_{k-1}, x_0 \rangle$ be a cycle of G, where $k \geq 3$ is an arbitrary integer. We introduce an operation called *cycle extension* which includes two aspects: First, augment G by replacing each edge in C with a path of odd length and, second, add a new cycle to the augmented graph in a specific manner. To be specific, we define cycle extension as follows:

Definition 4. *The cycle extension of G around C is a graph denoted by $Ext_C(G)$ and given as follows*:

$$V(Ext_C(G)) = \bigcup_{0 \leq i \leq k-1} \{p_{i,j}, q_{i,j} \mid \forall 1 \leq j \leq l_i\} \cup V(G),$$

and

$$\begin{aligned} E(Ext_C(G)) = {} & (E(G) - E(C)) \\ & \cup \bigcup_{0 \leq i \leq k-1} \{(p_{i,j}, q_{i,j}) \mid \forall 1 \leq j \leq l_i\} \\ & \cup \bigcup_{0 \leq i \leq k-1} [\{(x_i, p_{i,1}), (p_{i,l_i}, x_{i+1})\} \\ & \cup \{(p_{i,j}, p_{i,j+1}) \mid \forall 1 \leq j \leq l_i - 1\}] \\ & \cup \bigcup_{0 \leq i \leq k-1} [\{(q_{i,j}, q_{i,j+1}) \mid \forall 1 \leq j \leq l_i - 1\} \\ & \cup \{(q_{i,l_i}, q_{i+1,1})\}], \end{aligned}$$

where l_i is even for all i.

We call the cycle induced by the vertices $\cup_{0 \leq i \leq k-1} \{q_{i,j} \mid \forall 1 \leq j \leq l_i\}$ the *extended cycle* of C, denoted by $\mathscr{O}_C$. An example of $Ext_C(G)$ is illustrated in Figure 9, where C is represented by darkened edges, and $\mathscr{O}_C$, by dashed edges. Throughout this section, we adopt the following notion:

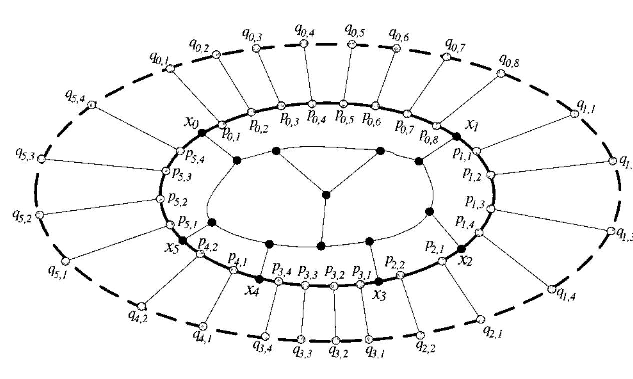


FIG. 9. An example of $Ext_C(G)$.

- $C = \langle x_0, x_1, \ldots, x_{k-1}, x_0 \rangle$,
- l_i: an even integer which is the number of vertices in $Ext_C(G)$ added between x_i and x_{i+1},
- $p_{i,j}$: a vertex in $Ext_C(G)$ added between x_i and x_{i+1}, and
- $q_{i,j}$: the vertex in $Ext_C(G)$ that is adjacent to $p_{i,j}$.

Many known families of 1-Hamiltonian graphs are trivalent graphs. It can be easily verified that if G is a trivalent graph then $Ext_C(G)$ is also a trivalent graph. Since we are interested in constructing 1-Hamiltonian graphs, we focus our discussion of cycle extensions on **trivalent graphs** only. Henceforth, we suppose that G is trivalent in the following discussion of cycle extensions.

In the following discussion, we use z_i to denote the unique neighbor of x_i that is not in C. The addition and subtraction involved in the subscript or index of a vertex in a cycle is taken modulo k, where k denotes the length of the cycle or k is clear from context without ambiguity. For example, (x_i, x_{i+1}) for $i = k-1$ in C is simply (x_{k-1}, x_0), and $(q_{i-1,l_{i-1}}, q_{i,1})$ in $\mathcal{O}_C$ for $i = 0$ is simply $(q_{k-1,l_{k-1}}, q_{0,1})$.

Let H be a cycle of G. We use H_C^* to denote the cycle in $Ext_C(G)$ obtained from H by replacing all (x_j, x_{j+1}) in $E(H) \cap E(C)$ with $\langle x_j,\ p_{j,1},\ p_{j,2}, \ldots, p_{j,l_j},\ x_{j+1} \rangle$. We use Ω_H to denote the cycle obtained from $\mathcal{O}_C$ by replacing every $\langle q_{i,1},\ q_{i,2},\ q_{i,3}, \ldots, q_{i,l_i} \rangle$ with $\langle q_{i,1},\ p_{i,1},\ p_{i,2},\ q_{i,2},\ q_{i,3},\ p_{i,3}, \ldots, p_{i,l_i},\ q_{i,l_i} \rangle$ if $\langle p_{i,1}, p_{i,2}, \ldots, p_{i,l_i} \rangle$ is not a subpath in H_C^*. In what follows, we introduce six operations $M_1(H,e,j), M_2(H,e)$, and $M_i(H,x)$ for $3 \leq i \leq 6$ that augment the cycle H of G to a cycle of $Ext_C(G)$ with respect to some edge $e = (x_i, x_{i+1})$ or vertex $x = x_i$ or $x = x_{i+1}$ in C and a specific j. We use $M_1(H,e,j), M_2(H,e)$, and $M_i(H,x)$ to mean the operation and the corresponding cycle interchangeably.

For ease of exposition, we define

$$\Omega_{q_i} = \langle q_{i,1}, p_{i,1}, p_{i,2}, q_{i,2}, q_{i,3}, p_{i,3}, \ldots, p_{i,l_i}, q_{i,l_i} \rangle \quad \text{and}$$
$$\Omega_{p_i} = \langle p_{i,1}, q_{i,1}, q_{i,2}, p_{i,2}, p_{i,3}, q_{i,3} \cdots, q_{i,l_i}, p_{i,l_i} \rangle,$$

which will be used in the definitions of operations M_2, M_3, M_4, M_5, and M_6. To be specific, the six operations are defined as follows:

Operation M_1. Given a cycle H of G which contains the edge e, we define an operation $M_1(H,e,j)$ to construct a cycle in $Ext_C(G) - \{(p_{i,j}, p_{i,j+1}), (q_{i,j}, q_{i,j+1})\}$ for some $1 \leq j \leq l_i - 1$.

