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# An ANOVA test for the equivalency of means under unequal variances

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**Abstract**

In this paper, we present a two-stage test procedure for testing the hypothesis that the normal means are falling into a practically indifferent zone. Both the level and the power associated with the proposed test are controllable and are completely independent of the unknown variances. Relation to a single-stage procedure is discussed when the two-stage sampling procedure cannot be completely carried through. An example and tables needed for implementation are given. © 2000 Elsevier Science B.V. All rights reserved.

*Keywords:* Analysis of variance; Two stage; Single stage; Power

## 1. Introduction

It is well known that, for a large sample size (100 observations will suffice in many applications), the null hypothesis of equal means $\mu_1 = \cdots = \mu_k$ $(k \geq 2)$ can almost surely be rejected if the underlying distribution is continuous. In applications, practitioners often wish to know whether the means of interest fall into some meaningful preference region under a hypothesis. This idea leads to the interval null hypothesis $\mathrm{H}_0$: $\sum_{i=1}^{k} (\mu_i - \bar{\mu})^2/k \leq \delta^2$ against the alternative $\mathrm{H}_\mathrm{a}$: $\sum_{i=1}^{k} (\mu_i - \bar{\mu})^2/k > \delta^2$, where $\bar{\mu}$ is the average of $\mu_1, \ldots, \mu_k$ and $\delta$ $(\geq 0)$ is a zone of indifference which must be specified in advance by an expert in his experiment. The null hypothesis $\mathrm{H}_0$ can be interpreted as saying that there is little deviation among the means and the constant $\delta$ can be interpreted as the amount of variation among means about which we are

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indifferent. Stating the null hypothesis as a region rather than as a point necessitates the use of an unbiased test (see, e.g., Lehmann, 1986) which requires the evaluation of percentage points of some noncentral distributions and, especially, the calculation of a power function for a specified alternative (see, e.g., Ding, 1999).

In a one-way layout fixed-effects model, let there be $k$ treatment populations $\pi_1,\dots,\pi_k$ such that observations obtained from population $\pi_i$ are independent and normally distributed with unknown mean $\mu_i$ and unknown variance $\sigma_i^2$ $(i=1,\dots,k)$. Even if the variances are equal but unknown, the power of the noncentral $F$ test for testing the hypothesis $\mathrm{H}_0\colon \sum(\mu_i-\bar{\mu})^2/\sigma^2 \le \delta^2$ depends upon the unknown common variance, which renders it difficult to plan an experiment reasonably. Moreover, when the variances are unknown and unequal, there does not exist an exact one-stage statistical procedure to solve the analysis of variance test problem (see, e.g., Bishop and Dudewicz, 1978) if one wishes to have both the level and the power of the test controllable at some fixed values. In this paper, we will employ the two-stage sampling procedure (see, e.g., Bishop and Dudewicz, 1978; Stein, 1945; Rasch et al., 1997) and propose an $\tilde{F}$ (analog of $F$) test for testing the null hypothesis $\mathrm{H}_0\colon \sum(\mu_i-\bar{\mu})^2/k \le \delta^2$ against the alternative $\mathrm{H}_\mathrm{a}\colon \sum(\mu_i-\bar{\mu})^2/k > \delta^2$. The distribution of the test $\tilde{F}$ by the two-stage procedure drops the assumptions of unknown variances, hence the test has the level and the power independent of the variances. We first introduce the two-stage sampling procedure for the one-way layout in Section 2 and then calculate the critical values and the power of the test in Section 3. Statistical tables to implement the procedure are provided. A numerical example for one-way layout to illustrate the use of the $\tilde{F}$ test is given in Section 4. When the required sample sizes cannot be reached in an experiment, the single-stage procedure (see Chen and Chen, 1998) can provide a useful solution which is discussed in Section 5. The two-stage procedure is a design-oriented procedure while the single-stage procedure is a data-analysis procedure with data being already available on hand. Their relation is also discussed. In Section 6, an extension to two-way fixed-effects model is investigated. Finally, in Section 7, a relation to the single-stage procedure in the two-way layout is outlined.

## 2. The $\tilde{F}$ test

In a one-way layout, the fixed-effects model is given by

$$X_{ij} = \mu_i + e_{ij}, \quad i=1,\dots,k,\ j=1,\dots,n_i,$$

where the $e_{ij}$'s are independent and normally distributed with mean zero and variance $\sigma_i^2$, denoted by $e_{ij} \sim \mathrm{N}(0,\sigma_i^2)$. We may denote $\mu_i$ by $\mu_i=\bar{\mu}+\alpha_i$, where $\bar{\mu}=\sum_{i=1}^k \mu_i/k$, and $\alpha_i$ is the treatment effect. The variances $\sigma_i^2$'s are unknown and possibly unequal. Our goal is to test the null hypothesis that the normal means fall into a zone of practical indifference of size $\delta^2$ $(\ge 0)$, i.e., to test the hypothesis

$$\mathrm{H}_0\colon \sum_{i=1}^k (\mu_i-\bar{\mu})^2/k \le \delta^2 \quad \text{vs.} \quad \mathrm{H}_\mathrm{a}\colon \sum_{i=1}^k (\mu_i-\bar{\mu})^2/k \ge \delta^{*2}\ (\delta^{*2} > \delta^2) \tag{1}$$

or equivalently to test the hypothesis with $\alpha_i = \mu_i - \bar{\mu}$

$$\mathrm{H_0}\colon \sum_{i=1}^{k} \alpha_i^2/k \le \delta^2 \quad \text{vs.} \quad \mathrm{H_a}\colon \sum_{i=1}^{k} \alpha_i^2/k \ge \delta^{*2}\ (\delta^{*2} > \delta^2)$$

in such a way that both the level and the power of the test are controllable and are not dependent upon the unknown variances. A two-stage sampling procedure ($\mathrm{P}_1$) for this problem is given as follows:

$\mathrm{P}_1$ : Choose a number $z > 0$ ($z$ is determined by the required power of the test to be discussed later), and take an initial random sample of size $n_0$ (at least 2, but 10 or more will give better results) from each of the $k$ populations. For the $i$th population let $S_i^2$ be the usual unbiased estimate of $\sigma_i^2$ based on the initial $n_0$ observations, and define

$$N_i = \max\left\{ n_0 + 1, \left[\frac{S_i^2}{z}\right] + 1 \right\}, \tag{2}$$

where $[x]$ denotes the greatest integer less than or equal to $x$. Then, take $N_i - n_0$ additional random observations (assuming no time trend) from the $i$th population so we have a total of $N_i$ observations denoted by $X_{i1}, \ldots, X_{in_0}, \ldots, X_{iN_i}$. For each $i$, set the coefficients $a_{i1}, \ldots, a_{in_0}, \ldots, a_{iN_i}$, so that

$$a_{i1} = \cdots = a_{in_0} = \frac{1 - (N_i - n_0)b_i}{n_0} = a_i,$$

$$a_{i,n_0+1} = \cdots = a_{iN_i} = \frac{1}{N_i}\left[1 + \sqrt{\frac{n_0(N_i z - S_i^2)}{(N_i - n_0)S_i^2}}\right] = b_i,$$

and then compute the weighted mean

$$\tilde{X}_{i.} = a_i \sum_{j=1}^{n_0} X_{ij} + b_i \sum_{j=n_0+1}^{N_i} X_{ij} \tag{3}$$

which is a linear combination of the first-stage sample data $(X_{i1}, \ldots, X_{in_0})$ and the second-stage sample data $(X_{in_0+1}, \ldots, X_{iN_i})$. Note that the coefficients $a_{ij}$'s are so determined to satisfy the equations

$$\text{(a)} \sum_{j=1}^{N_i} a_{ij} = 1, \quad \text{(b)}\ a_{i1} = \cdots = a_{in_0}, \quad \text{and} \quad \text{(c)}\ S_i^2 \sum_{j=1}^{N_i} a_{ij}^2 = z$$

that the random variables $t_i = (\tilde{X}_{i.} - \mu_i)/\sqrt{z},\ i = 1, \ldots, k$, have independent and identically distributed (i.i.d.) $t$ distributions each with $n_0 - 1$ d.f. (see, e.g., Dudewicz and Dalal, 1975; Chen and Chen, 1998). The condition (a) is to ensure the unbiasedness of $\tilde{X}_{i.}$ for $\mu_i$, (b) guarantees that the sample mean $\bar{X}_i$ and the sample variance $S_i^2$ based on the first-stage observations are independent, and (c) is the variance estimate of $\tilde{X}_{i.}$ controlled at a power-specified value $z$ which makes the choices of $a_{ij}$ possible and guarantees that the $\{t_i\}$ have independent $t$ distributions.

Finally, compute the test statistic

$$\tilde{F} = \sum_{i=1}^{k} \frac{(\tilde{X}_{i.} - \tilde{X}_{..})^2}{z} \tag{4}$$

where $\tilde{X}_{..}$ is the arithmetic mean of $\tilde{X}_{1.}, \dots, \tilde{X}_{k.}$, and we reject $\mathrm{H}_0$ at level $\alpha$ if and only if

$$\tilde{F} > \tilde{F}_\alpha, \tag{5}$$

where the level $\alpha$ critical value $\tilde{F}_\alpha = \tilde{F}_\alpha(\delta, z, k, n_0)$ and the $P^*$-power-related $z$ value are determined such that the following simultaneous equations are satisfied:

$$P\left(\tilde{F} > \tilde{F}_\alpha \,|\, \mathrm{H}_0\colon \sum_{i=1}^{k} (\mu_i - \bar{\mu})^2/k \le \delta^2\right) \le \alpha \tag{6}$$

and

$$P\left(\tilde{F} > \tilde{F}_\alpha \,|\, \mathrm{H}_\mathrm{a}\colon \sum_{i=1}^{k} (\mu_i - \bar{\mu})^2/k \ge \delta^{*2} > \delta^2\right) \ge P^*, \tag{7}$$

where $\alpha \in (0,1)$ and $P^* \in (0,1)$ are predetermined values.

We can rewrite the test statistic $\tilde{F}$ in (4) as

$$\begin{aligned} \tilde{F} &= \sum_{i=1}^{k} \left( \frac{\tilde{X}_{i.} - \mu_i}{\sqrt{z}} - \frac{\tilde{X}_{..} - \bar{\mu}}{\sqrt{z}} + \frac{\mu_i - \bar{\mu}}{\sqrt{z}} \right)^2 \\ &= \sum_{i=1}^{k} \left( t_i - \bar{t} + \frac{\alpha_i}{\sqrt{z}} \right)^2, \end{aligned} \tag{8}$$

where $\bar{t} = \sum_{i=1}^{k} t_i/k$.

It should be noted that if $\tilde{X}_{..}$ in (4) were taken to be a weighted average of $\tilde{X}_{i.}$'s of the form $\sum N_i \tilde{X}_{i.} / \sum N_i$, then the test statistic in (4) would be

$$\tilde{F} = \sum_{i=1}^{k} \left( t_i - \frac{\sum N_i t_i}{\sum N_i} + \frac{\mu_i - \sum N_i \mu_i / \sum N_i}{\sqrt{z}} \right)^2$$

which is a function of the unknown but random sample sizes $N_i$'s. This contradicts the two-stage procedure and it fails to determine the sample sizes in (2) with a prespecified power when the population variances are unknown and unequal. Furthermore, if $\tilde{X}_{..}$ were taken to be the weighted average of $\tilde{X}_{i.}$'s, the noncentrality parameter of $\tilde{F}$ would be a function of the unknown $N_i$'s and it would not consist of the form of the parameters to be tested in (1). Finally, if all $N_i$ are taken to be equal and are larger than or equal to $n_0 + 1$, either the required power $P^*$ cannot be met or the procedure is not economical for the design. Hence, the $\tilde{X}_{..}$ in (4) is taken to be the arithmetic mean in order to reach a satisfactory solution to the problem under investigation.

It is clear that the distribution of the test statistic $\tilde{F}$ in (8) is independent of the unknown variances $\sigma_i^2$'s. The power in (7) is controllable through $z$ since if

$\mu_i - \bar{\mu} \neq 0$ for some $i$ it is intuitively clear (for analytical proof, see Bishop and Dudewicz, 1978) that

$$\lim_{z \to 0} P(\tilde{F} > \tilde{F}_\alpha) = 1.$$

It is easy to see that the limiting distribution of $\tilde{F}$ is a noncentral chi-square with $k-1$ degrees of freedom and noncentrality parameter $\Delta = \sum_{i=1}^{k} (\mu_i - \bar{\mu})^2/z = \sum_{i=1}^{k} \alpha_i^2/z$. As discussed in Lehmann (1986), the noncentral $\chi^2$ has monotone likelihood ratio property in $\Delta$, thus, as $n_0$ is large, $\tilde{F}$ is an asymptotically UMP test for $H_0$ vs. $H_a$.

For the test procedure $\tilde{F}$ to be of practical usage for small $n_0$ the critical values $\tilde{F}_\alpha$ and its power-related design parameter $z$ must be determined, which will be discussed in the following section.

## 3. The critical values and the power of $\tilde{F}$

The critical values $\tilde{F}_\alpha$ and the power of $\tilde{F}$ were obtained by Monte Carlo simulation when $n_0$ is small ($n_0 = 5, 10, 15$). In our calculation we consider the asymptotically least-favorable configuration of the means for the power of $\tilde{F}$, subject to $\sum(\mu_i - \bar{\mu})^2 = c$ (see, e.g., Bishop and Dudewicz, 1978).

It can be seen from David et al. (1972) that, for fixed $\delta$, the minimum range of the $\mu_i$ (or $\alpha_i$) under $H_0 : \sum_{i=1}^{k} (\mu_i - \bar{\mu})^2/k \leq \delta^2$ occurs at the asymptotically least-favorable configuration, for even $k$

$$\boldsymbol{\mu}^0 = (-\delta, \ldots, -\delta, \delta, \ldots, \delta) \tag{9}$$

with half of the $\mu$'s being $-\delta$ and half being $\delta$, and for odd $k$,

$$\boldsymbol{\mu}^0 = \left( -\delta\sqrt{\frac{k-1}{k+1}}, \ldots, -\delta\sqrt{\frac{k-1}{k+1}}, \delta\sqrt{\frac{k+1}{k-1}}, \ldots, \delta\sqrt{\frac{k+1}{k-1}} \right) \tag{10}$$

with $(k+1)/2$ of the $\mu$'s being $-\delta\sqrt{(k-1)/(k+1)}$ and the rest being $\delta\sqrt{(k+1)/(k-1)}$. Similarly, the maximum range of the $\mu_i$ under $H_a : \sum_{i=1}^{k} (\mu_i - \bar{\mu})^2/k \geq \delta^{*2} > \delta^2$, for fixed $\delta^*$, occurs at the asymptotically least-favorable configuration

$$\boldsymbol{\mu}^* = (-\delta^*\sqrt{k/2}, 0, \ldots, 0, \delta^*\sqrt{k/2}). \tag{11}$$

For each $k$ ($k = 2(1)6, 8$) and each $n_0$ ($n_0 = 5, 10, 15$), $k$ independent $t$ random variates, $t_1, \ldots, t_k$ were generated by the formula $t = Y/\sqrt{u/r}$, where $Y$ is the standard normal random variate generated from RANNOR (SAS Institute, Inc., 1990) and $u$ is the independent chi-square random variate with $r = n_0 - 1$ degrees of freedom generated from the gamma random number generator RANGAM. The quantity $\delta$ in (9) and (10) is replaced by $(\delta/\delta^*)(\delta^*)$. For selected $\delta, \delta^*$ and $z$, we formulate the ratios $\delta/\delta^*$ and $\delta^*/\sqrt{z}$ in the calculation of (8) according to $\boldsymbol{\mu}_0$ in (9) or (10) and according to $\boldsymbol{\mu}^*$ in (11). The reason to use the ratio $\delta/\delta^*$ instead of $\delta$ and the ratio $\delta^*/\sqrt{z}$ is to reduce the huge size of statistical tables and to include more choices of

$\delta$ and $\delta^*$. Similarly when $\boldsymbol{\mu}^*$ in (11) is substituted for $\mu$'s in (8) we have seen the ratio $\delta^*/\sqrt{z}$ in $\tilde{F}$. For example, when $k=2$, we use

$$\tilde{F}=[t_1-\bar{t}-(\delta/\delta^*)(\delta^*/\sqrt{z})]^2+[t_2-\bar{t}+(\delta/\delta^*)(\delta^*/\sqrt{z})]^2$$

for calculating the critical values of $\tilde{F}$ under (9), and

$$\tilde{F}=(t_1-\bar{t}-\delta^*/\sqrt{z})^2+(t_2-\bar{t}+\delta^*/\sqrt{z})^2$$

for calculating the power of $\tilde{F}$ under (11); when $k=3$, we use

$$\begin{aligned}\tilde{F}=&[t_1-\bar{t}-(1/\sqrt{2})(\delta/\delta^*)(\delta^*/\sqrt{z})]^2\\&+[t_2-\bar{t}-(1/\sqrt{2})(\delta/\delta^*)(\delta^*/\sqrt{z})]^2\\&+[t_3-\bar{t}+\sqrt{2}(\delta/\delta^*)(\delta^*/\sqrt{z})]^2\end{aligned}$$

for calculating the critical values of $\tilde{F}$ under (10), and

$$\tilde{F}=(t_1-\bar{t}-\sqrt{1.5}\delta^*/\sqrt{z})^2+(t_2-\bar{t})^2+(t_3-\bar{t}+\sqrt{1.5}\delta^*/\sqrt{z})^2$$

for calculating the power of $\tilde{F}$ under (11). In each simulation run, for a specified pair of values of $\delta/\delta^*$ and $\delta^*/\sqrt{z}$, $k$ i.i.d. $t$ random variates were generated and $\tilde{F}$ in (8) under (9) or (10) for $H_0$ was calculated. After 20,000 simulation runs, all $\tilde{F}$ values were ranked in ascending order. Then the 99th, 95th and 90th percentiles were used to estimate the level 1%, 5% and 10% critical values $\tilde{F}_{0.01}$, $\tilde{F}_{0.05}$ and $\tilde{F}_{0.10}$, respectively. Similarly, for given $\tilde{F}_\alpha$ and $\delta^*/\sqrt{z}$, $\tilde{F}$ in (8) under (11) for $H_a$ was calculated. This process was repeated 20,000 times and the power of (7) was estimated by

$$P^*\cong\frac{\text{No. times}(\tilde{F}>\tilde{F}_\alpha)}{20,000}. \qquad (12)$$

The estimated critical values and the estimated power are given in Table 3 in the appendix for $\alpha=0.01,0.05,0.10$, $k=2(1)6,8$, $\delta/\delta^*=0,0.2,0.4,0.5,0.6$, $\delta^*/\sqrt{z}=1(.5)10$, and $n_0=5,10,15$. To reduce the table size without losing practical usefulness, we delete the cases of $P^*$ greater than 0.99 (using $\alpha=0.05$ as the guideline). The critical values are reported to the first decimal place, and the power are accurate to the second decimal place. An example of how to use Table 3 is illustrated as follows: If one has $k=4$ treatments in his experiments, and the initial sample available is $n_0=10$ observations, at the price of $\alpha=5\%$ risk, he will feel comfortably indifferent among these treatments if they are within a one-half unit of variation ($\delta=0.5$) among the means; on the other hand, if these treatments have variation larger than one unit ($\delta^*=1.0$) among means, he would like to detect such a difference with a required power, say, $P^*=0.82$. From Table 3, he can find the ratio $\delta^*/\sqrt{z}=3.0$ corresponding to the ratio $\delta/\delta^*=0.5$ and the required power $P^*=0.82$. Then, the design constant is found to be $z=(\delta^*/3.0)^2$ or $z=0.1111$ which will be employed in (2) to determine the required total sample size $N_i$ in the experiment. Simulation study shows that linear interpolation in $\delta^*/\sqrt{z}$ would give satisfactory results for values of $P^*$ being not tabulated.