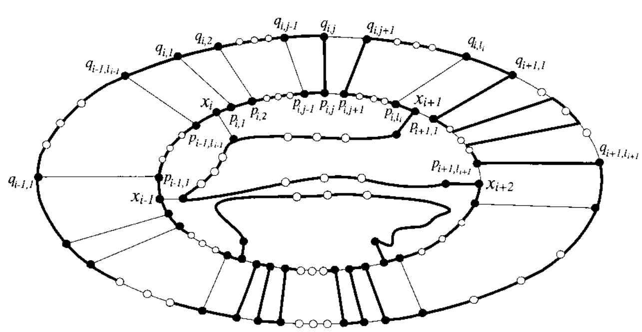


FIG. 10. Illustration for operation M_1.

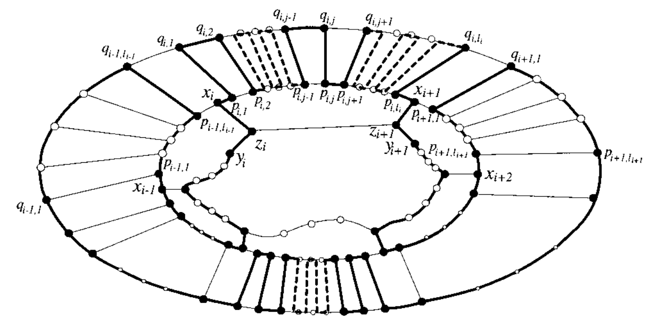


FIG. 11. Illustration for operation M_2.

Let Q be the path $\Omega_H - (q_{i,j}, q_{i,j+1})$ and P be the path $H_C^* - (p_{i,j}, p_{i,j+1})$. We define

$$M_1(H,e,j) = \langle p_{i,j} \to P \to p_{i,j+1}, q_{i,j+1} \to Q \to q_{i,j}, p_{i,j} \rangle,$$

as illustrated in Figure 10.

Operation M_2. Given that z_i and z_{i+1} are adjacent in G and given a Hamiltonian cycle H of G containing $\langle x_{i-1}, x_i, x_{i+1}, x_{i+2} \rangle$ as a subpath, we define an operation $M_2(H,e)$ to construct a Hamiltonian cycle of $Ext_C(G) - \{(q_{i-1,l_{i-1}}, q_{i,1}), (q_{i,l_i}, q_{i+1,1})\}$.

Let y_i be the unique neighbor of z_i different from x_i and z_{i+1}, and y_{i+1} be the unique neighbor of z_{i+1} different from x_{i+1} and z_i. Since G is trivalent and H contains $\langle x_{i-1}, x_i, x_{i+1}, x_{i+2} \rangle$ as a subpath, $\langle y_i, z_i, z_{i+1}, y_{i+1} \rangle$ is a subpath of H in G. Moreover, Ω_H contains $\langle q_{i-1,l_{i-1}}, q_{i,1}, \ldots, q_{i,l_i}, q_{i+1,1} \rangle$ as a subpath and H_C^* contains $\langle p_{i-1,l_{i-1}}, x_i, p_{i,1}, p_{i,2}, \ldots, p_{i,l_i}, x_{i+1}, p_{i+1,1} \rangle$ as a subpath. Hence, we can obtain a path P from H_C^* by replacing $\langle p_{i-1,l_{i-1}}, x_i, p_{i,1}, p_{i,2}, \ldots, p_{i,l_i}, x_{i+1}, p_{i+1,1} \rangle$ with $\langle x_i, p_{i,1} \to \Omega_{p_i} \to p_{i,l_i}, x_{i+1} \rangle$ and replacing (z_i, z_{i+1}) with (z_i, x_i) and (z_{i+1}, x_{i+1}). Deleting $\langle q_{i-1,l_{i-1}}, q_{i,1}, \ldots, q_{i,l_i}, q_{i+1,1} \rangle$ from Ω_H yields a path Q. Then, we define

$$M_2(H,e) = \langle p_{i-1,l_{i-1}}, q_{i-1,l_{i-1}} \to Q \to q_{i+1,1}, p_{i+1,1} \to P \to p_{i-1,l_{i-1}} \rangle,$$

as illustrated in Figure 11.

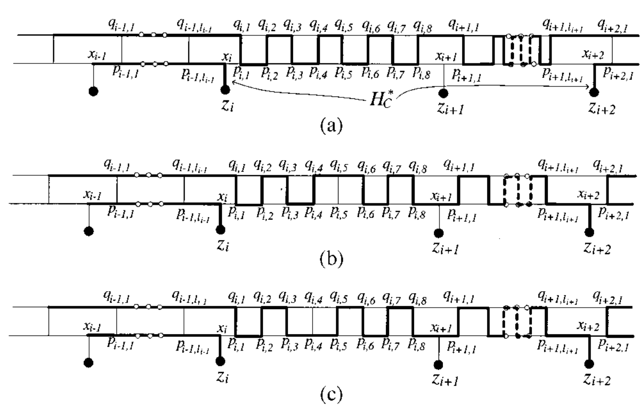


FIG. 12. Illustration for operations M_3 and M_4.

Operations M_3 and M_4. Given a Hamiltonian cycle H of $G - x_{i+1}$, we define an operation $M_3(H, x_{i+1})$ to construct a Hamiltonian cycle of $Ext_C(G) - p_{i,j}$ with j odd and an operation $M_4(H, x_{i+1})$ to construct a Hamiltonian cycle of $Ext_C(G) - q_{i,j}$ with j even.

Since x_{i+1} is not in H, H contains $\langle x_{i-1}, x_i, z_i \rangle$ and $\langle x_{i+3}, x_{i+2}, z_{i+2} \rangle$ as subpaths. See Figure 12(a) for an illustration. Moreover, $\langle q_{i,1} \to \Omega_{q_i} \to q_{i,l_i}, q_{i+1,1} \to \Omega_{q_{i+1}} \to q_{i+1,l_{i+1}} \rangle$ is a subpath of Ω_H. Let $Q = \Omega_H - \langle q_{i,1} \to \Omega_{q_i} \to q_{i,l_i}, q_{i+1,1} \to \Omega_{q_{i+1}} \to q_{i+1,l_{i+1}}, q_{i+2,1} \rangle$, which is a path from $q_{i+2,1}$ to $q_{i,1}$. Let $P = H_C^* - (x_{i+2}, p_{i+2,1})$, which is a path from x_{i+2} to $p_{i+2,1}$. Then, we define

$$\begin{aligned} M_3(H,e) &= \langle p_{i+2,1}, q_{i+2,1} \to Q \to q_{i,1}, p_{i,1}, p_{i,2}, q_{i,2}, \ldots, \\ &\quad p_{i,j-1}, q_{i,j-1}, q_{i,j}, q_{i,j+1}, p_{i,j+1}, p_{i,j+2}, \\ &\quad q_{i,j+2}, \ldots, q_{i,l_i}, p_{i,l_i}, x_{i+1}, p_{i+1,1} \to \Omega_{p_{i+1}} \to \\ &\quad p_{i+1,l_{i+1}}, x_{i+2} \to P \to p_{i+2,1} \rangle, \\ M_4(H,e) &= \langle p_{i+2,1}, q_{i+2,1} \to Q \to q_{i,1}, p_{i,1}, p_{i,2}, q_{i,2}, \ldots, \\ &\quad q_{i,j-1}, p_{i,j-1}, p_{i,j}, p_{i,j+1}, q_{i,j+1}, q_{i,j+2}, \\ &\quad p_{i,j+2}, \ldots, p_{i,l_i}, x_{i+1}, p_{i+1,1} \to \Omega_{p_{i+1}} \to \\ &\quad p_{i+1,l_{i+1}}, x_{i+2} \to P \to p_{i+2,1} \rangle, \end{aligned}$$

as illustrated in Figure 12(b) and (c).

Operations M_5 and M_6. Given a Hamiltonian cycle H of $G - x_i$, as illustrated in Figure 13(a), we define an operation $M_5(H, x_i)$ to construct a Hamiltonian cycle of $Ext_C(G) - p_{i,j}$ with j even and an operation $M_6(H, x_i)$ to construct a Hamiltonian cycle of $Ext_C(G) - q_{i,j}$ with j odd.

Let $Q = \Omega_H - \langle q_{i-2,l_{i-2}}, q_{i-1,1} \to \Omega_{q_{i-1}} \to q_{i-1,l_{i-1}}, q_{i,1} \to \Omega_{q_i} \to q_{i,l_i} \rangle$, which is a path from q_{i,l_i} to $q_{i-2,l_{i-2}}$. Let $P = H_C^* - (x_{i-1}, p_{i-2,l_{i-2}})$, which is a path from $p_{i-2,l_{i-2}}$ to x_{i-1}. We define

$$\begin{aligned} M_5(H,x) &= \langle x_{i-1}, p_{i-1,1} \to \Omega_{p_{i-1}} \to p_{i-1,l_{i-1}}, x_i, p_{i,1}, q_{i,1}, \\ &\quad q_{i,2}, p_{i,2}, p_{i,3}, \ldots, p_{i,j-1}, q_{i,j-1}, q_{i,j}, q_{i,j+1}, \\ &\quad p_{i,j+1}, p_{i,j+2}, q_{i,j+2}, \ldots, q_{i,l_i} \to Q \to q_{i-2,l_{i-2}}, \\ &\quad p_{i-2,l_{i-2}} \to P \to x_{i-1} \rangle, \\ M_6(H,x) &= \langle x_{i-1}, p_{i-1,1} \to \Omega_{p_{i-1}} \to p_{i-1,l_{i-1}}, x_i, p_{i,1}, \\ &\quad q_{i,1}, q_{i,2}, p_{i,2}, p_{i,3}, \ldots, q_{i,j-1}, p_{i,j-1}, p_{i,j}, p_{i,j+1}, \\ &\quad q_{i,j+1}, q_{i,j+2}, p_{i,j+2}, \ldots, q_{i,l_i} \to Q \to q_{i-2,l_{i-2}}, \\ &\quad p_{i-2,l_{i-2}} \to P \to x_{i-1} \rangle, \end{aligned}$$

as illustrated in Figure 13(b) and (c).

These six operations are used in the proofs of the following lemmas and theorems:

Lemma 1. *Let G be a trivalent 1-vertex-Hamiltonian graph and $C = \langle x_0, x_1, \ldots, x_{k-1}, x_0 \rangle$ be a cycle of G. Then, $Ext_C(G)$ is trivalent 1-vertex-Hamiltonian.*

Proof. Consider $f \in V(G) - V(C)$. Since G is trivalent 1-vertex-Hamiltonian, there is a Hamiltonian cycle H of $G - f$ such that at least an edge in C, say (x_i, x_{i+1}), is in H. Therefore, $M_1(H, (x_i, x_{i+1}), j), 1 \le j \le l_i - 1$, forms a Hamiltonian cycle of $Ext_C(G) - f$.

Consider $f = x_i$ for $0 \le i \le k-1$. Since G is trivalent and $k \ge 3$, two vertices x_{i-1} and x_{i+1} in C have degree 2 in $G - f$. It follows that we can always find a Hamiltonian cycle H of $G - f$ such that H contains an edge in C, say (x_j, x_{j+1}), for $j \ne i-1, i$. On the other hand, Ω_H contains the subpath $\langle q_{i-2,l_{i-2}}, q_{i-1,1} \to \Omega_{q_{i-1}} \to q_{i-1,l_{i-1}}, q_{i,1} \to \Omega_{q_i} \to q_{i,l_i}, q_{i+1,1} \rangle$ which does not contain the vertex x_i. Therefore, $M_1(H, (x_j, x_{j+1}), j'), 1 \le j' \le l_j - 1$, is a Hamiltonian cycle of $Ext_C(G) - f$.

Consider $f \in \{p_{i,j}, p_{i,j'}, q_{i,j}, q_{i,j'}\}$ for some $1 \le j, j' \le l_i$ with j odd and j' even. Since G is 1-vertex-Hamiltonian, there are Hamiltonian cycles H_1 and H_2 of $G - x_{i+1}$ and $G - x_i$, respectively. It follows that the operations $M_3(H_1, x_{i+1}), M_4(H_1, x_{i+1}), M_5(H_2, x_i)$, and $M_6(H_2, x_i)$ can be applied and they are, indeed, Hamiltonian cycles of $Ext_C(G) - p_{i,j}, Ext_C(G) - q_{i,j'}, Ext_C(G) - p_{i,j'}$, and $Ext_C(G) - q_{i,j}$, respectively. Hence, the lemma follows. ∎

Lemma 2. *Let G be a trivalent 1-edge-Hamiltonian graph and $C = \langle x_0, x_1, \ldots, x_{k-1}, x_0 \rangle$ be a cycle of G. If $f \in E(Ext_C(G)) - \cup_{0 \le i \le k-1} \{(q_{i,l_i}, q_{i+1,1})\}$, then $Ext_C(G) - f$ is Hamiltonian.*

Proof. Consider that f is an edge in $E(G) - E(C)$. Since G is trivalent 1-edge-Hamiltonian, there is a Hamiltonian cycle H of $G - f$ such that H contains at least an edge in C, say (x_i, x_{i+1}), with $0 \le i \le k-1$. It follows that $M_1(H, (x_i, x_{i+1}), j)$ with $1 \le j \le l_i - 1$ can be applied and yields a Hamiltonian cycle of $Ext_C(G) - f$.