Table 1
Bacterial killing ability example (first 15 observations) and intermediate statistics

| | Solvent 1 | Solvent 2 | Solvent 3 | Solvent 4 |
|---|---|---|---|---|
| | 96.44 | 93.63 | 93.58 | 97.18 |
| | 96.87 | 93.99 | 93.02 | 97.42 |
| | 97.24 | 94.61 | 93.86 | 97.65 |
| | 95.41 | 91.69 | 92.90 | 95.90 |
| | 95.29 | 93.00 | 91.43 | 96.35 |
| | 95.61 | 94.17 | 92.68 | 97.13 |
| | 95.28 | 92.62 | 91.57 | 96.06 |
| | 94.63 | 93.41 | 92.87 | 96.33 |
| | 95.58 | 94.67 | 92.65 | 96.71 |
| | 98.20 | 95.28 | 95.31 | 98.11 |
| | 98.29 | 95.13 | 95.33 | 98.38 |
| | 98.30 | 95.68 | 95.17 | 98.35 |
| | 98.65 | 97.52 | 98.59 | 98.05 |
| | 98.43 | 97.52 | 98.00 | 98.25 |
| | 98.41 | 97.37 | 98.79 | 98.12 |
| *Intermediate statistics* | | | | |
| $S_i^2$ | 2.10995 | 3.17085 | 5.88428 | 0.77969 |
| $a_i$ | 0.05200 | 0.03024 | 0.01803 | 0.04424 |
| $b_i$ | 0.05501 | 0.03902 | 0.01920 | 0.33637 |
| $N_i$ | 19 | 29 | 53 | 16 |
| $\tilde{X}_{i\cdot}$ | 97.192 | 95.381 | 95.391 | 97.547 |
| $z = 0.1111$ | | $\tilde{F} = 35.981$ | | |

For moderate or large $n_0$, the critical values and the power of the $\tilde{F}$ test can be obtained by using the noncentral chi-square approximation with $k-1$ degrees of freedom and noncentrality parameter $\Delta = k\delta^2/z = k[(\delta/\delta^*)(\delta^*/\sqrt{z})]^2$ under $H_0$ and $\Delta = k(\delta^*/\sqrt{z})^2$ under $H_a$. The critical values and the power of the chi-square test can be computed by using the CINV and PROBCHI functions (SAS Institute, Inc., 1990). They are given in Table 4 in the appendix for $\alpha = 0.01, 0.05, 0.10$, $k = 2(1)6, 8$, $\delta/\delta^* = 0, 0.2, 0.4, 0.5, 0.6$, and $\delta^*/\sqrt{z} = 1(.5)7$.

## 4. A numerical example

The data in Table 1 is from an experiment reported in Bishop and Dudewicz (1978) for studying the bacterial killing ability of four solvents. The percentage of fungus destroyed was recorded. Let $\mu_i$ denote the mean percentage of fungus destroyed by solvent $i$. If the experimenter regards a difference of $\delta = 0.5$ unit of variation among the means to be irrelevant, and he wishes to detect a difference of at least $\delta^* = 1.0$ unit of variation among the means, then he can translate it into the null hypothesis

$$H_0\colon \sum_{i=1}^{4} (\mu_i - \bar{\mu})^2/4 \le (0.5)^2$$

Table 2
Bacterial killing ability (second-stage observations)

| Solvent 1 | Solvent 2 | Solvent 3 | | | Solvent 4 |
|---|---|---|---|---|---|
| 98.59 | 96.97 | 96.36 | 93.43 | 98.15 | 97.97 |
| 98.20 | 97.21 | 96.69 | 92.72 | 96.73 | |
| 98.37 | 97.44 | 96.89 | 93.56 | 97.55 | |
| 98.57 | 96.86 | 96.13 | 94.13 | 94.44 | |
| | 97.26 | 97.65 | 93.57 | 93.61 | |
| | 98.27 | 97.81 | 96.27 | 93.61 | |
| | 97.57 | 97.71 | 98.05 | 94.20 | |
| | 97.81 | 97.48 | 97.67 | 94.20 | |
| | 98.20 | 97.96 | 98.93 | | |
| | 93.92 | 94.30 | 97.23 | | |
| | 93.86 | 93.29 | 95.95 | | |
| | 92.57 | 94.21 | 97.79 | | |
| | 93.32 | 92.90 | 97.41 | | |
| | 92.15 | 93.02 | 96.94 | | |
| | | 93.43 | 97.08 | | |

against the alternative hypothesis

$$H_a: \sum_{i=1}^{4}(\mu_i - \bar{\mu})^2/4 \geq (1.0)^2.$$

In the first stage of experiment $n_0 = 15$ observations (a random sample of size 15) were run with each solvent. Wen and Chen (1994) discovered that these data are not normally distributed. So, we have conducted a robust Levene's test for homogeneity of variances (see, e.g., Conover et al., 1981) and found a significant difference among the variances ($p$ value $< 0.001$). Further, Dudewicz and van der Meulen (1983) have also shown robustness results which applies to the two-stage procedure for general non-normal distributions. If the experimenter decides the level of the test to be 5% and a power of at least 0.85, he can use the two-stage test procedure by taking the initial sample of size $n_0 = 15$ observations from each population. The critical value $\tilde{F}_\alpha = 25.9$ and $\delta^*/\sqrt{z} = 3.0$ at $\delta/\delta^* = 0.5$ are found using Table 3, so $z = (1.0/3.0)^2 = 0.1111$. The initial sample variances based on the first-stage samples, the coefficients for calculating the weighted sample means, and the final weighted sample means, $S_i^2, a_i, b_i, N_i$ and $\tilde{X}_{i.}$ defined in (2) and (3) are given at the bottom of Table 1. The remaining $N_i - 15$ observations taken at the second stage are given in Table 2. Using formula (4) we found the test statistic $\tilde{F} = 35.981$, which exceeds the critical value of 25.9, so $H_0$ is rejected.

## 5. Relation to the single-stage procedure

The two-stage procedure discussed in Section 2 is a design-oriented method which determines the necessary sample sizes $N_i$ in order to meet a prespecified power requirement. In situations where the two-stage experiment is terminated earlier due to budget shortage or some other uncontrollable factors, the required total sample size $N_i$ in (2) cannot be reached, one may have to use the available $n_i$ ($n_i \geq (n_0+1)$) observations on hand and recalculate the coefficients $a_{ij}$'s according to the so-called single-stage sampling procedure (see Chen and Chen, 1998) such that the statistical inference theory can still work. The general single-stage procedure ($P_2$) is described below.

$P_2$: Given a random sample of size $n_i$ from normal population (or treatment) $\pi_i$ with unknown mean $\mu_i$ and unknown variance $\sigma_i^2$ ($1 \leq i \leq k$). Employ the first (or randomly chosen) $n_0$ ($2 \leq n_0 < n_i$) observations and calculate the usual unbiased sample mean and unbiased sample variance, respectively, by

$$\bar{X}_i = \sum_{j=1}^{n_0} X_{ij}/n_0$$

and

$$S_i^2 = \sum_{j=1}^{n_0} (X_{ij} - \bar{X}_i)^2/(n_0 - 1).$$

Then, calculate the coefficients

$$U_i = \frac{1}{n_i} + \frac{1}{n_i}\sqrt{\frac{n_i - n_0}{n_0}(n_i z^*/S_i^2 - 1)},$$

$$V_i = \frac{1}{n_i} - \frac{1}{n_i}\sqrt{\frac{n_0}{n_i - n_0}(n_i z^*/S_i^2 - 1)},$$

where $z^*$ is the maximum of $\{S_j^2/n_j,\ j = 1,\ldots,k\}$. Let the final weighted sample mean be defined by

$$\tilde{X}_{i.} = \sum_{j=1}^{n_i} W_{ij} X_{ij}, \tag{13}$$

where

$$W_{ij} = \begin{cases} U_i & \text{for } 1 \leq j \leq n_0, \\ V_i & \text{for } (n_0 + 1) \leq j \leq n_i, \end{cases}$$

and $W_{ij}$ satisfy the following conditions:

$$\sum_{j=1}^{n_i} W_{ij} = 1, \quad W_{i1} = \cdots = W_{in_0}, \quad S_i^2 \sum_{j=1}^{n_i} W_{ij}^2 = z^*.$$

It is well known (see Chen and Chen, 1998) that given the sample variances $S_i^2$, $i = 1,\ldots,k$, the weighted sample mean $\tilde{X}_i$ has a conditional normal distribution with mean $\mu_i$ and variance $\sum_j W_{ij}^2 \sigma_i^2$. Furthermore, the transformations

$$t_i = \frac{\tilde{X}_{i.} - \mu_i}{\sqrt{S_i^2 \sum_{j=1}^{n_i} W_{ij}^2}} = \frac{\tilde{X}_{i.} - \mu_i}{\sqrt{z^*}}, \quad i = 1,\ldots,k$$

have i.i.d. $t$ distributions each with $n_0 - 1$ degrees of freedom. Note that in the single-stage procedure, the data-dependent $z^*$ is used to replace the design constant

$z$ for the two-stage procedure. Thus, the power of the single-stage procedure is not controllable.

The statistic

$$\tilde{F}^1 = \sum_{i=1}^{k} \frac{(\tilde{X}_{i.} - \tilde{X}_{..})^2}{z^*}, \tag{14}$$

where $\tilde{X}_{..}$ is the arithmetic mean of the $\tilde{X}_{i.}$'s, is used as a test statistic for testing the hypothesis $H_0$ vs. $H_a$ in (1). Further, $\tilde{F}^1$ can be written as

$$\tilde{F}^1 = \sum_{i=1}^{k} \left( \frac{\tilde{X}_{i.} - \mu_i}{\sqrt{z^*}} - \frac{\tilde{X}_{..} - \bar{\mu}}{\sqrt{z^*}} + \frac{\mu_i - \bar{\mu}}{\sqrt{z^*}} \right)^2$$
$$= \sum_{i=1}^{k} \left( t_i - \bar{t} + \frac{\mu_i - \bar{\mu}}{\sqrt{z^*}} \right)^2.$$

Note that if $\tilde{X}_{..}$ in (14) were taken to be the weighted average of $\tilde{X}_i$'s, it would lead to testing different hypotheses (could be meaningless ones) rather than (1).

The critical values of $\tilde{F}^1$ for testing $H_0$ and its power against $H_a$ can be obtained at the asymptotically least-favorable configurations given in (9)–(11) by using the tables in which $\delta^*/\sqrt{z}$ is replaced by $\delta^*/\sqrt{z^*}$. For example, if $k=4$, $n_0=10$, $\alpha=0.05$, $\delta=1$, $\delta^*=2$ and $z^* = \max(S_j^2/n_j) = 0.64$. Thus, $\delta/\delta^* = 0.5$ and $\delta^*/\sqrt{z^*} = 2/\sqrt{0.64} = 2.5$. From Table 3, we find the critical value $\tilde{F}_{0.05} = 23.0$ and the power $P^* = 0.67$.

The actual power of the test using the single-stage procedure is data-dependent. Its power could be larger than, equal to, or smaller than the required one using the two-stage procedure whose sample sizes are determined by the prespecified power. This point is elaborated as follows: If the sample size $n_i > n_0 + 1$, $i = 1, \ldots, k$, were given by the single-stage procedure and the following cases.

*Case* 1. If $S_i^2/n_i = S_j^2/n_j$ for all $i \neq j$, then the two- and single-stage procedures have the same power because $S_i^2/n_i = z$, except for a rounding error in sample size by definition (2) and $S_i^2/n_i = \max\{S_j^2/n_j, j = 1, \ldots, k\} = z^*$ defined by single-stage procedure. Thus, $z = z^*$ gives the same power.

*Case* 2. If $z^* = \max_{1 \le j \le k}(S_j^2/n_j) < z$, then the single-stage procedure has a power larger than that of the two-stage one. A smaller $z^*$-value means a larger sample size $n_i$ than the required one by two-stage procedure and hence, it carries a larger power.

*Case* 3. If $\min_{1 \le j \le k}(S_j^2/n_j) > z$, then the power of the single-stage test is smaller than that of the two-stage test.

*Case* 4. All other situations, the single-stage procedure could have power larger than, equal to, or smaller than that of the two-stage test depending on the actual sample data and the true population variances.

## 6. The two-way layout

The two-way fixed-effects model in the analysis of variance is usually defined by

$$X_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \quad (i = 1, \ldots, I;\ j = 1, \ldots, J;\ k = 1, \ldots, n_{ij})$$

where the random errors $e_{ijk}$'s are independently and normally distributed with mean zero and unknown (and possibly unequal) variances $\sigma_{ij}^2$, and by convention,

$$\sum_{i=1}^{I} \alpha_i = \sum_{j=1}^{J} \beta_j = 0,$$
$$\sum_{i=1}^{I} (\alpha\beta)_{ij} = 0 \quad \text{for every } j \tag{15}$$

and

$$\sum_{j=1}^{J} (\alpha\beta)_{ij} = 0 \quad \text{for every } i.$$

The null and alternative hypotheses under consideration are

$$\begin{aligned} &\mathrm{H}_0^1\colon \sum_{i=1}^{I} \alpha_i^2/I \le \delta_1^2 \quad \text{vs.} \quad \mathrm{H}_a^1\colon \sum_{i=1}^{I} \alpha_i^2/I \ge \delta_1^{*2} > \delta_1^2, \\ &\mathrm{H}_0^2\colon \sum_{j=1}^{J} \beta_j^2/J \le \delta_2^2 \quad \text{vs.} \quad \mathrm{H}_a^2\colon \sum_{j=1}^{J} \beta_j^2/J \ge \delta_2^{*2} > \delta_2^2, \end{aligned} \tag{16}$$

and

$$\mathrm{H}_0^3\colon \sum_{i=1}^{I}\sum_{j=1}^{J} (\alpha\beta)_{ij}^2/IJ \le \delta_3^2 \quad \text{vs.} \quad \mathrm{H}_a^3\colon \sum_{i=1}^{I}\sum_{j=1}^{J} (\alpha\beta)_{ij}^2/IJ \ge \delta_3^{*2} > \delta_3^2.$$

The purpose is to seek tests of these hypotheses based on statistics whose distributions are independent of the unknown variances and the unknown means. In the two-way layout, there are $I * J$ possible treatment combinations. We refer cell $(i,j)$ to the treatment combination of level $i$ of the first factor and level $j$ of the second factor. In each cell $(i,j)$, the two-stage sampling procedure ($\mathrm{P}_3$) is given below.

$\mathrm{P}_3$: Choose a number $z > 0$ (to be determined by the power), and in each cell $(i,j)$ take an initial sample of size $n_0$, $X_{ij1},\ldots,X_{ijn_0}$. Compute the usual unbiased variance estimate $S_{ij}^2$ of $\sigma_{ij}^2$ based on the first $n_0$ random observations, and define

$$N_{ij} = \max\left\{ n_0 + 1, \left[\frac{S_{ij}^2}{z}\right] + 1 \right\}. \tag{17}$$

Then, take $N_{ij} - n_0$ additional random observations from cell $(i,j)$ so we have a total of $N_{ij}$ observations denoted by $X_{ij1},\ldots,X_{ijn_0},\ldots,X_{ijN_{ij}}$. For each cell $(i,j)$, set the coefficients $a_{ij1},\ldots,a_{ijn_0},\ldots,a_{ijN_{ij}}$, so that

$$\begin{aligned} &a_{ij1} = \cdots = a_{ijn_0} = \frac{1-(N_{ij}-n_0)b_{ij}}{n_0} = a_{ij}, \\ &a_{ij,n_0+1} = \cdots = a_{ijN_{ij}} = b_{ij}, \end{aligned}$$

where

$$b_{ij} = \frac{1}{N_{ij}}\left[1 + \sqrt{\frac{n_0(N_{ij}z - S_{ij}^2)}{(N_{ij}-n_0)S_{ij}^2}}\right],$$

and then compute the weighted sample mean

$$\tilde{X}_{ij.} = \sum_{k=1}^{N_{ij}} a_{ij} X_{ijk} = a_{ij} \sum_{k=1}^{n_0} X_{ijk} + b_{ij} \sum_{k=n_0+1}^{N_{ij}} X_{ijk}.$$

As in Section 2, it can be shown that the random variables

$$t_{ij} = \frac{\tilde{X}_{ij.} - (\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij})}{\sqrt{z}} \tag{18}$$

have independent $t$ distribution with $n_0 - 1$ degrees of freedom, denoted by $t_{n_0-1}$.

Finally, compute

$$\tilde{X}_{i..} = \frac{1}{J} \sum_{j=1}^{J} \tilde{X}_{ij.}, \qquad \tilde{X}_{.j.} = \frac{1}{I} \sum_{i=1}^{I} \tilde{X}_{ij.}, \qquad \tilde{X}_{...} = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \tilde{X}_{ij.} \tag{19}$$

These means in (19) are taken to be the unweighted ones carrying the same argument raised immediately after expression (8).

Similar to the usual two-way layout argued by Bishop and Dudewicz (1978, p. 422), our test statistic for $\mathrm{H}_0^1$ vs. $\mathrm{H}_a^1$ is

$$\tilde{F}_1 = J \sum_{i=1}^{I} \frac{(\tilde{X}_{i..} - \tilde{X}_{...})^2}{z}. \tag{20}$$

At level $\alpha$ the hypothesis $\mathrm{H}_0^1$: $\sum_{i=1}^{I} \alpha_i^2 / I \le \delta_1^2$ is rejected if and only if

$$\tilde{F}_1 > \tilde{F}_1^{\alpha},$$

where the level $\alpha$ critical value $\tilde{F}_1^{\alpha} = \tilde{F}_1^{\alpha}(\delta_1, z, I, J, n_0)$ and the $P^*$-power-related $z$ value are determined such that

$$P\left(\tilde{F}_1 > \tilde{F}_1^{\alpha} \,|\, \mathrm{H}_0^1\text{: } \sum_{i=1}^{I} \alpha_i^2 / I \le \delta_1^2\right) \le \alpha$$

and

$$P\left(\tilde{F}_1 > \tilde{F}_1^{\alpha} \,|\, \mathrm{H}_a^1\text{: } \sum_{i=1}^{I} \alpha_i^2 / I \ge \delta_1^{*2} > \delta_1^2\right) \ge P^*. \tag{21}$$

We can rewrite the test statistic $\tilde{F}_1$ in (20) by applying conditions (15) and the definition (19) as

$$\tilde{F}_1 = J \sum_{i=1}^{I} (\bar{t}_{i.} - \bar{t}_{..} + \alpha_i / \sqrt{z})^2, \tag{22}$$

where $\bar{t}_{i.} = \sum_{j=1}^{J} t_{ij} / J, \bar{t}_{..} = \sum_{i=1}^{I} \sum_{j=1}^{J} t_{ij} / IJ$.

Similarly, the hypothesis $\mathrm{H}_0^2$: $\sum_{j=1}^{J} \beta_j^2 / J \le \delta_2^2$ is tested using the statistic

$$\tilde{F}_2 = I \sum_{j=1}^{J} \frac{(\tilde{X}_{.j.} - \tilde{X}_{...})^2}{z}. \tag{23}$$

The null hypothesis $H_0^2$ is rejected at level $\alpha$ if and only if

$$\tilde{F}_2 > \tilde{F}_2^{\alpha},$$

where the level $\alpha$ critical value $\tilde{F}_2^{\alpha} = \tilde{F}_2^{\alpha}(\delta_2, z, I, J, n_0)$ and the $P^*$-power-related $z$ value are determined by the simultaneous equations

$$P\left(\tilde{F}_2 > \tilde{F}_2^{\alpha} \,|\, H_0^2\colon \sum_{j=1}^{J} \beta_j^2/J \le \delta_2^2\right) \le \alpha$$

and

$$P\left(\tilde{F}_2 > \tilde{F}_2^{\alpha} \,|\, H_a^2\colon \sum_{j=1}^{J} \beta_j^2/J \ge \delta_2^{*2} > \delta_2^2\right) \ge P^*. \tag{24}$$

The test statistic $\tilde{F}_2$ in (23) using conditions (15) can be rewritten as

$$\tilde{F}_2 = I \sum_{j=1}^{J} (\bar{t}_{.j} - \bar{t}_{..} + \beta_j/\sqrt{z})^2, \tag{25}$$

where $\bar{t}_{.j} = \sum_{i=1}^{I} t_{ij}/I$.

Finally, $H_0^3$ is tested using

$$\tilde{F}_3 = \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{(\tilde{X}_{ij.} - \tilde{X}_{i..} - \tilde{X}_{.j.} + \tilde{X}_{...})^2}{z}. \tag{26}$$

The hypothesis $H_0^3$: $\sum_{i=1}^{I} \sum_{j=1}^{J} (\alpha\beta)_{ij}^2/IJ \le \delta_3^2$ is rejected if and only if

$$\tilde{F}_3 > \tilde{F}_3^{\alpha}$$

where $\tilde{F}_3^{\alpha}$ and $z$ value are determined by the simultaneous equations

$$P\left(\tilde{F}_3 > \tilde{F}_3^{\alpha} \,|\, H_0^3\colon \sum_{i=1}^{I} \sum_{j=1}^{J} (\alpha\beta)_{ij}^2/IJ \le \delta_3^2\right) \le \alpha$$

and

$$P\left(\tilde{F}_3 > \tilde{F}_3^{\alpha} \,|\, H_a^3\colon \sum_{i=1}^{I} \sum_{j=1}^{J} (\alpha\beta)_{ij}^2/IJ \ge \delta_3^{*2} > \delta_3^2\right) \ge P^*. \tag{27}$$

As in the previous case the statistic $\tilde{F}_3$ in (26) can be rewritten as

$$\tilde{F}_3 = \sum_{i=1}^{I} \sum_{j=1}^{J} (t_{ij} - \bar{t}_{i.} - \bar{t}_{.j} + \bar{t}_{..} + (\alpha\beta)_{ij}/\sqrt{z})^2. \tag{28}$$

The statistic $\tilde{F}_1$ in (22), $\tilde{F}_2$ in (25) and $\tilde{F}_3$ in (28), respectively, are used to simulate the critical values $\tilde{F}_1^{\alpha}, \tilde{F}_2^{\alpha}, \tilde{F}_3^{\alpha}$ and their powers at their asymptotically least-favorable configurations of $\alpha_i$'s, $\beta$'s and $(\alpha\beta)_{ij}$'s similar to (9)–(11).

It is easy to see that the limiting distributions of $\tilde{F}_1, \tilde{F}_2$ and $\tilde{F}_3$ are noncentral chi-square with degrees of freedom $I-1, J-1$ and $(I-1)(J-1)$, and with noncentrality parameters $\Delta_1 = \sum_{i=1}^{I} \alpha_i^2/z, \Delta_2 = \sum_{j=1}^{J} \beta_j^2/z$ and $\Delta_3 = \sum_{i=1}^{I} \sum_{j=1}^{J} (\alpha\beta)_{ij}^2/z$, respectively. The tables of the critical values and the power can be produced by using the noncentral chi-square distribution for moderate or large $n_0$. The $r$-way model of analysis of variance and its associated hypotheses can be similarly extended by the analogue of the two-way model.

## 7. The single-stage procedure for two-way ANOVA

When the required sample sizes $N_{ij}$ (17) in the two-way layout cannot be reached by the two-stage sampling procedure, one may employ the feasible single-stage procedure for a reasonable solution. The single-stage sampling procedure ($P_4$) for testing the hypotheses of (16) proceeds as follows.

$P_4$: Initially, we employ the first (or randomly chosen) $n_0$ observations within each cell and compute the usual sample mean and unbiased sample variance, respectively,

$$\bar{X}_{ij} = \sum_{k=1}^{n_0} X_{ijk}/n_0$$

and

$$S_{ij}^2 = \sum_{k=1}^{n_0} (X_{ijk} - \bar{X}_{ij})^2/(n_0 - 1).$$

Then the weights of the observations in cell $(i,j)$ are

$$\begin{aligned} U_{ij} &= \frac{1}{n_{ij}} + \frac{1}{n_{ij}}\sqrt{\frac{n_{ij}-n_0}{n_0}(n_{ij}z^*/S_{ij}^2 - 1)}, \\ V_{ij} &= \frac{1}{n_{ij}} - \frac{1}{n_{ij}}\sqrt{\frac{n_0}{n_{ij}-n_0}(n_{ij}z^*/S_{ij}^2 - 1)}, \end{aligned} \tag{29}$$

where $z^*$ is the maximum value of $\{S_{ij}^2/n_{ij},\ i = 1,\ldots,I,\ j = 1,\ldots,J\}$. Let the final weighted sample mean for cell $(i,j)$ be defined by

$$\tilde{X}_{ij.} = \sum_{k=1}^{n_{ij}} W_{ijk}X_{ijk}, \tag{30}$$

where

$$W_{ijk} = \begin{cases} U_{ij} & \text{for } 1 \le k \le n_0, \\ V_{ij} & \text{for } (n_0+1) \le k \le n_{ij}. \end{cases}$$

Therefore, we compute

$$\tilde{X}_{i..} = \frac{1}{J}\sum_{j=1}^{J}\tilde{X}_{ij.}, \qquad \tilde{X}_{.j.} = \frac{1}{I}\sum_{i=1}^{I}\tilde{X}_{ij.}, \qquad \tilde{X}_{...} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}\tilde{X}_{ij.}$$

The test statistic we consider to use for $H^1$ is

$$\begin{aligned} \tilde{F}_1^1 &= \sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{\tilde{X}_{i..} - \tilde{X}_{...}}{\sqrt{z^*}}\right)^2 \\ &= J\sum_{i=1}^{I}\left(\bar{t}_{i.} - \bar{t}_{..} + \frac{\alpha_i}{\sqrt{z^*}}\right)^2, \end{aligned} \tag{31}$$

for $H^2$:

$$\tilde{F}_2^1 = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{\tilde{X}_{.j.} - \tilde{X}_{...}}{\sqrt{z^*}}\right)^2 = I\sum_{j=1}^{J}\left(\bar{t}_{.j} - \bar{t}_{..} + \frac{\beta_j}{\sqrt{z^*}}\right)^2, \quad (32)$$

and for $H^3$:

$$\tilde{F}_3^1 = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{\tilde{X}_{ij.} - \tilde{X}_{i..} - \tilde{X}_{.j.} + \tilde{X}_{...}}{\sqrt{z^*}}\right)^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(t_{ij} - \bar{t}_{i.} - \bar{t}_{.j} + \bar{t}_{..} + \frac{(\alpha\beta)_{ij}}{\sqrt{z^*}}\right)^2, \quad (33)$$

where

$$\bar{t}_{i.} = \frac{1}{J}\sum_{j=1}^{J} t_{ij}, \quad \bar{t}_{.j} = \frac{1}{I}\sum_{i=1}^{I} t_{ij}, \quad \bar{t}_{..} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} t_{ij}.$$

It can be shown (see Chen and Chen, 1998) that

$$t_{ij} = \frac{\tilde{X}_{ij.} - (\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij})}{\sqrt{z^*}}, \quad (34)$$

for $i=1,\dots,I,\ j=1,\dots,J$, are distributed as independent Student's $t$ each with $n_0 - 1$ degrees of freedom. This result is due to the fact that given the sample variances $S_{ij}^2$'s, the weighted sample mean $\tilde{X}_{ij.}$ has a conditional normal distribution with mean $\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$ and variance $\sum_k W_{ijk}^2 \sigma_{ij}^2$, as described in Section 5.

Similar to the case of one-way layout, the critical values and the power for the single-stage procedure can be obtained by using the tables prepared for the two-stage procedure.The relationship between the single- and two-stage procedure is similar to the argument in Section 5.

In the situation where all $n_{ij}$'s are equal to $n$, i.e., a balanced design, and the $\delta_1, \delta_2$ and $\delta_3$ are equal to zero, the critical values of $\tilde{F}_1^1, \tilde{F}_2^1$, and $\tilde{F}_3^1$ for small $n_0$ and selected numbers of $I$ and $J$ were calculated by Chen and Chen (1998).

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## Appendix

The critical values and the power of the $\tilde{F}$ test and chi-test are given in Tables 3 and 4, respectively.

Table 3
Critical values and power of the $\tilde{F}$ test

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\tilde{F}_{0.01}, P^*)$ | | $(\tilde{F}_{0.05}, P^*)$ | | $(\tilde{F}_{0.10}, P^*)$ | |
|---|---|---|---|---|---|---|---|
| | | $k=2,\ n_0=5$ | | | | | |
| 0.0 | 1.0 | 19.0 | 0.02 | 7.7 | 0.14 | 4.8 | 0.26 |
| | 1.5 | 19.0 | 0.05 | 7.7 | 0.29 | 4.8 | 0.48 |
| | 2.0 | 19.0 | 0.11 | 7.7 | 0.52 | 4.8 | 0.70 |
| | 2.5 | 19.0 | 0.25 | 7.7 | 0.74 | 4.8 | 0.86 |
| | 3.0 | 19.0 | 0.46 | 7.7 | 0.88 | 4.8 | 0.94 |
| | 3.5 | 19.0 | 0.70 | 7.7 | 0.95 | 4.8 | 0.97 |
| | 4.0 | 19.0 | 0.85 | 7.7 | 0.98 | 4.8 | 0.99 |
| | 4.5 | 19.0 | 0.93 | 7.7 | 0.99 | 4.8 | 0.99 |
| 0.2 | 1.0 | 19.2 | 0.02 | 7.9 | 0.14 | 5.0 | 0.26 |
| | 1.5 | 19.5 | 0.04 | 8.4 | 0.26 | 5.3 | 0.44 |
| | 2.0 | 20.1 | 0.09 | 9.0 | 0.44 | 5.7 | 0.65 |
| | 2.5 | 20.4 | 0.22 | 9.4 | 0.65 | 6.2 | 0.81 |
| | 3.0 | 21.3 | 0.34 | 9.9 | 0.82 | 6.7 | 0.91 |
| | 3.5 | 22.3 | 0.58 | 10.5 | 0.91 | 7.2 | 0.96 |
| | 4.0 | 23.5 | 0.75 | 11.5 | 0.95 | 7.9 | 0.98 |
| | 4.5 | 26.1 | 0.85 | 12.5 | 0.98 | 8.7 | 0.99 |
| | 5.0 | 26.6 | 0.93 | 13.4 | 0.99 | 9.3 | 0.99 |
| 0.4 | 1.0 | 20.4 | 0.02 | 8.9 | 0.11 | 5.7 | 0.22 |
| | 1.5 | 21.3 | 0.04 | 10.0 | 0.20 | 6.7 | 0.35 |
| | 2.0 | 24.7 | 0.06 | 11.6 | 0.31 | 8.0 | 0.50 |
| | 2.5 | 25.5 | 0.11 | 13.4 | 0.46 | 9.4 | 0.65 |
| | 3.0 | 29.4 | 0.17 | 15.7 | 0.60 | 11.3 | 0.77 |
| | 3.5 | 33.0 | 0.25 | 17.9 | 0.73 | 13.1 | 0.86 |
| | 4.0 | 34.6 | 0.42 | 19.8 | 0.84 | 15.1 | 0.92 |
| | 4.5 | 38.1 | 0.57 | 22.3 | 0.90 | 17.1 | 0.95 |
| | 5.0 | 38.4 | 0.77 | 24.6 | 0.95 | 19.5 | 0.97 |
| | 5.5 | 46.9 | 0.79 | 28.9 | 0.96 | 22.8 | 0.98 |
| | 6.0 | 49.0 | 0.88 | 31.2 | 0.98 | 25.0 | 0.99 |
| | 6.5 | 52.7 | 0.93 | 34.6 | 0.99 | 28.0 | 0.99 |
| 0.5 | 1.0 | 21.5 | 0.02 | 9.3 | 0.10 | 6.2 | 0.19 |
| | 1.5 | 23.9 | 0.03 | 11.1 | 0.17 | 7.4 | 0.31 |
| | 2.0 | 26.3 | 0.05 | 13.5 | 0.24 | 9.4 | 0.42 |
| | 2.5 | 29.1 | 0.07 | 15.6 | 0.36 | 11.5 | 0.55 |
| | 3.0 | 32.2 | 0.12 | 18.7 | 0.47 | 14.1 | 0.66 |
| | 3.5 | 37.1 | 0.18 | 22.1 | 0.59 | 16.8 | 0.76 |
| | 4.0 | 41.7 | 0.25 | 25.1 | 0.71 | 19.9 | 0.84 |
| | 4.5 | 46.3 | 0.36 | 29.2 | 0.78 | 23.3 | 0.89 |
| | 5.0 | 52.0 | 0.45 | 33.0 | 0.86 | 26.4 | 0.93 |
| | 5.5 | 57.7 | 0.56 | 37.4 | 0.91 | 30.5 | 0.95 |
| | 6.0 | 63.3 | 0.68 | 41.7 | 0.94 | 34.4 | 0.97 |
| | 6.5 | 68.0 | 0.78 | 46.7 | 0.96 | 38.7 | 0.98 |
| | 7.0 | 73.3 | 0.86 | 50.9 | 0.98 | 42.8 | 0.99 |
| | 7.5 | 78.6 | 0.91 | 56.6 | 0.98 | 48.1 | 0.99 |
| | 8.0 | 84.4 | 0.94 | 61.8 | 0.99 | 53.0 | 0.99 |
| 0.6 | 1.0 | 23.6 | 0.01 | 10.3 | 0.09 | 6.8 | 0.17 |
| | 1.5 | 23.7 | 0.02 | 12.3 | 0.13 | 8.6 | 0.25 |
| | 2.0 | 31.1 | 0.03 | 15.8 | 0.18 | 11.4 | 0.33 |
| | 2.5 | 32.9 | 0.05 | 18.2 | 0.27 | 13.9 | 0.44 |
| 0.6 | 3.0 | 37.3 | 0.08 | 22.2 | 0.34 | 17.3 | 0.53 |
| | 3.5 | 43.6 | 0.10 | 27.3 | 0.40 | 21.4 | 0.61 |
| | 4.0 | 49.6 | 0.13 | 31.1 | 0.53 | 25.1 | 0.71 |
| | 4.5 | 55.8 | 0.18 | 36.0 | 0.62 | 29.4 | 0.78 |
| | 5.0 | 63.0 | 0.24 | 41.8 | 0.70 | 34.4 | 0.84 |
| | 5.5 | 67.7 | 0.35 | 47.2 | 0.77 | 39.7 | 0.88 |
| | 6.0 | 77.8 | 0.38 | 53.2 | 0.83 | 45.1 | 0.92 |
| | 6.5 | 84.4 | 0.50 | 59.2 | 0.89 | 50.9 | 0.94 |
| | 7.0 | 91.7 | 0.61 | 66.1 | 0.92 | 57.1 | 0.96 |
| | 7.5 | 97.5 | 0.73 | 72.5 | 0.94 | 63.4 | 0.97 |
| | 8.0 | 109.1 | 0.76 | 80.0 | 0.96 | 70.2 | 0.98 |
| | 8.5 | 115.4 | 0.85 | 87.7 | 0.97 | 77.8 | 0.99 |
| | 9.0 | 126.1 | 0.88 | 96.6 | 0.98 | 85.6 | 0.99 |
| | 9.5 | 140.5 | 0.90 | 106.0 | 0.99 | 94.8 | 0.99 |
| | | $k=2,\ n_0=10$ | | | | | |
| 0.0 | 1.0 | 9.7 | 0.07 | 5.1 | 0.22 | 3.4 | 0.35 |
| | 1.5 | 9.7 | 0.18 | 5.1 | 0.46 | 3.4 | 0.60 |
| | 2.0 | 9.7 | 0.39 | 5.1 | 0.70 | 3.4 | 0.82 |
| | 2.5 | 9.7 | 0.65 | 5.1 | 0.88 | 3.4 | 0.93 |
| | 3.0 | 9.7 | 0.85 | 5.1 | 0.96 | 3.4 | 0.98 |
| | 3.5 | 9.7 | 0.95 | 5.1 | 0.99 | 3.4 | 0.99 |
| 0.2 | 1.0 | 9.9 | 0.06 | 5.5 | 0.20 | 3.7 | 0.32 |
| | 1.5 | 10.6 | 0.15 | 5.7 | 0.40 | 4.0 | 0.54 |
| | 2.0 | 11.5 | 0.31 | 6.3 | 0.63 | 4.2 | 0.76 |
| | 2.5 | 12.2 | 0.51 | 6.8 | 0.80 | 4.8 | 0.89 |
| | 3.0 | 12.8 | 0.74 | 7.5 | 0.91 | 5.3 | 0.96 |
| | 3.5 | 14.4 | 0.86 | 8.3 | 0.97 | 5.9 | 0.99 |
| | 4.0 | 14.8 | 0.95 | 8.9 | 0.99 | 6.6 | 0.99 |
| 0.4 | 1.0 | 11.6 | 0.04 | 6.3 | 0.16 | 4.3 | 0.27 |
| | 1.5 | 12.7 | 0.10 | 7.4 | 0.29 | 5.3 | 0.44 |
| | 2.0 | 15.2 | 0.17 | 8.9 | 0.44 | 6.5 | 0.60 |
| | 2.5 | 17.6 | 0.27 | 10.9 | 0.59 | 8.1 | 0.74 |
| | 3.0 | 19.8 | 0.42 | 12.4 | 0.75 | 9.7 | 0.85 |
| | 3.5 | 22.0 | 0.59 | 14.5 | 0.85 | 11.4 | 0.92 |
| | 4.0 | 24.7 | 0.74 | 16.9 | 0.92 | 13.6 | 0.96 |
| | 4.5 | 28.9 | 0.82 | 19.5 | 0.96 | 15.7 | 0.98 |
| | 5.0 | 30.6 | 0.92 | 21.7 | 0.98 | 18.1 | 0.99 |
| 0.5 | 1.0 | 12.0 | 0.04 | 6.7 | 0.14 | 4.6 | 0.25 |
| | 1.5 | 14.9 | 0.06 | 8.6 | 0.23 | 6.1 | 0.37 |
| | 2.0 | 17.8 | 0.11 | 10.7 | 0.35 | 8.0 | 0.50 |
| | 2.5 | 19.9 | 0.20 | 13.1 | 0.47 | 10.1 | 0.63 |
| | 3.0 | 23.8 | 0.27 | 15.8 | 0.60 | 12.6 | 0.74 |
| | 3.5 | 26.9 | 0.41 | 18.5 | 0.72 | 15.1 | 0.83 |
| | 4.0 | 31.3 | 0.53 | 22.0 | 0.81 | 18.2 | 0.89 |
| | 4.5 | 34.6 | 0.67 | 25.4 | 0.88 | 21.1 | 0.94 |
| | 5.0 | 39.5 | 0.77 | 28.9 | 0.94 | 24.4 | 0.97 |
| | 5.5 | 43.8 | 0.85 | 32.9 | 0.97 | 28.0 | 0.98 |
| | 6.0 | 48.8 | 0.91 | 37.1 | 0.98 | 31.9 | 0.99 |