Consider $f = (x_i, p_{i,1})$ or (p_{i,l_i}, x_{i+1}) for some $0 \le i \le k-1$. Since G is 1-edge-Hamiltonian, there is a Hamiltonian cycle H of $G - (x_i, x_{i+1})$. It follows from the trivalence of G that H contains (x_{i-1}, x_i) and (x_{i+1}, x_{i+2}). Furthermore, Ω_H contains $\langle q_{i-1,l_{i-1}}, q_{i,1} \to \Omega_{q_i} \to q_{i,l_i}, q_{i+1,1} \rangle$ as a subpath. On the other hand, both subpaths $\langle x_{i-1}, x_i, z_i \rangle$ and $\langle z_{i+1}, x_{i+1}, x_{i+2} \rangle$ are in H. Then, we can apply the operation M_1 on (x_{i-1}, x_i) and j for

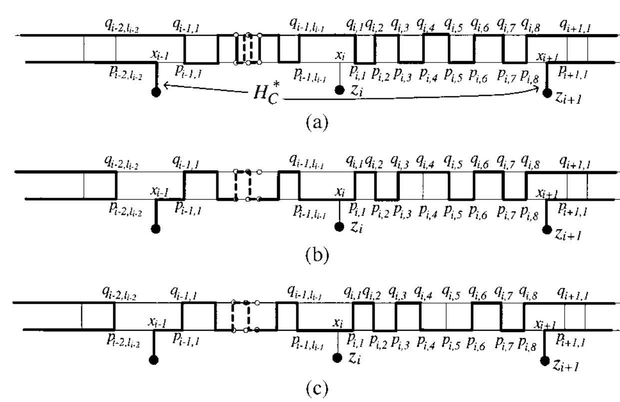


FIG. 13. Illustration for Operations M_5 and M_6.

$1 \le j \le l_{i-1}-1$. It follows that $M_1(H,(x_{i-1},x_i),j)$ forms a Hamiltonian cycle of $Ext_C(G) - \{(x_i,p_{i,1}),(p_{i,l_i},x_{i+1})\}$.

Consider $f = (p_{i,j},p_{i,j+1})$ or $(q_{i,j},q_{i,j+1})$ for some $0 \le i \le k-1$ and some $1 \le j \le l_i - 1$. Let H be a Hamiltonian cycle of $G - (x_{i-1},x_i)$. It follows that (x_i,x_{i+1}) is in H. Therefore, $M_1(H,(x_i,x_{i+1}),j)$ with $1 \le j \le l_i - 1$ forms a Hamiltonian cycle of $Ext_C(G) - \{(p_{i,j},p_{i,j+1}),(q_{i,j},q_{i,j+1})\}$.

Consider $f = (p_{i,j},q_{i,j})$ for some $0 \le i \le k-1$ and some $1 \le j \le l_i$. Let H be a Hamiltonian cycle of $G - (x_{i-1},x_i)$. It follows that both (x_i,x_{i+1}) and (x_{i-2},x_{i-1}) are in H. Moreover, H_C^* contains $\langle x_i,p_{i,1},p_{i,2},\ldots,p_{i,l_i},x_{i+1}\rangle$ as a subpath and Ω_H contains $\langle q_{i-1,l_{i-1}},q_{i,1},q_{i,2},\ldots,q_{i,l_i},q_{i+1,1}\rangle$ as a subpath. Hence, $M_1(H,(x_{i-2},x_{i-1}),j')$ for $1 \le j' \le l_{i-2}-1$ forms a Hamiltonian cycle of $Ext_C(G) - (p_{i,j},q_{i,j})$. The lemma is proved. ∎

One may ask whether $Ext_C(G)$ is 1-Hamiltonian if G is trivalent 1-Hamiltonian. The answer is no and it can be verified by a counterexample shown in Figure 14. The graphs shown in Figure 14, proposed by Wang et al. [6], are called *eye networks* and denoted by $Eye(n)$ for $n \ge 1$. The graph $Eye(1)$ shown in Figure 14 is a trivalent 1-Hamiltonian graph. Let O_1 be the cycle indicated by darken edges in $Eye(1)$ as shown in Figure 14. Note that $Eye(2) = Ext_{O_1}(Eye(1))$. Although $Eye(1)$ is trivalent 1-Hamiltonian, $Eye(2)$ is not 1-Hamiltonian since we cannot find a Hamiltonian cycle in $Eye(2) - ((2,0),(2,3)), Eye(2) - ((2,6),(2,9))$, and $Eye(2) - ((2,12),(2,15))$.

Although that G is trivalent 1-Hamiltonian does not imply that $Ext_C(G)$ is 1-Hamiltonian, we are interested in finding a sufficient condition on cycle C for $Ext_C(G)$ to be 1-Hamiltonian. To this end, we define the *recoverable set* $R(C)$ of cycle C as follows:

$$R(C) = \{(x_i,x_{i+1}) \mid \langle x_{i-1},x_i,x_{i+1},x_{i+2}\rangle \text{ is a subpath of some Hamiltonian cycle of } G \text{ and } (z_i,z_{i+1}) \text{ is an edge in } G \text{ for } 0 \le i \le k-1\}.$$

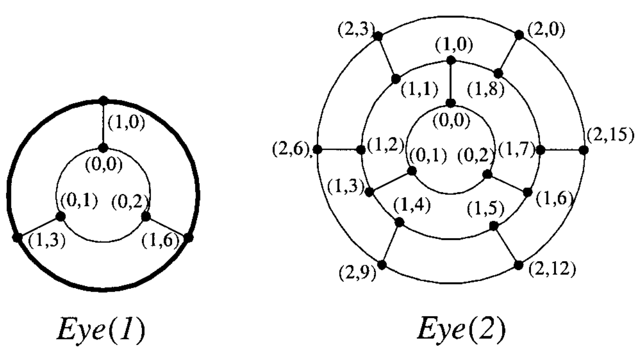


FIG. 14. Eye networks.