Table 3 (Continued.)

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\tilde{F}_{0.01}, P^*)$ | | $(\tilde{F}_{0.05}, P^*)$ | | $(\tilde{F}_{0.10}, P^*)$ | |
|---|---|---|---|---|---|---|---|
| 0.6 | 1.0 | 13.1 | 0.03 | 7.5 | 0.12 | 5.4 | 0.21 |
| | 1.5 | 16.1 | 0.04 | 9.7 | 0.18 | 7.2 | 0.29 |
| | 2.0 | 19.7 | 0.07 | 12.5 | 0.26 | 9.8 | 0.39 |
| | 2.5 | 24.3 | 0.10 | 15.7 | 0.35 | 12.4 | 0.51 |
| | 3.0 | 27.4 | 0.18 | 19.2 | 0.45 | 15.6 | 0.60 |
| | 3.5 | 32.0 | 0.26 | 22.8 | 0.57 | 19.0 | 0.71 |
| | 4.0 | 36.8 | 0.35 | 27.3 | 0.66 | 23.1 | 0.79 |
| | 4.5 | 42.6 | 0.44 | 32.0 | 0.74 | 27.5 | 0.85 |
| | 5.0 | 48.8 | 0.53 | 37.1 | 0.81 | 31.9 | 0.90 |
| | 5.5 | 54.5 | 0.65 | 42.5 | 0.88 | 36.9 | 0.94 |
| | 6.0 | 61.7 | 0.72 | 48.4 | 0.91 | 42.5 | 0.96 |
| | 6.5 | 67.9 | 0.81 | 54.0 | 0.95 | 47.8 | 0.98 |
| | 7.0 | 75.0 | 0.87 | 60.5 | 0.97 | 54.0 | 0.99 |
| | 7.5 | 84.4 | 0.90 | 67.9 | 0.98 | 60.7 | 0.99 |
| | 8.0 | 89.7 | 0.95 | 73.9 | 0.99 | 67.1 | 1.00 |

$k = 2,\ n_0 = 15$

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\tilde{F}_{0.01}, P^*)$ | | $(\tilde{F}_{0.05}, P^*)$ | | $(\tilde{F}_{0.10}, P^*)$ | |
|---|---|---|---|---|---|---|---|
| 0.0 | 1.0 | 8.3 | 0.08 | 4.5 | 0.25 | 3.1 | 0.37 |
| | 1.5 | 8.3 | 0.23 | 4.5 | 0.50 | 3.1 | 0.64 |
| | 2.0 | 8.3 | 0.47 | 4.5 | 0.74 | 3.1 | 0.84 |
| | 2.5 | 8.3 | 0.73 | 4.5 | 0.91 | 3.1 | 0.95 |
| | 3.0 | 8.3 | 0.90 | 4.5 | 0.97 | 3.1 | 0.99 |
| 0.2 | 1.0 | 9.1 | 0.07 | 5.0 | 0.22 | 3.4 | 0.34 |
| | 1.5 | 9.3 | 0.19 | 5.1 | 0.44 | 3.6 | 0.59 |
| | 2.0 | 9.7 | 0.40 | 5.6 | 0.67 | 3.9 | 0.80 |
| | 2.5 | 11.2 | 0.57 | 6.2 | 0.84 | 4.4 | 0.91 |
| | 3.0 | 11.6 | 0.79 | 6.8 | 0.94 | 5.0 | 0.97 |
| | 3.5 | 12.5 | 0.91 | 7.6 | 0.98 | 5.6 | 0.99 |
| 0.4 | 1.0 | 9.9 | 0.05 | 5.7 | 0.18 | 4.0 | 0.29 |
| | 1.5 | 11.9 | 0.11 | 7.0 | 0.30 | 5.0 | 0.45 |
| | 2.0 | 13.7 | 0.21 | 8.5 | 0.46 | 6.2 | 0.62 |
| | 2.5 | 16.0 | 0.33 | 10.3 | 0.62 | 7.8 | 0.76 |
| | 3.0 | 17.8 | 0.51 | 12.0 | 0.78 | 9.3 | 0.88 |
| | 3.5 | 20.6 | 0.65 | 14.0 | 0.87 | 11.2 | 0.93 |
| | 4.0 | 23.8 | 0.77 | 16.0 | 0.94 | 13.1 | 0.97 |
| | 4.5 | 26.0 | 0.89 | 18.4 | 0.97 | 15.1 | 0.99 |
| | 5.0 | 29.6 | 0.94 | 21.2 | 0.99 | 17.6 | 0.99 |
| 0.5 | 1.0 | 11.0 | 0.04 | 6.3 | 0.15 | 4.4 | 0.26 |
| | 1.5 | 13.3 | 0.08 | 8.1 | 0.25 | 6.0 | 0.38 |
| | 2.0 | 15.8 | 0.14 | 10.2 | 0.36 | 7.8 | 0.51 |
| | 2.5 | 19.1 | 0.21 | 12.5 | 0.50 | 9.8 | 0.66 |
| | 3.0 | 21.8 | 0.35 | 15.1 | 0.63 | 12.2 | 0.77 |
| | 3.5 | 25.2 | 0.47 | 18.0 | 0.75 | 14.8 | 0.85 |
| | 4.0 | 29.1 | 0.60 | 21.3 | 0.84 | 17.6 | 0.91 |
| | 4.5 | 33.6 | 0.70 | 24.7 | 0.91 | 20.7 | 0.95 |
| | 5.0 | 37.8 | 0.81 | 28.3 | 0.95 | 24.0 | 0.98 |
| | 5.5 | 41.9 | 0.89 | 32.1 | 0.98 | 27.5 | 0.99 |
| | 6.0 | 47.1 | 0.94 | 36.0 | 0.99 | 31.3 | 1.00 |

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\tilde{F}_{0.01}, P^*)$ | | $(\tilde{F}_{0.05}, P^*)$ | | $(\tilde{F}_{0.10}, P^*)$ | |
|---|---|---|---|---|---|---|---|
| 0.6 | 1.0 | 12.2 | 0.03 | 7.0 | 0.12 | 5.0 | 0.22 |
| | 1.5 | 14.7 | 0.06 | 9.1 | 0.20 | 6.9 | 0.32 |
| | 2.0 | 18.0 | 0.10 | 11.9 | 0.27 | 9.3 | 0.41 |
| | 2.5 | 22.2 | 0.14 | 15.1 | 0.38 | 12.2 | 0.52 |
| | 3.0 | 26.0 | 0.21 | 18.7 | 0.47 | 15.3 | 0.62 |
| | 3.5 | 31.5 | 0.26 | 22.8 | 0.57 | 19.1 | 0.71 |
| | 4.0 | 35.9 | 0.38 | 26.9 | 0.68 | 23.0 | 0.80 |
| | 4.5 | 40.4 | 0.51 | 31.0 | 0.78 | 26.7 | 0.87 |
| | 5.0 | 46.7 | 0.59 | 36.1 | 0.84 | 31.4 | 0.92 |
| | 5.5 | 53.5 | 0.67 | 41.5 | 0.90 | 36.5 | 0.95 |
| | 6.0 | 57.8 | 0.80 | 46.8 | 0.94 | 41.3 | 0.97 |
| | 6.5 | 65.3 | 0.85 | 53.3 | 0.96 | 47.2 | 0.98 |
| | 7.0 | 72.7 | 0.90 | 59.6 | 0.97 | 53.5 | 0.99 |
| | 7.5 | 81.1 | 0.94 | 66.7 | 0.99 | 60.0 | 0.99 |

$k = 3,\ n_0 = 5$

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\tilde{F}_{0.01}, P^*)$ | | $(\tilde{F}_{0.05}, P^*)$ | | $(\tilde{F}_{0.10}, P^*)$ | |
|---|---|---|---|---|---|---|---|
| 0.0 | 1.0 | 30.5 | 0.02 | 13.4 | 0.12 | 8.9 | 0.25 |
| | 1.5 | 30.5 | 0.03 | 13.4 | 0.26 | 8.9 | 0.47 |
| | 2.0 | 30.5 | 0.08 | 13.4 | 0.51 | 8.9 | 0.74 |
| | 2.5 | 30.5 | 0.20 | 13.4 | 0.77 | 8.9 | 0.90 |
| | 3.0 | 30.5 | 0.43 | 13.4 | 0.91 | 8.9 | 0.96 |
| | 3.5 | 30.5 | 0.71 | 13.4 | 0.97 | 8.9 | 0.99 |
| 0.2 | 1.0 | 30.7 | 0.02 | 13.5 | 0.11 | 9.0 | 0.23 |
| | 1.5 | 31.0 | 0.04 | 13.8 | 0.24 | 9.3 | 0.45 |
| | 2.0 | 31.7 | 0.08 | 14.3 | 0.47 | 9.8 | 0.69 |
| | 2.5 | 32.3 | 0.18 | 15.2 | 0.70 | 10.4 | 0.86 |
| | 3.0 | 33.6 | 0.38 | 16.1 | 0.87 | 11.1 | 0.95 |
| | 3.5 | 35.3 | 0.59 | 16.6 | 0.95 | 11.7 | 0.98 |
| | 4.0 | 36.1 | 0.82 | 17.8 | 0.98 | 12.8 | 0.99 |
| 0.4 | 1.0 | 31.1 | 0.02 | 14.4 | 0.10 | 9.9 | 0.21 |
| | 1.5 | 32.0 | 0.03 | 15.4 | 0.20 | 10.9 | 0.36 |
| | 2.0 | 35.7 | 0.05 | 17.5 | 0.34 | 12.5 | 0.56 |
| | 2.5 | 39.9 | 0.08 | 19.9 | 0.51 | 14.2 | 0.74 |
| | 3.0 | 41.2 | 0.19 | 22.3 | 0.70 | 17.0 | 0.85 |
| | 3.5 | 42.1 | 0.40 | 25.2 | 0.83 | 19.4 | 0.93 |
| | 4.0 | 50.2 | 0.48 | 29.2 | 0.91 | 22.3 | 0.96 |
| | 4.5 | 52.8 | 0.70 | 31.9 | 0.96 | 24.9 | 0.98 |
| | 5.0 | 57.9 | 0.83 | 35.6 | 0.98 | 28.8 | 0.99 |
| | 5.5 | 63.3 | 0.90 | 39.4 | 0.99 | 32.1 | 0.99 |
| 0.5 | 1.0 | 32.1 | 0.02 | 15.2 | 0.09 | 10.4 | 0.19 |
| | 1.5 | 35.0 | 0.02 | 17.1 | 0.16 | 12.2 | 0.31 |
| | 2.0 | 37.3 | 0.04 | 19.8 | 0.26 | 14.5 | 0.46 |
| | 2.5 | 42.5 | 0.07 | 23.1 | 0.40 | 17.6 | 0.60 |
| | 3.0 | 48.1 | 0.11 | 27.1 | 0.55 | 21.0 | 0.74 |
| | 3.5 | 51.4 | 0.21 | 31.0 | 0.70 | 24.5 | 0.85 |
| | 4.0 | 57.3 | 0.33 | 35.5 | 0.82 | 28.5 | 0.91 |
| | 4.5 | 65.2 | 0.44 | 40.2 | 0.90 | 33.4 | 0.95 |
| | 5.0 | 70.0 | 0.63 | 46.3 | 0.93 | 38.3 | 0.97 |

Table 3 (Continued.)

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\tilde{F}_{0.01}, P^*)$ | | $(\tilde{F}_{0.05}, P^*)$ | | $(\tilde{F}_{0.10}, P^*)$ | |
|---|---|---|---|---|---|---|---|
| 0.5 | 5.5 | 74.4 | 0.79 | 51.4 | 0.96 | 43.3 | 0.98 |
| | 6.0 | 84.1 | 0.86 | 57.8 | 0.98 | 48.8 | 0.99 |
| | 6.5 | 94.9 | 0.90 | 64.7 | 0.99 | 55.5 | 0.99 |
| 0.6 | 1.0 | 33.5 | 0.01 | 15.9 | 0.08 | 11.2 | 0.16 |
| | 1.5 | 37.6 | 0.02 | 18.6 | 0.13 | 13.4 | 0.26 |
| | 2.0 | 43.9 | 0.03 | 22.7 | 0.19 | 16.9 | 0.36 |
| | 2.5 | 46.8 | 0.05 | 27.0 | 0.28 | 20.7 | 0.49 |
| | 3.0 | 55.1 | 0.06 | 32.4 | 0.38 | 25.4 | 0.60 |
| | 3.5 | 62.2 | 0.09 | 37.9 | 0.51 | 30.9 | 0.70 |
| | 4.0 | 66.6 | 0.18 | 43.5 | 0.64 | 35.7 | 0.81 |
| | 4.5 | 75.9 | 0.25 | 50.5 | 0.75 | 42.0 | 0.87 |
| | 5.0 | 83.7 | 0.37 | 58.6 | 0.82 | 49.2 | 0.92 |
| | 5.5 | 93.3 | 0.49 | 66.4 | 0.88 | 56.5 | 0.95 |
| | 6.0 | 102.5 | 0.62 | 74.0 | 0.93 | 63.6 | 0.97 |
| | 6.5 | 111.2 | 0.75 | 83.1 | 0.95 | 72.5 | 0.98 |
| | 7.0 | 124.4 | 0.80 | 93.6 | 0.97 | 82.3 | 0.98 |
| | 7.5 | 135.1 | 0.88 | 102.4 | 0.98 | 91.0 | 0.99 |
| | 8.0 | 147.6 | 0.92 | 113.6 | 0.99 | 101.4 | 0.99 |
| $k = 3,\ n_0 = 10$ | | | | | | | |
| 0.0 | 1.0 | 14.3 | 0.06 | 8.2 | 0.22 | 6.0 | 0.35 |
| | 1.5 | 14.3 | 0.18 | 8.2 | 0.48 | 6.0 | 0.64 |
| | 2.0 | 14.3 | 0.44 | 8.2 | 0.77 | 6.0 | 0.87 |
| | 2.5 | 14.3 | 0.74 | 8.2 | 0.93 | 6.0 | 0.97 |
| | 3.0 | 14.3 | 0.92 | 8.2 | 0.99 | 6.0 | 0.99 |
| 0.2 | 1.0 | 14.8 | 0.05 | 8.5 | 0.21 | 6.3 | 0.33 |
| | 1.5 | 15.4 | 0.15 | 8.9 | 0.44 | 6.6 | 0.60 |
| | 2.0 | 15.8 | 0.37 | 9.5 | 0.71 | 7.0 | 0.83 |
| | 2.5 | 16.3 | 0.66 | 10.1 | 0.89 | 7.7 | 0.94 |
| | 3.0 | 17.6 | 0.85 | 10.9 | 0.97 | 8.3 | 0.99 |
| 0.4 | 1.0 | 15.8 | 0.04 | 9.5 | 0.17 | 7.1 | 0.28 |
| | 1.5 | 17.9 | 0.10 | 11.0 | 0.32 | 8.3 | 0.48 |
| | 2.0 | 19.9 | 0.22 | 12.8 | 0.52 | 9.8 | 0.69 |
| | 2.5 | 22.8 | 0.39 | 14.9 | 0.72 | 11.7 | 0.84 |
| | 3.0 | 25.9 | 0.59 | 17.3 | 0.86 | 14.0 | 0.93 |
| | 3.5 | 29.2 | 0.76 | 20.4 | 0.93 | 16.6 | 0.97 |
| | 4.0 | 33.2 | 0.88 | 23.2 | 0.98 | 19.3 | 0.99 |
| 0.5 | 1.0 | 16.4 | 0.04 | 10.1 | 0.15 | 7.6 | 0.25 |
| | 1.5 | 19.3 | 0.08 | 12.4 | 0.25 | 9.5 | 0.40 |
| | 2.0 | 23.4 | 0.13 | 15.0 | 0.41 | 11.7 | 0.58 |
| | 2.5 | 26.5 | 0.27 | 17.9 | 0.59 | 14.5 | 0.74 |
| | 3.0 | 31.3 | 0.40 | 21.8 | 0.73 | 17.6 | 0.85 |
| | 3.5 | 35.7 | 0.57 | 25.6 | 0.85 | 21.3 | 0.92 |
| | 4.0 | 41.7 | 0.70 | 30.2 | 0.92 | 25.5 | 0.96 |
| | 4.5 | 46.1 | 0.84 | 34.8 | 0.96 | 29.7 | 0.98 |
| | 5.0 | 52.3 | 0.91 | 39.8 | 0.98 | 34.5 | 0.99 |
| 0.6 | 1.0 | 18.0 | 0.03 | 11.0 | 0.12 | 8.4 | 0.21 |
| | 1.5 | 21.9 | 0.05 | 14.0 | 0.19 | 10.9 | 0.33 |

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\tilde{F}_{0.01}, P^*)$ | | $(\tilde{F}_{0.05}, P^*)$ | | $(\tilde{F}_{0.10}, P^*)$ | |
|---|---|---|---|---|---|---|---|
| 0.6 | 2.0 | 26.8 | 0.08 | 17.6 | 0.30 | 14.0 | 0.46 |
| | 2.5 | 30.2 | 0.17 | 21.5 | 0.44 | 17.6 | 0.60 |
| | 3.0 | 37.0 | 0.24 | 26.5 | 0.56 | 22.2 | 0.71 |
| | 3.5 | 42.7 | 0.36 | 31.6 | 0.69 | 26.9 | 0.82 |
| | 4.0 | 49.9 | 0.48 | 37.7 | 0.79 | 32.5 | 0.89 |
| | 4.5 | 56.3 | 0.63 | 44.0 | 0.87 | 38.4 | 0.94 |
| | 5.0 | 66.0 | 0.71 | 51.4 | 0.92 | 45.3 | 0.96 |
| | 5.5 | 73.0 | 0.84 | 58.4 | 0.96 | 52.1 | 0.98 |
| | 6.0 | 83.6 | 0.88 | 67.0 | 0.98 | 60.2 | 0.99 |
| | 6.5 | 93.0 | 0.93 | 75.9 | 0.99 | 68.7 | 0.99 |
| $k = 3,\ n_0 = 15$ | | | | | | | |
| 0.0 | 1.0 | 12.0 | 0.08 | 7.3 | 0.26 | 5.4 | 0.38 |
| | 1.5 | 12.0 | 0.26 | 7.3 | 0.54 | 5.4 | 0.68 |
| | 2.0 | 12.0 | 0.56 | 7.3 | 0.82 | 5.4 | 0.89 |
| | 2.5 | 12.0 | 0.83 | 7.3 | 0.95 | 5.4 | 0.98 |
| 0.2 | 1.0 | 12.7 | 0.07 | 7.6 | 0.24 | 5.8 | 0.36 |
| | 1.5 | 12.9 | 0.21 | 8.0 | 0.49 | 6.0 | 0.64 |
| | 2.0 | 13.9 | 0.47 | 8.5 | 0.76 | 6.4 | 0.85 |
| | 2.5 | 14.5 | 0.74 | 9.2 | 0.91 | 7.1 | 0.96 |
| | 3.0 | 15.6 | 0.90 | 10.1 | 0.98 | 7.7 | 0.99 |
| 0.4 | 1.0 | 13.8 | 0.05 | 8.5 | 0.20 | 6.5 | 0.31 |
| | 1.5 | 15.8 | 0.13 | 10.1 | 0.36 | 7.7 | 0.52 |
| | 2.0 | 18.1 | 0.27 | 12.0 | 0.56 | 9.4 | 0.71 |
| | 2.5 | 20.7 | 0.47 | 14.0 | 0.76 | 11.2 | 0.86 |
| | 3.0 | 23.7 | 0.66 | 16.8 | 0.87 | 13.6 | 0.94 |
| | 3.5 | 26.3 | 0.83 | 19.4 | 0.95 | 15.8 | 0.98 |
| | 4.0 | 31.4 | 0.90 | 22.1 | 0.98 | 18.5 | 0.99 |
| 0.5 | 1.0 | 14.4 | 0.05 | 9.3 | 0.16 | 7.0 | 0.27 |
| | 1.5 | 17.9 | 0.09 | 11.6 | 0.28 | 9.0 | 0.43 |
| | 2.0 | 20.3 | 0.20 | 14.2 | 0.45 | 11.3 | 0.60 |
| | 2.5 | 24.7 | 0.32 | 17.3 | 0.62 | 14.0 | 0.75 |
| | 3.0 | 28.5 | 0.49 | 20.7 | 0.77 | 17.2 | 0.87 |
| | 3.5 | 33.5 | 0.63 | 24.9 | 0.87 | 21.0 | 0.93 |
| | 4.0 | 37.7 | 0.80 | 28.9 | 0.94 | 24.7 | 0.97 |
| | 4.5 | 43.3 | 0.89 | 33.4 | 0.97 | 29.0 | 0.99 |
| | 5.0 | 48.5 | 0.95 | 38.8 | 0.99 | 33.8 | 1.00 |
| 0.6 | 1.0 | 15.4 | 0.04 | 10.0 | 0.13 | 7.7 | 0.22 |
| | 1.5 | 19.2 | 0.07 | 12.8 | 0.23 | 10.2 | 0.36 |
| | 2.0 | 24.5 | 0.10 | 16.6 | 0.33 | 13.4 | 0.48 |
| | 2.5 | 29.0 | 0.19 | 20.7 | 0.46 | 17.2 | 0.62 |
| | 3.0 | 34.3 | 0.30 | 25.4 | 0.60 | 21.6 | 0.73 |
| | 3.5 | 40.7 | 0.42 | 30.8 | 0.72 | 26.2 | 0.84 |
| | 4.0 | 47.2 | 0.56 | 36.7 | 0.82 | 31.9 | 0.90 |
| | 4.5 | 53.3 | 0.70 | 42.9 | 0.89 | 37.8 | 0.94 |
| | 5.0 | 61.9 | 0.80 | 50.0 | 0.94 | 44.4 | 0.97 |
| | 5.5 | 69.3 | 0.88 | 57.9 | 0.97 | 51.8 | 0.99 |
| | 6.0 | 78.9 | 0.93 | 65.8 | 0.98 | 59.4 | 0.99 |

Table 3 (Continued.)

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\tilde{F}_{0.01}, P^*)$ | | $(\tilde{F}_{0.05}, P^*)$ | | $(\tilde{F}_{0.10}, P^*)$ | |
|---|---|---|---|---|---|---|---|
| | | $k=4,\ n_0=5$ | | | | | |
| 0.0 | 1.0 | 41.0 | 0.02 | 18.6 | 0.12 | 12.7 | 0.24 |
| | 1.5 | 41.0 | 0.03 | 18.6 | 0.25 | 12.7 | 0.48 |
| | 2.0 | 41.0 | 0.07 | 18.6 | 0.52 | 12.7 | 0.77 |
| | 2.5 | 41.0 | 0.19 | 18.6 | 0.80 | 12.7 | 0.93 |
| | 3.0 | 41.0 | 0.46 | 18.6 | 0.94 | 12.7 | 0.98 |
| | 3.5 | 41.0 | 0.76 | 18.6 | 0.99 | 12.7 | 1.00 |
| 0.2 | 1.0 | 41.4 | 0.01 | 18.8 | 0.11 | 12.8 | 0.23 |
| | 1.5 | 42.3 | 0.03 | 19.2 | 0.24 | 13.4 | 0.44 |
| | 2.0 | 43.1 | 0.07 | 20.1 | 0.47 | 14.0 | 0.72 |
| | 2.5 | 43.7 | 0.15 | 20.8 | 0.74 | 14.8 | 0.89 |
| | 3.0 | 44.4 | 0.37 | 21.5 | 0.91 | 15.4 | 0.97 |
| | 3.5 | 45.9 | 0.65 | 22.4 | 0.97 | 16.0 | 0.99 |
| 0.4 | 1.0 | 41.5 | 0.01 | 19.5 | 0.10 | 13.6 | 0.21 |
| | 1.5 | 42.7 | 0.03 | 20.8 | 0.20 | 15.0 | 0.37 |
| | 2.0 | 46.3 | 0.05 | 23.7 | 0.35 | 17.2 | 0.58 |
| | 2.5 | 48.8 | 0.10 | 25.9 | 0.58 | 19.3 | 0.79 |
| | 3.0 | 52.1 | 0.22 | 29.5 | 0.76 | 22.2 | 0.90 |
| | 3.5 | 56.4 | 0.41 | 33.1 | 0.89 | 25.8 | 0.95 |
| | 4.0 | 62.2 | 0.60 | 36.8 | 0.95 | 28.9 | 0.98 |
| | 4.5 | 66.1 | 0.81 | 41.1 | 0.98 | 33.0 | 0.99 |
| 0.5 | 1.0 | 41.6 | 0.01 | 20.1 | 0.09 | 14.2 | 0.19 |
| | 1.5 | 44.3 | 0.02 | 23.1 | 0.14 | 17.2 | 0.30 |
| | 2.0 | 49.0 | 0.03 | 26.5 | 0.27 | 19.6 | 0.49 |
| | 2.5 | 53.9 | 0.07 | 29.7 | 0.45 | 23.0 | 0.67 |
| | 3.0 | 58.1 | 0.14 | 35.0 | 0.62 | 27.7 | 0.80 |
| | 3.5 | 65.8 | 0.24 | 40.7 | 0.76 | 32.0 | 0.90 |
| | 4.0 | 69.4 | 0.46 | 45.6 | 0.88 | 37.3 | 0.95 |
| | 4.5 | 81.6 | 0.56 | 52.5 | 0.93 | 43.6 | 0.97 |
| | 5.0 | 84.8 | 0.79 | 58.4 | 0.97 | 49.2 | 0.99 |
| | 5.5 | 98.5 | 0.85 | 66.8 | 0.98 | 56.3 | 0.99 |
| 0.6 | 1.0 | 43.3 | 0.01 | 21.1 | 0.09 | 15.1 | 0.17 |
| | 1.5 | 48.6 | 0.02 | 25.3 | 0.12 | 18.6 | 0.25 |
| | 2.0 | 52.8 | 0.03 | 30.1 | 0.19 | 22.7 | 0.38 |
| | 2.5 | 59.7 | 0.05 | 34.6 | 0.32 | 27.2 | 0.54 |
| | 3.0 | 68.5 | 0.07 | 41.3 | 0.44 | 33.3 | 0.66 |
| | 3.5 | 74.9 | 0.14 | 47.8 | 0.61 | 39.3 | 0.79 |
| | 4.0 | 83.5 | 0.23 | 55.9 | 0.73 | 46.8 | 0.87 |
| | 4.5 | 93.2 | 0.34 | 64.4 | 0.83 | 54.9 | 0.92 |
| | 5.0 | 104.3 | 0.49 | 73.6 | 0.90 | 63.3 | 0.96 |
| | 5.5 | 116.1 | 0.63 | 83.9 | 0.94 | 73.0 | 0.97 |
| | 6.0 | 130.4 | 0.73 | 95.7 | 0.96 | 83.7 | 0.98 |
| | 6.5 | 140.2 | 0.85 | 107.1 | 0.98 | 94.3 | 0.99 |
| | 7.0 | 155.5 | 0.91 | 119.2 | 0.99 | 106.5 | 0.99 |
| | | $k=4,\ n_0=10$ | | | | | |
| 0.0 | 1.0 | 17.8 | 0.06 | 10.9 | 0.23 | 8.3 | 0.37 |
| | 1.5 | 17.8 | 0.22 | 10.9 | 0.54 | 8.3 | 0.69 |
| 0.0 | 2.0 | 17.8 | 0.53 | 10.9 | 0.83 | 8.3 | 0.91 |
| | 2.5 | 17.8 | 0.83 | 10.9 | 0.97 | 8.3 | 0.99 |
| 0.2 | 1.0 | 18.3 | 0.05 | 11.4 | 0.21 | 8.7 | 0.35 |
| | 1.5 | 19.0 | 0.18 | 11.8 | 0.49 | 8.9 | 0.65 |
| | 2.0 | 19.6 | 0.45 | 12.3 | 0.77 | 9.5 | 0.88 |
| | 2.5 | 20.9 | 0.74 | 13.4 | 0.93 | 10.3 | 0.97 |
| | 3.0 | 21.8 | 0.92 | 14.0 | 0.99 | 11.0 | 0.99 |
| 0.4 | 1.0 | 19.6 | 0.04 | 12.6 | 0.16 | 9.6 | 0.29 |
| | 1.5 | 21.8 | 0.11 | 14.0 | 0.36 | 10.9 | 0.53 |
| | 2.0 | 24.4 | 0.27 | 16.0 | 0.61 | 12.9 | 0.75 |
| | 2.5 | 27.9 | 0.48 | 18.8 | 0.80 | 15.3 | 0.90 |
| | 3.0 | 31.9 | 0.70 | 22.0 | 0.92 | 18.0 | 0.96 |
| | 3.5 | 37.0 | 0.85 | 25.6 | 0.97 | 21.4 | 0.99 |
| 0.5 | 1.0 | 20.9 | 0.03 | 13.2 | 0.15 | 10.3 | 0.26 |
| | 1.5 | 24.5 | 0.08 | 15.7 | 0.29 | 12.5 | 0.44 |
| | 2.0 | 27.6 | 0.19 | 18.8 | 0.49 | 15.3 | 0.64 |
| | 2.5 | 32.6 | 0.33 | 23.0 | 0.67 | 18.8 | 0.81 |
| | 3.0 | 37.7 | 0.52 | 27.3 | 0.82 | 22.7 | 0.91 |
| | 3.5 | 43.4 | 0.71 | 32.4 | 0.91 | 27.4 | 0.96 |
| | 4.0 | 49.6 | 0.85 | 37.9 | 0.97 | 32.4 | 0.99 |
| | 4.5 | 57.4 | 0.92 | 44.3 | 0.98 | 38.4 | 0.99 |
| 0.6 | 1.0 | 22.8 | 0.02 | 14.1 | 0.12 | 11.0 | 0.23 |
| | 1.5 | 26.0 | 0.06 | 17.7 | 0.22 | 14.2 | 0.36 |
| | 2.0 | 31.6 | 0.11 | 21.9 | 0.36 | 18.0 | 0.53 |
| | 2.5 | 37.8 | 0.20 | 27.3 | 0.50 | 22.9 | 0.66 |
| | 3.0 | 45.1 | 0.31 | 33.2 | 0.66 | 28.5 | 0.79 |
| | 3.5 | 52.9 | 0.46 | 40.4 | 0.78 | 35.0 | 0.88 |
| | 4.0 | 61.7 | 0.60 | 47.5 | 0.88 | 41.6 | 0.94 |
| | 4.5 | 69.8 | 0.77 | 56.0 | 0.94 | 49.5 | 0.97 |
| | 5.0 | 80.7 | 0.85 | 65.4 | 0.97 | 58.3 | 0.99 |
| | 5.5 | 91.8 | 0.92 | 75.5 | 0.98 | 67.8 | 0.99 |
| | | $k=4,\ n_0=15$ | | | | | |
| 0.0 | 1.0 | 15.0 | 0.09 | 9.6 | 0.28 | 7.5 | 0.41 |
| | 1.5 | 15.0 | 0.31 | 9.6 | 0.60 | 7.5 | 0.74 |
| | 2.0 | 15.0 | 0.66 | 9.6 | 0.88 | 7.5 | 0.94 |
| | 2.5 | 15.0 | 0.90 | 9.6 | 0.98 | 7.5 | 0.99 |
| 0.2 | 1.0 | 15.5 | 0.08 | 9.9 | 0.26 | 7.7 | 0.40 |
| | 1.5 | 16.3 | 0.25 | 10.2 | 0.57 | 8.2 | 0.70 |
| | 2.0 | 17.0 | 0.57 | 11.2 | 0.82 | 8.8 | 0.90 |
| | 2.5 | 18.3 | 0.82 | 12.1 | 0.95 | 9.4 | 0.98 |
| 0.4 | 1.0 | 17.3 | 0.06 | 11.0 | 0.21 | 8.7 | 0.33 |
| | 1.5 | 19.0 | 0.16 | 12.8 | 0.41 | 10.4 | 0.55 |
| | 2.0 | 22.3 | 0.33 | 15.2 | 0.64 | 12.2 | 0.77 |
| | 2.5 | 25.5 | 0.57 | 17.9 | 0.83 | 14.7 | 0.91 |
| | 3.0 | 28.4 | 0.80 | 20.9 | 0.94 | 17.2 | 0.97 |
| | 3.5 | 32.8 | 0.91 | 24.1 | 0.98 | 20.4 | 0.99 |
| 0.5 | 1.0 | 17.8 | 0.05 | 11.7 | 0.18 | 9.3 | 0.29 |
| | 1.5 | 21.1 | 0.12 | 14.3 | 0.34 | 11.6 | 0.48 |

Table 3 (Continued.)