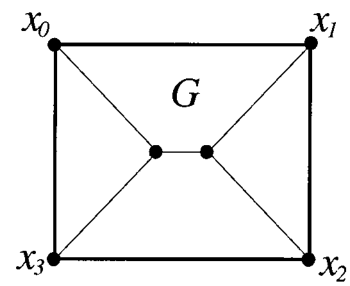


FIG. 15. The cycle $\langle x_0,x_1,x_2,x_3,x_0\rangle$ is recoverable with respect to the graph G.

The definition of $R(C)$, indeed, arises from the given condition for the operation M_2 to be applicable. A cycle C of G is said to be *recoverable* with respect to G if no two edges of $E(C) - R(C)$ are adjacent. For example, the cycle $C = \langle x_0,x_1,x_2,x_3,x_0\rangle$ shown in Figure 15 is recoverable with respect to G since $\{(x_0,x_1),(x_2,x_3)\}$ is the recoverable set of C.

Theorem 7. *Let G be a trivalent 1-Hamiltonian graph and $C = \langle x_0,x_1,\ldots,x_{k-1},x_0\rangle$ be recoverable with respect to G. Then, $Ext_C(G)$ is trivalent 1-Hamiltonian.*

Proof. It follows from Lemma 1 that $Ext_C(G)$ is a 1-vertex-Hamiltonian graph. Furthermore, it follows from Lemma 2 that $Ext_C(G) - f$ is Hamiltonian for $f \in E(Ext_C(G)) - \cup_{0\le i\le k-1}\{(q_{i,l_i},q_{i+1,1})\}$. Now, it suffices to show that $Ext_C(G) - f$ is Hamiltonian for $f \in \cup_{0\le i\le k-1}\{(q_{i,l_i},q_{i+1,1})\}$. Equivalently, we will construct a Hamiltonian cycle in $Ext_C(G)$ without using $(q_{i,l_i},q_{i+1,1})$ for $0 \le i \le k-1$.

Since C is recoverable with respect to G, it follows that $R(C) \ne \emptyset$ and we have either $(x_i,x_{i+1}) \in R(C)$ or $(x_i,x_{i+1}) \notin R(C)$. For $(x_j,x_{j+1}) \in R(C)$, we use H_j to denote a Hamiltonian cycle of G such that H_j contains $\langle x_{j-1},x_j,x_{j+1},x_{j+2}\rangle$ as a subpath. First, consider $(x_i,x_{i+1}) \in R(C)$. Then, the operation M_2 can be applied and $M_2(H_i,(x_i,x_{i+1}))$ is a Hamiltonian cycle of $Ext_C(G) - \{(q_{i-1,l_{i-1}},q_{i,1}),(q_{i,l_i},q_{i+1,1})\}$. Next, consider $(x_i,x_{i+1}) \notin R(C)$. Since any two edges in $E(C) - R(C)$ are not adjacent, it follows that (x_{i+1},x_{i+2}) is in $R(C)$. Then, $M_2(H_{i+1},(x_{i+1},x_{i+2}))$ forms a Hamiltonian cycle of $Ext_C(G) - \{(q_{i,l_i},q_{i+1,1}),(q_{i+1,l_{i+1}},q_{i+2,1})\}$ for $(x_{i+1},x_{i+2}) \in R(C)$. Therefore, when G is 1-Hamiltonian and C is recoverable with respect to $G, Ext_C(G)$ is 1-edge-Hamiltonian. Thus, the theorem follows. ∎

Although it is, in general, hard to verify whether a cycle C is recoverable with respect to G, we show that $\mathcal{O}_C$ is recoverable with respect to $Ext_C(G)$ in the following lemma:

Lemma 3. *Let G be a trivalent graph and $C = \langle x_0,x_1,\ldots,x_{k-1},x_0\rangle$ be a cycle of G. If $G - (x_i,x_{i+1})$ is Hamiltonian for every $0 \le i \le k-1$, then $\mathcal{O}_C$ is recoverable with respect to $Ext_C(G)$.*

Proof. It suffices to show that $R(\mathcal{O}_C)$ is a collection of $(q_{i,j}, q_{i,j+1})$ for all $0 \le i \le k-1$ and $1 \le j \le l_i - 1$ since each $p_{i,j}$ is adjacent to $p_{i,j+1}$ in $Ext_C(G)$ but p_{i,l_i} is not adjacent to $p_{i+1,1}$.

Let H be a Hamiltonian cycle of $G - (x_{i-1}, x_i)$. Since G is trivalent, both (x_{i-2}, x_{i-1}) and (x_i, x_{i+1}) are in H. It follows that Ω_H contains $\langle q_{i-3,l_{i-3}},\ q_{i-2,1}, q_{i-2,2}, \ldots, q_{i-2,l_{i-2}},\ q_{i-1,1}\rangle$ and $\langle q_{i-1,l_{i-1}},\ q_{i,1},\ q_{i,2}, \ldots, q_{i,l_i}, q_{i+1,1}\rangle$ as subpaths. Consequently, the Hamiltonian cycle $M_1(H, (x_{i-2}, x_{i-1}), j')$, $1 \le j' \le l_{i-2} - 1$, of $Ext_C(G)$ contains $\langle q_{i-1,l_{i-1}}, q_{i,1}, q_{i,2}, \ldots, q_{i,l_i}, q_{i+1,1}\rangle$ as a subpath and, furthermore, $(q_{i,j}, q_{i,j+1})$ satisfies the definition of recoverable set $R(\mathcal{O}_C)$. Since (x_{i-1}, x_i) is an arbitrary edge and $(q_{i,l_i}, q_{i+1,1})$ and $(q_{i',l_{i'}}, q_{i'+1,1})$ are not adjacent for $i \ne i'$, $R(\mathcal{O}_C)$ is a collection of $(q_{i,j}, q_{i,j+1})$ for all $0 \le i \le k-1$ and $1 \le j \le l_i - 1$ and, moreover, $\mathcal{O}_C$ is recoverable with respect to $Ext_C(G)$. Hence, the lemma is proved. ∎

The cycle extension operation can be recursively performed: Let $Ext^0_C(G) = G, \mathcal{O}^0_C = C, Ext^1_C(G) = Ext_C(G)$, and $\mathcal{O}^1_C = \mathcal{O}_C$. Then, $Ext^n_C(G)$ for $n \ge 2$ can be recursively defined by setting $Ext^n_C(G) = Ext_{\mathcal{O}^{n-1}_C}(Ext^{n-1}_C(G))$ and $\mathcal{O}^n_C$ being the extended cycle of $\mathcal{O}^{n-1}_C$ in $Ext^n_C(G)$.