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\tilde{F}_{0.01}, P^*)$ | | $(\tilde{F}_{0.05}, P^*)$ | | $(\tilde{F}_{0.10}, P^*)$ | |
|---|---|---|---|---|---|---|---|
| 0.5 | 2.0 | 25.4 | 0.23 | 17.8 | 0.53 | 14.5 | 0.68 |
| | 2.5 | 29.9 | 0.40 | 21.6 | 0.71 | 17.9 | 0.83 |
| | 3.0 | 35.3 | 0.60 | 25.9 | 0.85 | 22.0 | 0.92 |
| | 3.5 | 40.3 | 0.78 | 30.9 | 0.94 | 26.6 | 0.97 |
| | 4.0 | 46.2 | 0.90 | 36.1 | 0.98 | 31.6 | 0.99 |
| 0.6 | 1.0 | 19.5 | 0.03 | 13.0 | 0.14 | 10.4 | 0.24 |
| | 1.5 | 23.5 | 0.08 | 16.3 | 0.25 | 13.2 | 0.39 |
| | 2.0 | 28.8 | 0.14 | 20.8 | 0.38 | 17.4 | 0.53 |
| | 2.5 | 35.0 | 0.25 | 25.8 | 0.55 | 21.8 | 0.70 |
| | 3.0 | 41.5 | 0.40 | 32.1 | 0.69 | 27.6 | 0.82 |
| | 3.5 | 49.6 | 0.55 | 38.3 | 0.83 | 33.5 | 0.91 |
| | 4.0 | 57.8 | 0.70 | 46.2 | 0.90 | 40.6 | 0.95 |
| | 4.5 | 66.5 | 0.82 | 54.1 | 0.95 | 48.3 | 0.98 |
| | 5.0 | 76.4 | 0.90 | 62.9 | 0.98 | 56.8 | 0.99 |
| $k = 5,\ n_0 = 5$ | | | | | | | |
| 0.0 | 1.0 | 49.4 | 0.02 | 23.2 | 0.11 | 16.2 | 0.24 |
| | 1.5 | 49.4 | 0.03 | 23.2 | 0.26 | 16.2 | 0.50 |
| | 2.0 | 49.4 | 0.07 | 23.2 | 0.56 | 16.2 | 0.80 |
| | 2.5 | 49.4 | 0.21 | 23.2 | 0.85 | 16.2 | 0.95 |
| | 3.0 | 49.4 | 0.51 | 23.2 | 0.96 | 16.2 | 0.99 |
| | 3.5 | 49.4 | 0.82 | 23.2 | 0.99 | 16.2 | 1.00 |
| 0.2 | 1.0 | 49.6 | 0.02 | 23.4 | 0.11 | 16.7 | 0.23 |
| | 1.5 | 49.9 | 0.04 | 24.0 | 0.24 | 17.1 | 0.46 |
| | 2.0 | 50.6 | 0.07 | 24.5 | 0.52 | 17.6 | 0.76 |
| | 2.5 | 51.2 | 0.17 | 25.5 | 0.80 | 18.4 | 0.93 |
| | 3.0 | 52.1 | 0.48 | 26.9 | 0.94 | 19.2 | 0.98 |
| | 3.5 | 54.4 | 0.75 | 27.4 | 0.99 | 20.2 | 1.00 |
| 0.4 | 1.0 | 51.9 | 0.02 | 24.5 | 0.10 | 17.4 | 0.21 |
| | 1.5 | 52.5 | 0.03 | 26.2 | 0.19 | 19.2 | 0.38 |
| | 2.0 | 56.2 | 0.05 | 29.4 | 0.36 | 21.7 | 0.62 |
| | 2.5 | 57.9 | 0.11 | 32.4 | 0.61 | 24.3 | 0.83 |
| | 3.0 | 61.9 | 0.26 | 36.3 | 0.81 | 27.9 | 0.93 |
| | 3.5 | 67.7 | 0.47 | 40.0 | 0.93 | 31.6 | 0.97 |
| | 4.0 | 70.9 | 0.75 | 44.5 | 0.97 | 35.9 | 0.99 |
| 0.5 | 1.0 | 53.2 | 0.01 | 26.1 | 0.08 | 18.6 | 0.18 |
| | 1.5 | 54.8 | 0.02 | 27.9 | 0.16 | 20.8 | 0.33 |
| | 2.0 | 58.1 | 0.04 | 32.3 | 0.29 | 24.8 | 0.50 |
| | 2.5 | 63.6 | 0.08 | 36.7 | 0.49 | 28.4 | 0.72 |
| | 3.0 | 68.6 | 0.17 | 41.6 | 0.70 | 33.4 | 0.86 |
| | 3.5 | 75.0 | 0.33 | 47.9 | 0.84 | 38.9 | 0.94 |
| | 4.0 | 85.2 | 0.50 | 55.9 | 0.92 | 45.9 | 0.97 |
| | 4.5 | 95.6 | 0.68 | 63.5 | 0.97 | 53.0 | 0.99 |
| | 5.0 | 100.7 | 0.87 | 70.5 | 0.98 | 60.2 | 0.99 |
| 0.6 | 1.0 | 52.5 | 0.01 | 26.5 | 0.08 | 19.4 | 0.16 |
| | 1.5 | 59.4 | 0.02 | 30.8 | 0.13 | 23.2 | 0.26 |
| | 2.0 | 64.2 | 0.03 | 36.0 | 0.21 | 27.6 | 0.41 |
| | 2.5 | 70.3 | 0.05 | 41.4 | 0.36 | 33.1 | 0.59 |
| | 3.0 | 79.6 | 0.08 | 49.4 | 0.51 | 40.4 | 0.73 |
| 0.6 | 3.5 | 91.3 | 0.14 | 57.8 | 0.67 | 48.2 | 0.84 |
| | 4.0 | 100.2 | 0.26 | 67.9 | 0.80 | 57.1 | 0.91 |
| | 4.5 | 112.2 | 0.40 | 78.1 | 0.88 | 66.7 | 0.95 |
| | 5.0 | 123.6 | 0.60 | 88.3 | 0.94 | 77.1 | 0.97 |
| | 5.5 | 141.0 | 0.70 | 101.0 | 0.96 | 89.1 | 0.98 |
| | 6.0 | 152.7 | 0.85 | 114.8 | 0.98 | 101.5 | 0.99 |
| $k = 5,\ n_0 = 10$ | | | | | | | |
| 0.0 | 1.0 | 21.1 | 0.07 | 13.3 | 0.25 | 10.4 | 0.40 |
| | 1.5 | 21.1 | 0.25 | 13.3 | 0.59 | 10.4 | 0.75 |
| | 2.0 | 21.1 | 0.60 | 13.3 | 0.88 | 10.4 | 0.94 |
| | 2.5 | 21.1 | 0.89 | 13.3 | 0.98 | 10.4 | 0.99 |
| 0.2 | 1.0 | 21.3 | 0.06 | 13.8 | 0.23 | 10.9 | 0.37 |
| | 1.5 | 22.3 | 0.21 | 14.2 | 0.55 | 11.2 | 0.70 |
| | 2.0 | 23.0 | 0.52 | 14.9 | 0.83 | 12.0 | 0.91 |
| | 2.5 | 24.0 | 0.83 | 15.8 | 0.96 | 12.5 | 0.99 |
| 0.4 | 1.0 | 23.3 | 0.05 | 15.0 | 0.19 | 11.8 | 0.31 |
| | 1.5 | 26.1 | 0.12 | 16.9 | 0.41 | 13.5 | 0.58 |
| | 2.0 | 29.5 | 0.30 | 19.7 | 0.66 | 15.9 | 0.80 |
| | 2.5 | 32.5 | 0.57 | 22.5 | 0.86 | 18.6 | 0.93 |
| | 3.0 | 36.8 | 0.80 | 26.2 | 0.95 | 22.0 | 0.98 |
| | 3.5 | 41.8 | 0.93 | 30.5 | 0.99 | 25.8 | 0.99 |
| 0.5 | 1.0 | 23.7 | 0.04 | 15.8 | 0.17 | 12.5 | 0.28 |
| | 1.5 | 28.7 | 0.08 | 19.0 | 0.32 | 15.4 | 0.48 |
| | 2.0 | 32.2 | 0.23 | 22.8 | 0.54 | 18.8 | 0.70 |
| | 2.5 | 37.2 | 0.43 | 27.0 | 0.75 | 22.8 | 0.86 |
| | 3.0 | 43.7 | 0.64 | 32.6 | 0.88 | 27.7 | 0.95 |
| | 3.5 | 51.0 | 0.80 | 38.8 | 0.95 | 33.5 | 0.98 |
| | 4.0 | 57.0 | 0.93 | 44.8 | 0.99 | 39.4 | 0.99 |
| 0.6 | 1.0 | 24.9 | 0.04 | 16.9 | 0.14 | 13.6 | 0.24 |
| | 1.5 | 29.8 | 0.07 | 21.2 | 0.24 | 17.2 | 0.39 |
| | 2.0 | 37.1 | 0.13 | 26.7 | 0.39 | 22.0 | 0.57 |
| | 2.5 | 44.0 | 0.25 | 32.8 | 0.57 | 27.8 | 0.73 |
| | 3.0 | 52.5 | 0.41 | 40.2 | 0.73 | 34.6 | 0.85 |
| | 3.5 | 61.5 | 0.58 | 47.9 | 0.85 | 42.1 | 0.92 |
| | 4.0 | 72.4 | 0.73 | 57.6 | 0.92 | 50.6 | 0.97 |
| | 4.5 | 82.3 | 0.86 | 67.1 | 0.97 | 59.9 | 0.99 |
| | 5.0 | 95.2 | 0.92 | 78.0 | 0.99 | 70.6 | 0.99 |
| $k = 5,\ n_0 = 15$ | | | | | | | |
| 0.0 | 1.0 | 17.5 | 0.10 | 11.7 | 0.31 | 9.3 | 0.45 |
| | 1.5 | 17.5 | 0.36 | 11.7 | 0.67 | 9.3 | 0.79 |
| | 2.0 | 17.5 | 0.75 | 11.7 | 0.92 | 9.3 | 0.96 |
| | 2.5 | 17.5 | 0.95 | 11.7 | 0.99 | 9.3 | 0.99 |
| 0.2 | 1.0 | 17.6 | 0.10 | 12.1 | 0.28 | 9.6 | 0.42 |
| | 1.5 | 18.7 | 0.32 | 12.5 | 0.62 | 10.1 | 0.75 |
| | 2.0 | 19.6 | 0.65 | 13.4 | 0.88 | 10.8 | 0.94 |
| | 2.5 | 21.3 | 0.89 | 14.2 | 0.98 | 11.6 | 0.99 |

Table 3 (Continued.)

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\tilde{F}_{0.01}, P^*)$ | | $(\tilde{F}_{0.05}, P^*)$ | | $(\tilde{F}_{0.10}, P^*)$ | |
|---|---|---|---|---|---|---|---|
| 0.4 | 1.0 | 19.9 | 0.07 | 13.4 | 0.22 | 10.7 | 0.36 |
| | 1.5 | 22.1 | 0.20 | 15.6 | 0.45 | 12.6 | 0.61 |
| | 2.0 | 25.0 | 0.44 | 17.9 | 0.73 | 14.7 | 0.84 |
| | 2.5 | 28.8 | 0.69 | 21.2 | 0.89 | 17.7 | 0.95 |
| | 3.0 | 33.6 | 0.87 | 24.9 | 0.97 | 21.0 | 0.99 |
| 0.5 | 1.0 | 20.6 | 0.06 | 14.4 | 0.19 | 11.6 | 0.31 |
| | 1.5 | 24.6 | 0.14 | 17.4 | 0.37 | 14.2 | 0.53 |
| | 2.0 | 28.9 | 0.30 | 21.2 | 0.60 | 17.6 | 0.74 |
| | 2.5 | 34.5 | 0.50 | 25.5 | 0.79 | 21.6 | 0.89 |
| | 3.0 | 41.4 | 0.70 | 31.3 | 0.91 | 26.9 | 0.96 |
| | 3.5 | 47.6 | 0.86 | 36.9 | 0.97 | 32.1 | 0.99 |
| 0.6 | 1.0 | 22.8 | 0.04 | 15.6 | 0.15 | 12.6 | 0.26 |
| | 1.5 | 27.6 | 0.09 | 19.6 | 0.28 | 16.2 | 0.42 |
| | 2.0 | 33.7 | 0.18 | 24.5 | 0.46 | 20.7 | 0.61 |
| | 2.5 | 40.8 | 0.32 | 31.1 | 0.62 | 26.8 | 0.76 |
| | 3.0 | 48.6 | 0.49 | 38.2 | 0.77 | 33.2 | 0.87 |
| | 3.5 | 56.6 | 0.70 | 45.7 | 0.89 | 41.0 | 0.94 |
| | 4.0 | 68.9 | 0.79 | 55.6 | 0.95 | 49.7 | 0.97 |
| | 4.5 | 78.4 | 0.90 | 65.1 | 0.98 | 58.9 | 0.99 |
| $k=6,\ n_0=5$ | | | | | | | |
| 0.0 | 1.0 | 57.7 | 0.01 | 27.6 | 0.11 | 19.7 | 0.24 |
| | 1.5 | 57.7 | 0.03 | 27.6 | 0.27 | 19.7 | 0.51 |
| | 2.0 | 57.7 | 0.08 | 27.6 | 0.60 | 19.7 | 0.83 |
| | 2.5 | 57.7 | 0.23 | 27.6 | 0.88 | 19.7 | 0.96 |
| | 3.0 | 57.7 | 0.56 | 27.6 | 0.98 | 19.7 | 0.99 |
| 0.2 | 1.0 | 57.9 | 0.02 | 28.0 | 0.11 | 19.9 | 0.24 |
| | 1.5 | 58.6 | 0.03 | 28.9 | 0.26 | 20.3 | 0.49 |
| | 2.0 | 60.1 | 0.06 | 29.8 | 0.52 | 21.1 | 0.78 |
| | 2.5 | 60.8 | 0.23 | 30.3 | 0.84 | 21.7 | 0.95 |
| | 3.0 | 61.1 | 0.50 | 31.4 | 0.96 | 23.4 | 0.99 |
| | 3.5 | 63.4 | 0.80 | 33.2 | 0.99 | 24.5 | 1.00 |
| 0.4 | 1.0 | 58.2 | 0.02 | 29.4 | 0.10 | 21.2 | 0.21 |
| | 1.5 | 61.0 | 0.03 | 31.1 | 0.20 | 23.1 | 0.39 |
| | 2.0 | 62.1 | 0.05 | 33.7 | 0.41 | 25.5 | 0.66 |
| | 2.5 | 67.6 | 0.11 | 38.0 | 0.66 | 29.5 | 0.85 |
| | 3.0 | 69.5 | 0.33 | 41.6 | 0.87 | 32.6 | 0.95 |
| | 3.5 | 76.2 | 0.58 | 46.9 | 0.95 | 37.3 | 0.98 |
| | 4.0 | 83.5 | 0.80 | 51.0 | 0.99 | 41.9 | 0.99 |
| 0.5 | 1.0 | 60.0 | 0.01 | 30.3 | 0.09 | 22.3 | 0.19 |
| | 1.5 | 61.7 | 0.02 | 32.8 | 0.17 | 24.8 | 0.34 |
| | 2.0 | 68.5 | 0.04 | 37.5 | 0.31 | 28.8 | 0.56 |
| | 2.5 | 72.8 | 0.09 | 42.7 | 0.53 | 33.9 | 0.76 |
| | 3.0 | 78.6 | 0.20 | 48.6 | 0.76 | 39.4 | 0.89 |
| | 3.5 | 88.1 | 0.36 | 56.5 | 0.88 | 46.4 | 0.95 |
| | 4.0 | 95.6 | 0.62 | 63.9 | 0.95 | 53.5 | 0.98 |
| | 4.5 | 106.5 | 0.80 | 73.3 | 0.98 | 62.0 | 0.99 |
| 0.6 | 1.0 | 61.0 | 0.01 | 31.0 | 0.08 | 23.0 | 0.17 |
| | 1.5 | 65.4 | 0.02 | 36.1 | 0.13 | 27.4 | 0.28 |
| 0.6 | 2.0 | 71.5 | 0.03 | 41.7 | 0.24 | 32.8 | 0.44 |
| | 2.5 | 80.0 | 0.06 | 48.8 | 0.39 | 39.1 | 0.63 |
| | 3.0 | 91.7 | 0.09 | 58.4 | 0.55 | 47.9 | 0.77 |
| | 3.5 | 99.6 | 0.21 | 66.8 | 0.74 | 56.4 | 0.88 |
| | 4.0 | 113.3 | 0.34 | 79.0 | 0.85 | 67.4 | 0.94 |
| | 4.5 | 126.9 | 0.53 | 91.4 | 0.92 | 79.5 | 0.97 |
| | 5.0 | 141.8 | 0.70 | 105.1 | 0.96 | 91.7 | 0.98 |
| | 5.5 | 158.1 | 0.82 | 119.3 | 0.98 | 104.9 | 0.99 |
| | 6.0 | 177.8 | 0.90 | 134.2 | 0.99 | 120.5 | 0.99 |
| $k=6,\ n_0=10$ | | | | | | | |
| 0.0 | 1.0 | 24.2 | 0.07 | 15.6 | 0.26 | 12.4 | 0.42 |
| | 1.5 | 24.2 | 0.28 | 15.6 | 0.63 | 12.4 | 0.78 |
| | 2.0 | 24.2 | 0.67 | 15.6 | 0.92 | 12.4 | 0.96 |
| | 2.5 | 24.2 | 0.93 | 15.6 | 0.99 | 12.4 | 0.99 |
| 0.2 | 1.0 | 25.1 | 0.06 | 16.1 | 0.25 | 12.8 | 0.40 |
| | 1.5 | 25.6 | 0.23 | 16.8 | 0.58 | 13.5 | 0.73 |
| | 2.0 | 26.6 | 0.59 | 17.6 | 0.87 | 14.1 | 0.94 |
| | 2.5 | 28.5 | 0.86 | 18.4 | 0.98 | 15.0 | 0.99 |
| 0.4 | 1.0 | 25.9 | 0.05 | 17.4 | 0.20 | 13.9 | 0.34 |
| | 1.5 | 29.3 | 0.15 | 19.8 | 0.44 | 16.0 | 0.61 |
| | 2.0 | 32.5 | 0.38 | 22.6 | 0.73 | 18.7 | 0.85 |
| | 2.5 | 37.2 | 0.65 | 26.5 | 0.90 | 22.0 | 0.96 |
| | 3.0 | 42.0 | 0.87 | 30.2 | 0.98 | 25.6 | 0.99 |
| 0.5 | 1.0 | 28.1 | 0.04 | 18.6 | 0.17 | 15.1 | 0.29 |
| | 1.5 | 31.4 | 0.11 | 21.9 | 0.35 | 17.9 | 0.53 |
| | 2.0 | 37.1 | 0.26 | 26.7 | 0.58 | 21.9 | 0.75 |
| | 2.5 | 41.8 | 0.52 | 31.6 | 0.80 | 26.8 | 0.90 |
| | 3.0 | 50.2 | 0.71 | 38.2 | 0.92 | 32.6 | 0.96 |
| | 3.5 | 58.3 | 0.88 | 44.9 | 0.97 | 39.0 | 0.99 |
| 0.6 | 1.0 | 29.5 | 0.03 | 19.8 | 0.14 | 16.1 | 0.24 |
| | 1.5 | 34.8 | 0.07 | 24.5 | 0.27 | 20.1 | 0.43 |
| | 2.0 | 41.0 | 0.17 | 30.4 | 0.46 | 25.7 | 0.63 |
| | 2.5 | 50.3 | 0.30 | 37.8 | 0.64 | 32.3 | 0.79 |
| | 3.0 | 59.6 | 0.48 | 46.1 | 0.80 | 40.5 | 0.89 |
| | 3.5 | 69.5 | 0.68 | 56.0 | 0.90 | 49.3 | 0.95 |
| | 4.0 | 82.3 | 0.82 | 66.8 | 0.95 | 59.4 | 0.98 |
| | 4.5 | 95.4 | 0.91 | 78.2 | 0.98 | 70.9 | 0.99 |
| $k=6,\ n_0=15$ | | | | | | | |
| 0.0 | 1.0 | 20.0 | 0.11 | 13.7 | 0.32 | 11.1 | 0.47 |
| | 1.5 | 20.0 | 0.40 | 13.7 | 0.70 | 11.1 | 0.82 |
| | 2.0 | 20.0 | 0.81 | 13.7 | 0.95 | 11.1 | 0.98 |
| 0.2 | 1.0 | 20.6 | 0.10 | 14.2 | 0.30 | 11.6 | 0.44 |
| | 1.5 | 21.8 | 0.34 | 15.0 | 0.65 | 12.3 | 0.78 |
| | 2.0 | 22.3 | 0.73 | 15.5 | 0.92 | 12.7 | 0.96 |
| | 2.5 | 23.2 | 0.95 | 16.7 | 0.99 | 13.7 | 0.99 |
| 0.4 | 1.0 | 22.2 | 0.08 | 15.5 | 0.25 | 12.6 | 0.38 |