Now, consider, in particular, $Ext^2_C(G)$, which is the cycle extension of $Ext_C(G)$ around $\mathcal{O}_C$. Given $0 \le i \le k-1$ and $1 \le j \le l_i - 1$, let $l_{i,j}$ and L_i denote the number of vertices added between $q_{i,j}$ and $q_{i,j+1}$ and between q_{i,l_i} and $q_{i+1,1}$, respectively. These vertices are denoted by $r^m_{i,j}$, where $1 \le m \le l_{i,j}$ or $1 \le m \le L_i$. The vertex in $\mathcal{O}^2_C$ which is adjacent to $r^m_{i,j}$ is denoted by $s^m_{i,j}$. Note that $l_{i,j}$ and L_i are even. For ease of exposition, let $L^* = l_{i,j}$ for $1 \le j \le l_i - 1$ and $L^* = L_i$ for $j = l_i$. To be specific, the graph $Ext^2_C(G)$ is given as follows (see Figure 16):

$$V(Ext^2_C(G)) = V(Ext_C(G)) \cup \bigcup_{0\le i\le k-1} \bigcup_{1\le j\le l_i} \{r^m_{i,j}, s^m_{i,j} \mid 1 \le m \le L^*\} \quad \text{and}$$

$$\begin{aligned} E(Ext^2_C(G)) = {} & (E(Ext_C(G)) - E(\mathcal{O}_C)) \cup \bigcup_{0\le i\le k-1} \bigcup_{1\le j\le l_i} (\{(r^m_{i,j}, r^{m+1}_{i,j}), (s^m_{i,j}, s^{m+1}_{i,j}) \mid 1 \le m \le L^* - 1\} \\ & \cup \bigcup_{0\le i\le k-1} \bigcup_{1\le j\le l_i} \{(r^m_{i,j}, s^m_{i,j}) \mid 1 \le m \le L^*\} \\ & \cup \bigcup_{0\le i\le k-1} \{(q_{i,l_i}, r^1_{i,l_i}), (r^{L_i}_{i,l_i}, q_{i+1,1}), (s^{L_i}_{i,l_i}, s^1_{i+1,1})\} \\ & \cup \bigcup_{0\le i\le k-1} \bigcup_{1\le j\le l_i-1} \{(q_{i,j}, r^1_{i,j}), (r^{l_{i,j}}_{i,j}, q_{i,j+1}), (s^{l_{i,j}}_{i,j}, s^1_{i,j+1})\}. \end{aligned}$$

Using the similar proof technique in Lemma 3, we can show the following corollary:

Corollary 3. *Let G be a trivalent graph and $C = \langle x_0, x_1, \ldots, x_{k-1}, x_0\rangle$ be a cycle of G. If $G - (x_i, x_{i+1})$ is Hamiltonian for all $0 \le i \le k-1$, then $\mathcal{O}^2_C$ is recoverable with respect to $Ext^2_C(G)$.*

Lemma 4. *Let G be a trivalent 1-Hamiltonian graph and $C = \langle x_0, x_1, x_2, \ldots, x_{k-1}, x_0\rangle$ be a cycle of G. Then, $Ext^2_C(G)$ is trivalent 1-Hamiltonian.*

Proof. It follows from Lemma 1 that $Ext^2_C(G)$ is 1-vertex-Hamiltonian. It suffices to show that $Ext^2_C(G)$ is 1-edge-Hamiltonian. We divide the edge set of $Ext^2_C(G)$ into four sets:

$$\begin{aligned} \mathscr{E}_1 = {} & E(G) - E(C), \\ \mathscr{E}_2 = {} & \bigcup_{0\le i\le k-1} \{(q_{i,l_i}, r^1_{i,l_i}), (r^{L_i}_{i,l_i}, q_{i+1,1}), (s^{L_i}_{i,l_i}, s^1_{i+1,1})\} \\ & \cup \bigcup_{0\le i\le k-1} \bigcup_{1\le j\le l_i-1} \{(s^{l_{i,j}}_{i,j}, s^1_{i,j+1})\} \\ \mathscr{E}_3 = {} & \bigcup_{0\le i\le k-1} \{(x_i, p_{i,1}), (p_{i,l_i}, x_{i+1})\} \\ & \cup \bigcup_{0\le i\le k-1} \bigcup_{1\le j\le l_i-1} \{(p_{i,j}, p_{i,j+1}), (q_{i,j}, r^1_{i,j}), (r^{l_{i,j}}_{i,j}, q_{i,j+1}) |\} \\ & \cup \bigcup_{0\le i\le k-1} \bigcup_{1\le j\le l_i} \{(r^m_{i,j}, r^{m+1}_{i,j}), (s^m_{i,j}, s^{m+1}_{i,j}) | \\ & 1 \le m \le L^* - 1\}, \quad \text{and} \\ \mathscr{E}_4 = {} & \bigcup_{0\le i\le k-1} \bigcup_{1\le j\le l_i} \{(p_{i,j}, q_{i,j})\} \\ & \cup \bigcup_{0\le i\le k-1} \bigcup_{1\le j\le l_i} \{(r^m_{i,j}, s^m_{i,j}) | 1 \le m \le L^*\}. \end{aligned}$$

Consider $f \in \mathscr{E}_1$. By Lemma 2, there is a Hamiltonian cycle H in $Ext_C(G) - f$. Since $Ext_C(G)$ is trivalent, it follows that H contains at least an edge e in $\mathcal{O}_C$, say $e = (q_{i,j}, q_{i,j+1})$, for some $0 \le i \le k-1$ and $1 \le j \le l_i - 1$. Then, $M_1(H, e, m)$, $1 \le m \le l_{i,j} - 1$, is a Hamiltonian cycle of $Ext^2_C(G) - f$ for $f \in \mathscr{E}_1$.

Consider $f \in \mathscr{E}_2$. Since G is 1-Hamiltonian, there is a Hamiltonian cycle in $G - (x_{i-1}, x_i)$. It follows from the proof of Lemma 3 that we have

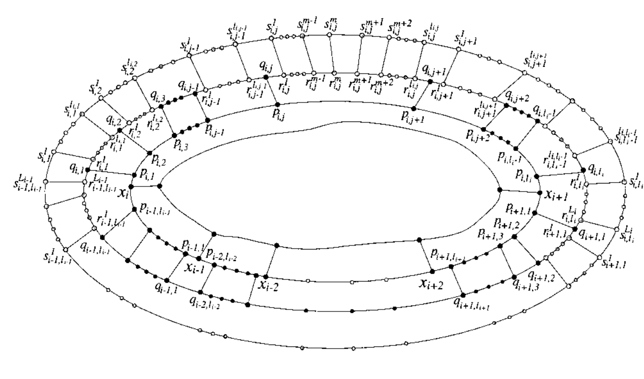


FIG. 16. An example of $Ext^2_C(G)$.

a Hamiltonian cycle, say H, in $Ext_C(G)$ containing $\langle q_{i-1,l_{i-1}}, q_{i,1}, q_{i,2}, \ldots, q_{i,l_i-1}, q_{i,l_i}, q_{i+1,1}\rangle$ as a subpath. Since $p_{i,j}$ is adjacent to $p_{i,j+1}$ for all $1 \leq j \leq l_i - 1$, we can apply the operation M_2 on any $(q_{i,j}, q_{i,j+1})$. Hence, $M_2(H, (q_{i,j}, q_{i,j+1}))$ forms a Hamiltonian cycle in $Ext_C^2(G) - \{(s_{i,j-1}^{l_{i,j-1}}, s_{i,j}^1), (s_{i,j}^{l_{i,j}}, s_{i,j+1}^1)\}$. Observing the Hamiltonian cycle $M_2(H, (q_{i,l_i-1}, q_{i,l_i}))$ which contains $\langle s_{i,l_i}^1, r_{i,l_i}^1, r_{i,l_i}^2, \ldots, r_{i,l_i}^{L_i}\rangle$ as a subpath, it follows that it is also a Hamiltonian cycle of $Ext_C^2(G) - (q_{i,l_i}, r_{i,l_i}^1)$. Similarly, we can use a Hamiltonian cycle of $G - (x_i, x_{i+1})$ to construct a Hamiltonian cycle H' of $Ext_C(G)$ which contains $\langle q_{i,l_i}, q_{i+1,1}, q_{i+1,2}, \ldots, q_{i+1,l_{i+1}}, q_{i+2,1}\rangle$ as a subpath. Therefore, $M_2(H', (q_{i+1,1}, q_{i+1,2}))$ forms a Hamiltonian cycle in $Ext_C^2(G) - \{(r_{i,l_i}^{L_i}, q_{i+1,1}), (s_{i,l_i}^{L_i}, s_{i+1,1}^1)\}$.

Consider $f \in \mathscr{E}_3$. We first consider $f = (x_i, p_{i,1})$ or (p_{i,l_i}, x_{i+1}). It follows from the proof of Lemma 2 that there is a Hamiltonian cycle H_1 in $Ext_C(G) - f$. Since $Ext_C(G)$ is trivalent, H_1 contains at least an edge $(q_{i',j}, q_{i',j+1})$ in $\mathscr{O}_C$ for some $0 \leq i' \leq k-1, i \neq i'$, and $1 \leq j \leq l_{i'} - 1$. Hence, $M_1(H_1, (q_{i',j}, q_{i',j+1}), m)$ is a Hamiltonian cycle of $Ext_C^2(G) - f$, where $1 \leq m \leq l_{i',j} - 1$. Second, we consider $f = (p_{i,j}, p_{i,j+1}), (q_{i,j}, r_{i,j}^1)$, or $(r_{i,j}^{l_{i,j}}, q_{i,j+1})$. It follows from the proof of Lemma 2 that there is a Hamiltonian cycle H_2 in $Ext_C(G) - \{(p_{i,j}, p_{i,j+1}), (q_{i,j}, q_{i,j+1})\}$. Since $Ext_C(G)$ is trivalent, H_2 contains $(q_{i,j+1}, q_{i,j+2})$. Hence, $M_1(H_2, (q_{i,j+1}, q_{i,j+2}), m)$ is a Hamiltonian cycle of $Ext_C^2(G) - \{(p_{i,j}, p_{i,j+1}), (q_{i,j}, r_{i,j}^1), (r_{i,j}^{l_{i,j}}, q_{i,j+1})\}$, where $1 \leq m \leq l_{i,j+1} - 1$. Finally, we consider $f = (r_{i,j}^m, r_{i,j}^{m+1})$ or $(s_{i,j}^m, s_{i,j}^{m+1})$. Similarly, there is a Hamiltonian cycle H_3 in $Ext_C(G) - (q_{i,j-1}, q_{i,j})$. Hence, we can apply the operation M_1 on $(q_{i,j}, q_{i,j+1})$ and m with $1 \leq m \leq l_{i,j} - 1$. Then, $M_1(H_3, (q_{i,j}, q_{i,j+1}), m)$ is a Hamiltonian cycle of $Ext_C^2(G) - f$.

Consider $f \in \mathscr{E}_4$. Since G is 1-Hamiltonian, there is a Hamiltonian cycle H of $G - (x_{i-1}, x_i)$. Then, both (x_{i-2}, x_{i-1}) and (x_i, x_{i+1}) are in H. Therefore, $H_1 = M_1(H, (x_{i-2}, x_{i-1}), j)$ for some $1 \leq j \leq l_{i-2} - 1$ is a Hamiltonian cycle of $Ext_C(G)$, which contains $\langle q_{i-1,l_{i-1}}, q_{i,1}, q_{i,2}, \ldots, q_{i,l_i}, q_{i+1,1}\rangle$ as a subpath. We then can apply the operation M_1 again on H_1 and $(q_{i-1,l_{i-1}}, q_{i,1})$. It follows that $M_1(H_1, (q_{i-1,l_{i-1}}, q_{i,1}), m)$ with $1 \leq m \leq L_{i-1} - 1$ is a Hamiltonian cycle of $Ext_C^2(G) - f$.