Table 3 (Continued.)

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\tilde{F}_{0.01}, P^*)$ | | $(\tilde{F}_{0.05}, P^*)$ | | $(\tilde{F}_{0.10}, P^*)$ | |
|---|---|---|---|---|---|---|---|
| 0.4 | 1.5 | 24.7 | 0.24 | 17.9 | 0.51 | 14.9 | 0.65 |
| | 2.0 | 28.8 | 0.49 | 20.9 | 0.78 | 17.4 | 0.88 |
| | 2.5 | 33.3 | 0.76 | 24.7 | 0.93 | 20.7 | 0.97 |
| | 3.0 | 38.4 | 0.92 | 28.9 | 0.99 | 24.4 | 0.99 |
| 0.5 | 1.0 | 23.4 | 0.06 | 16.7 | 0.20 | 13.6 | 0.33 |
| | 1.5 | 27.7 | 0.16 | 20.1 | 0.41 | 16.7 | 0.57 |
| | 2.0 | 32.8 | 0.36 | 24.5 | 0.66 | 20.9 | 0.78 |
| | 2.5 | 39.1 | 0.59 | 29.8 | 0.84 | 25.6 | 0.91 |
| | 3.0 | 46.5 | 0.79 | 35.9 | 0.94 | 31.2 | 0.98 |
| | 3.5 | 54.6 | 0.92 | 43.0 | 0.99 | 37.8 | 0.99 |
| 0.6 | 1.0 | 24.8 | 0.05 | 17.9 | 0.17 | 14.8 | 0.28 |
| | 1.5 | 30.0 | 0.12 | 22.4 | 0.31 | 19.0 | 0.46 |
| | 2.0 | 38.4 | 0.21 | 28.6 | 0.50 | 24.6 | 0.65 |
| | 2.5 | 46.3 | 0.38 | 35.6 | 0.70 | 30.9 | 0.82 |
| | 3.0 | 55.4 | 0.58 | 44.3 | 0.83 | 39.3 | 0.91 |
| | 3.5 | 66.3 | 0.75 | 53.9 | 0.93 | 48.2 | 0.96 |
| | 4.0 | 76.1 | 0.89 | 63.9 | 0.97 | 57.8 | 0.99 |
| | 4.5 | 89.5 | 0.95 | 76.2 | 0.99 | 68.6 | 1.00 |
| $k = 8,\ n_0 = 5$ | | | | | | | |
| 0.0 | 1.0 | 71.2 | 0.02 | 36.0 | 0.12 | 26.4 | 0.25 |
| | 1.5 | 71.2 | 0.03 | 36.0 | 0.30 | 26.4 | 0.55 |
| | 2.0 | 71.2 | 0.09 | 36.0 | 0.67 | 26.4 | 0.88 |
| | 2.5 | 71.2 | 0.30 | 36.0 | 0.93 | 26.4 | 0.98 |
| | 3.0 | 71.2 | 0.70 | 36.0 | 0.99 | 26.4 | 0.99 |
| 0.2 | 1.0 | 71.6 | 0.01 | 36.8 | 0.11 | 27.0 | 0.24 |
| | 1.5 | 72.1 | 0.03 | 37.2 | 0.27 | 27.6 | 0.53 |
| | 2.0 | 73.2 | 0.07 | 37.9 | 0.61 | 28.5 | 0.83 |
| | 2.5 | 73.7 | 0.27 | 38.6 | 0.90 | 29.2 | 0.97 |
| | 3.0 | 76.5 | 0.60 | 40.3 | 0.98 | 30.6 | 0.99 |
| 0.4 | 1.0 | 72.3 | 0.02 | 37.9 | 0.10 | 28.3 | 0.21 |
| | 1.5 | 76.6 | 0.03 | 40.0 | 0.22 | 30.8 | 0.42 |
| | 2.0 | 78.9 | 0.06 | 43.8 | 0.46 | 33.8 | 0.72 |
| | 2.5 | 80.6 | 0.18 | 47.6 | 0.77 | 37.2 | 0.92 |
| | 3.0 | 88.9 | 0.39 | 54.8 | 0.92 | 43.4 | 0.98 |
| | 3.5 | 95.0 | 0.70 | 58.1 | 0.98 | 48.6 | 0.99 |
| 0.5 | 1.0 | 74.3 | 0.02 | 39.9 | 0.08 | 29.2 | 0.19 |
| | 1.5 | 80.5 | 0.02 | 43.3 | 0.18 | 33.0 | 0.36 |
| | 2.0 | 90.4 | 0.03 | 48.4 | 0.36 | 37.5 | 0.62 |
| | 2.5 | 92.9 | 0.09 | 54.9 | 0.61 | 43.9 | 0.83 |
| | 3.0 | 100.3 | 0.23 | 62.5 | 0.84 | 51.7 | 0.94 |
| | 3.5 | 108.9 | 0.49 | 70.9 | 0.94 | 59.8 | 0.98 |
| | 4.0 | 120.1 | 0.75 | 82.8 | 0.98 | 69.8 | 0.99 |
| 0.6 | 1.0 | 73.9 | 0.01 | 40.6 | 0.08 | 30.5 | 0.18 |
| | 1.5 | 82.6 | 0.02 | 45.8 | 0.15 | 35.7 | 0.29 |
| | 2.0 | 90.0 | 0.03 | 54.0 | 0.26 | 42.9 | 0.48 |
| | 2.5 | 103.2 | 0.06 | 62.6 | 0.46 | 51.4 | 0.70 |
| | 3.0 | 113.4 | 0.12 | 74.0 | 0.65 | 62.2 | 0.83 |
| | 3.5 | 124.1 | 0.28 | 86.2 | 0.82 | 73.6 | 0.93 |
| | 4.0 | 139.2 | 0.49 | 100.1 | 0.92 | 86.4 | 0.97 |

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\tilde{F}_{0.01}, P^*)$ | | $(\tilde{F}_{0.05}, P^*)$ | | $(\tilde{F}_{0.10}, P^*)$ | |
|---|---|---|---|---|---|---|---|
| 0.6 | 4.5 | 160.3 | 0.66 | 116.8 | 0.96 | 102.4 | 0.99 |
| | 5.0 | 175.6 | 0.85 | 134.3 | 0.98 | 118.0 | 0.99 |
| $k = 8,\ n_0 = 10$ | | | | | | | |
| 0.0 | 1.0 | 29.4 | 0.08 | 20.0 | 0.30 | 16.3 | 0.46 |
| | 1.5 | 29.4 | 0.36 | 20.0 | 0.72 | 16.3 | 0.84 |
| | 2.0 | 29.4 | 0.80 | 20.0 | 0.96 | 16.3 | 0.98 |
| 0.2 | 1.0 | 29.9 | 0.08 | 20.8 | 0.27 | 17.0 | 0.43 |
| | 1.5 | 31.1 | 0.29 | 21.2 | 0.66 | 17.5 | 0.80 |
| | 2.0 | 31.3 | 0.75 | 22.1 | 0.94 | 18.4 | 0.97 |
| 0.4 | 1.0 | 32.0 | 0.06 | 22.0 | 0.23 | 18.2 | 0.37 |
| | 1.5 | 35.3 | 0.21 | 24.9 | 0.53 | 20.9 | 0.69 |
| | 2.0 | 39.3 | 0.51 | 28.7 | 0.81 | 24.1 | 0.91 |
| | 2.5 | 44.8 | 0.80 | 33.3 | 0.95 | 28.4 | 0.98 |
| 0.5 | 1.0 | 32.8 | 0.05 | 23.6 | 0.19 | 19.5 | 0.32 |
| | 1.5 | 38.3 | 0.15 | 27.5 | 0.43 | 23.0 | 0.60 |
| | 2.0 | 44.3 | 0.36 | 33.2 | 0.69 | 28.4 | 0.82 |
| | 2.5 | 53.0 | 0.61 | 39.7 | 0.88 | 34.3 | 0.95 |
| | 3.0 | 60.6 | 0.86 | 47.5 | 0.97 | 41.6 | 0.99 |
| 0.6 | 1.0 | 35.1 | 0.04 | 25.2 | 0.15 | 20.6 | 0.28 |
| | 1.5 | 42.1 | 0.09 | 30.9 | 0.31 | 26.0 | 0.48 |
| | 2.0 | 51.1 | 0.21 | 38.4 | 0.53 | 33.1 | 0.70 |
| | 2.5 | 60.2 | 0.43 | 47.3 | 0.75 | 41.5 | 0.86 |
| | 3.0 | 74.9 | 0.60 | 58.7 | 0.88 | 51.9 | 0.94 |
| | 3.5 | 86.9 | 0.82 | 71.0 | 0.95 | 63.7 | 0.98 |
| | 4.0 | 101.3 | 0.93 | 84.5 | 0.98 | 76.6 | 0.99 |
| $k = 8,\ n_0 = 15$ | | | | | | | |
| 0.0 | 1.0 | 24.4 | 0.13 | 17.4 | 0.37 | 14.5 | 0.52 |
| | 1.5 | 24.4 | 0.52 | 17.4 | 0.80 | 14.5 | 0.89 |
| | 2.0 | 24.4 | 0.91 | 17.4 | 0.98 | 14.5 | 0.99 |
| 0.2 | 1.0 | 24.8 | 0.13 | 18.0 | 0.35 | 15.0 | 0.50 |
| | 1.5 | 26.3 | 0.44 | 18.8 | 0.75 | 15.7 | 0.85 |
| | 2.0 | 27.0 | 0.85 | 19.8 | 0.96 | 16.6 | 0.98 |
| 0.4 | 1.0 | 26.9 | 0.09 | 19.8 | 0.28 | 16.6 | 0.42 |
| | 1.5 | 30.6 | 0.29 | 22.8 | 0.58 | 19.2 | 0.73 |
| | 2.0 | 35.0 | 0.63 | 26.2 | 0.87 | 22.3 | 0.94 |
| | 2.5 | 40.0 | 0.88 | 31.0 | 0.97 | 26.6 | 0.99 |
| 0.5 | 1.0 | 29.0 | 0.07 | 21.2 | 0.23 | 17.9 | 0.36 |
| | 1.5 | 33.2 | 0.22 | 25.3 | 0.49 | 21.6 | 0.65 |
| | 2.0 | 40.2 | 0.46 | 30.7 | 0.75 | 26.5 | 0.86 |
| | 2.5 | 46.9 | 0.75 | 37.0 | 0.92 | 32.5 | 0.96 |
| | 3.0 | 56.8 | 0.91 | 45.1 | 0.98 | 39.9 | 0.99 |
| 0.6 | 1.0 | 31.0 | 0.05 | 22.6 | 0.18 | 19.2 | 0.30 |
| | 1.5 | 37.3 | 0.14 | 28.7 | 0.36 | 24.5 | 0.52 |
| | 2.0 | 46.5 | 0.28 | 35.8 | 0.59 | 31.3 | 0.73 |
| | 2.5 | 57.0 | 0.50 | 44.5 | 0.80 | 39.9 | 0.89 |
| | 3.0 | 68.3 | 0.72 | 55.7 | 0.92 | 50.1 | 0.96 |
| | 3.5 | 80.9 | 0.89 | 67.8 | 0.97 | 61.5 | 0.99 |