Therefore, $Ext_C^2(G)$ is 1-edge-Hamiltonian, and the lemma follows. ∎

Theorem 8. *Let G be a trivalent 1-Hamiltonian graph and $C = \langle x_0, x_1, x_2, \ldots, x_{k-1}, x_0\rangle$ be a cycle of G. Then, $Ext_C^n(G)$ is trivalent 1-Hamiltonian for $n \geq 2$.*

Proof. Since G is 1-Hamiltonian, it follows from Lemma 4 and Corollary 3 that $Ext_C^2(G)$ is 1-Hamiltonian and that $\mathscr{O}_C^2$ is recoverable with respect to $Ext_C^2(G)$. Assume that $Ext_C^k(G)$ is 1-Hamiltonian and $\mathscr{O}_C^k$ is recoverable with respect to $Ext_C^k(G)$ for $2 \leq k \leq l$. Consider $Ext_C^{l+1}(G) = Ext_{\mathscr{O}_C^l}(Ext_C^l(G))$. Then, by Theorem 7, $Ext_{\mathscr{O}_C^l}(Ext_C^l(G)) = Ext_C^{l+1}(G)$ is 1-Hamiltonian. Furthermore, it follows from Lemma 3 that $\mathscr{O}_C^{l+1}$ is recoverable with respect to $Ext_C^{l+1}(G)$, which makes the induction applicable. Hence, the theorem follows. ∎

Recursively applying cycle extensions, we can construct a family of 1-Hamiltonian graphs from a known trivalent 1-Hamiltonian graph. In what follows, we discuss two such families of graphs, namely, eye networks and extended Petersen graphs. The eye networks proposed by Wang et al. [6] are, indeed, constructed by recursive cycle extensions from $Eye(1)$ as shown in the next theorem. The 1-Hamiltonicity of $Eye(n)$ is then a direct consequence of Theorem 8, whereas in [6], the authors showed the 1-Hamiltonicity of $Eye(n)$ using a different approach.

Theorem 9.

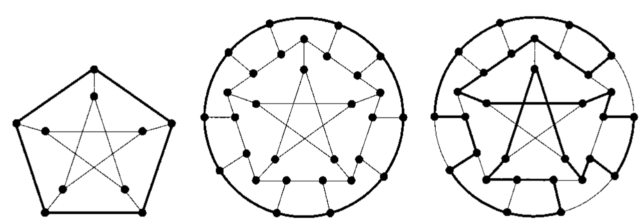
(*i*) *$Eye(n) = Ext_{O_1}^{n-1}(Eye(1))$ for $n \geq 2$, where O_1 be the outermost cycle of $Eye(1)$ (as shown by the darkened edges in Fig. 14).*
(*ii*) *$Eye(n)$ is 1-Hamiltonian for $n \geq 3$.*

Proof. It can be easily verified that $Eye(2)$ is obtained by performing a cycle extension on $Eye(1)$ around O_1, that is, $Eye(2) = Ext_{O_1}(Eye(1))$. Let O_2 be the outermost cycle of $Eye(2)$. Note that O_2 is also the extended cycle of O_1 in $Eye(2)$. Let O_n denote the outermost cycle of $Eye(n)$ for $n \geq 1$. It can be easily verified that for $n \geq 3, Eye(n) = Ext_{O_{n-1}}(Eye(n-1))$ and O_n is the extended cycle of $Ext_{O_{n-1}}(Eye(n-1))$. Thus, $Eye(n) = Ext_{O_1}^{n-1}(Eye(1))$ for $n \geq 2$.

Note that $Eye(1)$ is a 3-join of two K_4 and is also trivalent 1-Hamiltonian. It follows from Theorem 8 that $Ext_{O_1}^n(Eye(1))$, that is, $Eye(n+1)$, for $n \geq 2$, is 1-Hamiltonian. Hence, the theorem follows. ∎

By recursively applying cycle extensions, we define *extended Petersen graphs*, denoted by $EP(n)$ for $n \geq 0$, as follows:

(i) $EP(0)$ is the Petersen graph shown in Figure 17(a) with a specific cycle C indicated by the darkened edges. Define $\mathscr{O}_C^0 = C$.



(a) (b) (c)

FIG. 17. Extended Petersen graphs.

(ii) Define $EP(1) = Ext_{\mathcal{O}_C^0}(EP(0))$ as shown in Figure 17(b) and $\mathcal{O}_C^1 = \mathcal{O}_C$ shown by the darkened edges in this figure.

(iii) For $n \geq 2$, define $EP(n) = Ext_{\mathcal{O}_C^{n-1}}(EP(n-1))$ and $\mathcal{O}_C^n$ as the extended cycle of $\mathcal{O}_C^{n-1}$ in $EP(n)$. Equivalently, we write $EP(n) = Ext_C^n(EP(0))$.

It is known that the Petersen graph is 1-vertex-Hamiltonian but neither Hamiltonian nor 1-edge-Hamiltonian. We can use the properties of cycle extensions to show the 1-Hamiltonicity of $EP(n)$ as stated in the following theorem:

Theorem 10. *$EP(n)$ is 1-Hamiltonian for $n \geq 1$.*

Proof. Since the Petersen graph $EP(0)$ is 1-vertex-Hamiltonian, it follows from Lemma 1 that $EP(1)$ is also 1-vertex-Hamiltonian. Furthermore, $EP(1)$ is Hamiltonian, where a Hamiltonian cycle of $EP(1)$ is illustrated in Figure 17(c) by the darkened edges. Using proper rotation of $EP(1)$, we can always obtain a Hamiltonian cycle without containing any specific edge. It follows that $EP(1)$ is 1-edge-Hamiltonian and, thus, 1-Hamiltonian. It can also be easily verified that $\mathcal{O}_C^1$ is recoverable with respect to $EP(1)$. It follows from Theorems 7 and 8 that $EP(n)$ is 1-Hamiltonian for $n \geq 2$. ∎

4. CONCLUDING REMARKS

In this paper, we introduced three construction schemes, namely, 3-join, (3, 4)-join, and cycle extension, to construct families of 1-Hamiltonian graphs including several families already known in the literature and some new families. It follows from Corollary 1 that trivalent 1-edge-Hamiltonian graphs are closed under 3-joins. It follows from Lemma 1 that trivalent 1-vertex-Hamiltonian graphs are closed under cycle extensions. However, some families of 1-Hamiltonian graphs can be generated from performing 3-joins on 1-edge-Hamiltonian graphs and performing cycle extensions on 1-vertex-Hamiltonian graphs. For example, the 1-Hamiltonian graph shown in Figure 2(b) is obtained by performing a sequence of 3-joins on K_4 and $K_{3,3}$, where $K_{3,3}$ is 1-edge-Hamiltonian but not 1-vertex-Hamiltonian. In this paper, 3-joins and cycle extensions on trivalent graphs were studied, although these operations can be applied to arbitrary graphs. We wonder whether these operations can be applied to nontrivalent graphs and still preserve the good properties of the original graphs. On the other hand, we want to explore in the future what properties of graphs can be preserved under these operations.

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