Table 4
Critical values and power of the chi-square test

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\chi^2_{0.01}(\Delta), P^*)$ | | $(\chi^2_{0.05}(\Delta), P^*)$ | | $(\chi^2_{0.10}(\Delta), P^*)$ | |
|---|---|---|---|---|---|---|---|
| | | | | $k=2$ | | | |
| 0.0 | 1.0 | 6.63 | 0.12 | 3.84 | 0.29 | 2.71 | 0.41 |
| | 1.5 | 6.63 | 0.32 | 3.84 | 0.56 | 2.71 | 0.68 |
| | 2.0 | 6.63 | 0.60 | 3.84 | 0.81 | 2.71 | 0.88 |
| | 2.5 | 6.63 | 0.83 | 3.84 | 0.94 | 2.71 | 0.97 |
| | 3.0 | 6.63 | 0.95 | 3.84 | 0.99 | 2.71 | 1.00 |
| 0.2 | 1.0 | 7.14 | 0.10 | 4.14 | 0.27 | 2.92 | 0.38 |
| | 1.5 | 7.71 | 0.26 | 4.51 | 0.50 | 3.19 | 0.63 |
| | 2.0 | 8.42 | 0.47 | 5.00 | 0.72 | 3.57 | 0.83 |
| | 2.5 | 9.22 | 0.69 | 5.58 | 0.88 | 4.03 | 0.94 |
| | 3.0 | 10.09 | 0.86 | 6.24 | 0.96 | 4.57 | 0.98 |
| | 3.5 | 11.00 | 0.95 | 6.95 | 0.99 | 5.17 | 1.00 |
| 0.4 | 1.0 | 8.42 | 0.07 | 5.00 | 0.21 | 3.57 | 0.32 |
| | 1.5 | 10.09 | 0.15 | 6.24 | 0.35 | 4.57 | 0.49 |
| | 2.0 | 11.96 | 0.26 | 7.71 | 0.52 | 5.83 | 0.66 |
| | 2.5 | 13.99 | 0.42 | 9.36 | 0.68 | 7.27 | 0.80 |
| | 3.0 | 16.19 | 0.59 | 11.17 | 0.82 | 8.87 | 0.90 |
| | 3.5 | 18.54 | 0.74 | 13.14 | 0.91 | 10.64 | 0.95 |
| | 4.0 | 21.06 | 0.86 | 15.27 | 0.96 | 12.56 | 0.98 |
| | 4.5 | 23.74 | 0.93 | 17.56 | 0.99 | 14.65 | 0.99 |
| 0.5 | 1.0 | 9.22 | 0.05 | 5.58 | 0.17 | 4.03 | 0.28 |
| | 1.5 | 11.47 | 0.10 | 7.32 | 0.28 | 5.49 | 0.41 |
| | 2.0 | 13.99 | 0.18 | 9.36 | 0.41 | 7.27 | 0.55 |
| | 2.5 | 16.76 | 0.29 | 11.65 | 0.55 | 9.30 | 0.69 |
| | 3.0 | 19.78 | 0.42 | 14.18 | 0.68 | 11.58 | 0.80 |
| | 3.5 | 23.05 | 0.56 | 16.97 | 0.80 | 14.11 | 0.88 |
| | 4.0 | 26.57 | 0.69 | 20.01 | 0.88 | 16.89 | 0.94 |
| | 4.5 | 30.34 | 0.80 | 23.30 | 0.94 | 19.92 | 0.97 |
| | 5.0 | 34.36 | 0.89 | 26.84 | 0.97 | 23.20 | 0.99 |
| | 5.5 | 38.63 | 0.94 | 30.62 | 0.99 | 26.74 | 1.00 |
| 0.6 | 1.0 | 10.09 | 0.04 | 6.24 | 0.14 | 4.57 | 0.23 |
| | 1.5 | 12.95 | 0.07 | 8.51 | 0.21 | 6.53 | 0.33 |
| | 2.0 | 16.19 | 0.12 | 11.17 | 0.30 | 8.87 | 0.44 |
| | 2.5 | 19.78 | 0.18 | 14.18 | 0.41 | 11.58 | 0.55 |
| | 3.0 | 23.74 | 0.26 | 17.56 | 0.52 | 14.65 | 0.66 |
| | 3.5 | 28.05 | 0.36 | 21.30 | 0.63 | 18.07 | 0.76 |
| | 4.0 | 32.72 | 0.47 | 25.39 | 0.73 | 21.86 | 0.84 |
| | 4.5 | 37.76 | 0.59 | 29.85 | 0.82 | 26.01 | 0.90 |
| | 5.0 | 43.15 | 0.69 | 34.66 | 0.88 | 30.52 | 0.94 |
| | 5.5 | 48.91 | 0.78 | 39.84 | 0.93 | 35.38 | 0.97 |
| | 6.0 | 55.02 | 0.86 | 45.37 | 0.96 | 40.61 | 0.98 |
| | 6.5 | 61.49 | 0.91 | 51.27 | 0.98 | 46.20 | 0.99 |
| | 7.0 | 68.33 | 0.95 | 57.53 | 0.99 | 52.15 | 1.00 |
| | | | | $k=3$ | | | |
| 0.0 | 1.0 | 9.21 | 0.14 | 5.99 | 0.32 | 4.61 | 0.44 |
| | 1.5 | 9.21 | 0.40 | 5.99 | 0.64 | 4.61 | 0.75 |
| | 2.0 | 9.21 | 0.72 | 5.99 | 0.88 | 4.61 | 0.93 |
| | 2.5 | 9.21 | 0.92 | 5.99 | 0.98 | 4.61 | 0.99 |
| 0.2 | 1.0 | 9.74 | 0.12 | 6.35 | 0.29 | 4.88 | 0.42 |
| | 1.5 | 10.36 | 0.33 | 6.77 | 0.58 | 5.22 | 0.70 |
| | 2.0 | 11.15 | 0.61 | 7.34 | 0.82 | 5.68 | 0.90 |
| | 2.5 | 12.06 | 0.84 | 8.04 | 0.95 | 6.26 | 0.98 |
| 0.4 | 1.0 | 11.15 | 0.08 | 7.34 | 0.23 | 5.68 | 0.35 |
| | 1.5 | 13.09 | 0.20 | 8.83 | 0.42 | 6.93 | 0.56 |
| | 2.0 | 15.39 | 0.37 | 10.67 | 0.64 | 8.53 | 0.76 |
| | 2.5 | 17.98 | 0.58 | 12.81 | 0.81 | 10.42 | 0.89 |
| | 3.0 | 20.84 | 0.77 | 15.21 | 0.92 | 12.58 | 0.96 |
| | 3.5 | 23.96 | 0.90 | 17.86 | 0.97 | 14.98 | 0.99 |
| 0.5 | 1.0 | 12.06 | 0.06 | 8.04 | 0.19 | 6.26 | 0.30 |
| | 1.5 | 14.79 | 0.14 | 10.18 | 0.34 | 8.10 | 0.48 |
| | 2.0 | 17.98 | 0.26 | 12.81 | 0.51 | 10.42 | 0.65 |
| | 2.5 | 21.60 | 0.42 | 15.85 | 0.68 | 13.15 | 0.80 |
| | 3.0 | 25.62 | 0.59 | 19.28 | 0.82 | 16.28 | 0.90 |
| | 3.5 | 30.02 | 0.75 | 23.11 | 0.91 | 19.80 | 0.96 |
| | 4.0 | 34.82 | 0.87 | 27.32 | 0.96 | 23.70 | 0.98 |
| | 4.5 | 39.99 | 0.94 | 31.91 | 0.99 | 27.98 | 1.00 |
| 0.6 | 1.0 | 13.09 | 0.05 | 8.83 | 0.16 | 6.93 | 0.25 |
| | 1.5 | 16.65 | 0.09 | 11.71 | 0.26 | 9.44 | 0.38 |
| | 2.0 | 20.84 | 0.16 | 15.21 | 0.38 | 12.58 | 0.52 |
| | 2.5 | 25.62 | 0.27 | 19.28 | 0.52 | 16.28 | 0.66 |
| | 3.0 | 30.95 | 0.39 | 23.92 | 0.66 | 20.55 | 0.78 |
| | 3.5 | 36.84 | 0.53 | 29.11 | 0.77 | 25.36 | 0.87 |
| | 4.0 | 43.28 | 0.66 | 34.84 | 0.87 | 30.73 | 0.93 |
| | 4.5 | 50.26 | 0.78 | 41.12 | 0.93 | 36.63 | 0.97 |
| | 5.0 | 57.79 | 0.87 | 47.95 | 0.96 | 43.08 | 0.98 |
| | 5.5 | 65.86 | 0.93 | 55.31 | 0.98 | 50.07 | 0.99 |
| | | | | $k=4$ | | | |
| 0.0 | 1.0 | 11.34 | 0.16 | 7.81 | 0.36 | 6.25 | 0.48 |
| | 1.5 | 11.34 | 0.48 | 7.81 | 0.71 | 6.25 | 0.81 |
| | 2.0 | 11.34 | 0.82 | 7.81 | 0.93 | 6.25 | 0.97 |
| 0.2 | 1.0 | 11.93 | 0.14 | 8.23 | 0.33 | 6.58 | 0.46 |
| | 1.5 | 12.62 | 0.40 | 8.72 | 0.65 | 6.99 | 0.76 |
| | 2.0 | 13.51 | 0.72 | 9.39 | 0.89 | 7.55 | 0.94 |
| | 2.5 | 14.56 | 0.92 | 10.20 | 0.98 | 8.24 | 0.99 |
| 0.4 | 1.0 | 13.51 | 0.10 | 9.39 | 0.26 | 7.55 | 0.38 |
| | 1.5 | 15.75 | 0.25 | 11.15 | 0.49 | 9.05 | 0.63 |
| | 2.0 | 18.48 | 0.48 | 13.37 | 0.73 | 11.01 | 0.83 |
| | 2.5 | 21.62 | 0.71 | 15.98 | 0.89 | 13.35 | 0.94 |
| | 3.0 | 25.12 | 0.88 | 18.96 | 0.97 | 16.04 | 0.99 |
| 0.5 | 1.0 | 14.56 | 0.07 | 10.20 | 0.21 | 8.24 | 0.33 |
| | 1.5 | 17.76 | 0.18 | 12.77 | 0.40 | 10.48 | 0.54 |
| | 2.0 | 21.62 | 0.34 | 15.98 | 0.60 | 13.35 | 0.73 |
| | 2.5 | 26.04 | 0.54 | 19.75 | 0.78 | 16.77 | 0.87 |
| | 3.0 | 31.02 | 0.73 | 24.06 | 0.90 | 20.73 | 0.95 |
| | 3.5 | 36.51 | 0.87 | 28.90 | 0.96 | 25.21 | 0.98 |
| | 4.0 | 42.53 | 0.95 | 34.24 | 0.99 | 30.20 | 1.00 |

Table 4 (Continued.)

| $\frac{\delta}{\delta^*}$ | $\frac{\delta^*}{\sqrt{z}}$ | $(\chi^2_{0.01}(\Delta),$ | $P^*)$ | $(\chi^2_{0.05}(\Delta),$ | $P^*)$ | $(\chi^2_{0.10}(\Delta),$ | $P^*)$ |
|---|---|---|---|---|---|---|---|
| 0.6 | 1.0 | 15.75 | 0.05 | 11.15 | 0.17 | 9.05 | 0.28 |
| | 1.5 | 20.00 | 0.12 | 14.63 | 0.30 | 12.13 | 0.43 |
| | 2.0 | 25.12 | 0.22 | 18.96 | 0.46 | 16.04 | 0.60 |
| | 2.5 | 31.02 | 0.35 | 24.06 | 0.62 | 20.73 | 0.74 |
| | 3.0 | 37.67 | 0.51 | 29.92 | 0.76 | 26.17 | 0.86 |
| | 3.5 | 45.08 | 0.67 | 36.52 | 0.87 | 32.34 | 0.93 |
| | 4.0 | 53.21 | 0.80 | 43.86 | 0.94 | 39.24 | 0.97 |
| | 4.5 | 62.08 | 0.89 | 51.92 | 0.97 | 46.87 | 0.99 |
| | 5.0 | 71.68 | 0.95 | 60.70 | 0.99 | 55.23 | 1.00 |
| | | | | $k=5$ | | | |
| 0.0 | 1.0 | 13.28 | 0.19 | 9.49 | 0.40 | 7.78 | 0.52 |
| | 1.5 | 13.28 | 0.56 | 9.49 | 0.77 | 7.78 | 0.86 |
| | 2.0 | 13.28 | 0.89 | 9.49 | 0.96 | 7.78 | 0.98 |
| 0.2 | 1.0 | 13.92 | 0.17 | 9.96 | 0.36 | 8.17 | 0.49 |
| | 1.5 | 14.68 | 0.48 | 10.52 | 0.72 | 8.64 | 0.82 |
| | 2.0 | 15.68 | 0.81 | 11.28 | 0.93 | 9.29 | 0.97 |
| 0.4 | 1.0 | 15.68 | 0.11 | 11.28 | 0.28 | 9.29 | 0.41 |
| | 1.5 | 18.23 | 0.30 | 13.32 | 0.56 | 11.05 | 0.69 |
| | 2.0 | 21.38 | 0.57 | 15.91 | 0.80 | 13.36 | 0.88 |
| | 2.5 | 25.03 | 0.81 | 19.00 | 0.94 | 16.14 | 0.97 |
| | 3.0 | 29.15 | 0.94 | 22.53 | 0.99 | 19.38 | 1.00 |
| 0.5 | 1.0 | 16.87 | 0.08 | 12.22 | 0.24 | 10.10 | 0.36 |
| | 1.5 | 20.54 | 0.21 | 15.21 | 0.45 | 12.73 | 0.59 |
| | 2.0 | 25.03 | 0.42 | 19.00 | 0.68 | 16.14 | 0.79 |
| | 2.5 | 30.25 | 0.64 | 23.49 | 0.85 | 20.25 | 0.92 |
| | 3.0 | 36.14 | 0.82 | 28.65 | 0.95 | 25.02 | 0.98 |
| | 3.5 | 42.70 | 0.93 | 34.46 | 0.99 | 30.45 | 0.99 |
| 0.6 | 1.0 | 18.23 | 0.06 | 13.32 | 0.19 | 11.05 | 0.30 |
| | 1.5 | 23.15 | 0.14 | 17.40 | 0.34 | 14.69 | 0.48 |
| | 2.0 | 29.15 | 0.27 | 22.53 | 0.52 | 19.38 | 0.66 |
| | 2.5 | 36.14 | 0.44 | 28.65 | 0.70 | 25.02 | 0.81 |
| | 3.0 | 44.09 | 0.62 | 35.71 | 0.83 | 31.61 | 0.91 |
| | 3.5 | 52.96 | 0.77 | 43.69 | 0.92 | 39.12 | 0.96 |
| | 4.0 | 62.76 | 0.89 | 52.59 | 0.97 | 47.54 | 0.99 |
| | | | | $k=6$ | | | |
| 0.0 | 1.0 | 15.09 | 0.22 | 11.07 | 0.43 | 9.24 | 0.56 |
| | 1.5 | 15.09 | 0.63 | 11.07 | 0.82 | 9.24 | 0.89 |
| | 2.0 | 15.09 | 0.93 | 11.07 | 0.98 | 9.24 | 0.99 |
| 0.2 | 1.0 | 15.79 | 0.19 | 11.59 | 0.40 | 9.68 | 0.53 |
| | 1.5 | 16.63 | 0.55 | 12.23 | 0.77 | 10.22 | 0.86 |
| | 2.0 | 17.73 | 0.87 | 13.09 | 0.96 | 10.96 | 0.98 |
| 0.4 | 1.0 | 17.73 | 0.13 | 13.09 | 0.31 | 10.96 | 0.44 |
| | 1.5 | 20.58 | 0.36 | 15.39 | 0.61 | 12.97 | 0.74 |
| | 2.0 | 24.14 | 0.65 | 18.35 | 0.85 | 15.63 | 0.92 |
| | 2.5 | 28.30 | 0.88 | 21.90 | 0.97 | 18.85 | 0.99 |
| 0.5 | 1.0 | 19.06 | 0.10 | 14.15 | 0.26 | 11.88 | 0.38 |
| | 1.5 | 23.19 | 0.25 | 17.56 | 0.50 | 14.91 | 0.64 |
| | 2.0 | 28.30 | 0.49 | 21.90 | 0.74 | 18.85 | 0.84 |
| | 2.5 | 34.28 | 0.73 | 27.10 | 0.90 | 23.63 | 0.95 |
| | 3.0 | 41.08 | 0.89 | 33.10 | 0.97 | 29.21 | 0.99 |
| 0.6 | 1.0 | 20.58 | 0.07 | 15.39 | 0.20 | 12.97 | 0.32 |
| | 1.5 | 26.15 | 0.16 | 20.06 | 0.38 | 17.17 | 0.52 |
| | 2.0 | 33.02 | 0.32 | 25.99 | 0.58 | 22.61 | 0.71 |
| | 2.5 | 41.08 | 0.51 | 33.10 | 0.76 | 29.21 | 0.86 |
| | 3.0 | 50.28 | 0.70 | 41.33 | 0.89 | 36.93 | 0.94 |
| | 3.5 | 60.61 | 0.85 | 50.68 | 0.96 | 45.76 | 0.98 |
| | 4.0 | 72.04 | 0.94 | 61.13 | 0.99 | 55.69 | 0.99 |
| | | | | $k=8$ | | | |
| 0.0 | 1.0 | 18.48 | 0.27 | 14.07 | 0.50 | 12.02 | 0.63 |
| | 1.5 | 18.48 | 0.74 | 14.07 | 0.89 | 12.02 | 0.94 |
| | 2.0 | 18.48 | 0.98 | 14.07 | 0.99 | 12.02 | 1.00 |
| 0.2 | 1.0 | 19.30 | 0.24 | 14.70 | 0.46 | 12.56 | 0.59 |
| | 1.5 | 20.29 | 0.67 | 15.48 | 0.85 | 13.23 | 0.92 |
| | 2.0 | 21.60 | 0.94 | 16.52 | 0.99 | 14.15 | 0.99 |
| 0.4 | 1.0 | 21.60 | 0.16 | 16.52 | 0.36 | 14.15 | 0.50 |
| | 1.5 | 25.03 | 0.46 | 19.35 | 0.71 | 16.67 | 0.81 |
| | 2.0 | 29.39 | 0.78 | 23.04 | 0.92 | 20.01 | 0.96 |
| | 2.5 | 34.54 | 0.95 | 27.49 | 0.99 | 24.10 | 1.00 |
| 0.5 | 1.0 | 23.19 | 0.12 | 17.82 | 0.30 | 15.30 | 0.43 |
| | 1.5 | 28.22 | 0.33 | 22.04 | 0.59 | 19.10 | 0.72 |
| | 2.0 | 34.54 | 0.62 | 27.49 | 0.83 | 24.10 | 0.91 |
| | 2.5 | 42.01 | 0.85 | 34.07 | 0.96 | 30.20 | 0.98 |
| 0.6 | 1.0 | 25.03 | 0.08 | 19.35 | 0.24 | 16.67 | 0.36 |
| | 1.5 | 31.87 | 0.22 | 25.17 | 0.45 | 21.96 | 0.59 |
| | 2.0 | 40.42 | 0.42 | 32.67 | 0.68 | 28.89 | 0.80 |
| | 2.5 | 50.56 | 0.65 | 41.71 | 0.85 | 37.36 | 0.92 |
| | 3.0 | 62.23 | 0.83 | 52.27 | 0.95 | 47.33 | 0.98 |
| | 3.5 | 75.38 | 0.94 | 64.30 | 0.99 | 58.77 | 1.00 |

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