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Linear birth and death processes under the influence of disasters with time-dependent killing probabilities

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Supercritical linear birth-and-death processes are considered under the influence of disasters that arrive as a renewal process independently of the population size. The novelty of this paper lies in assuming that the killing probability in a disaster is a function of the time that has elapsed since the last disaster. A necessary and sufficient condition for a.s. extinction is found. When catastrophes form a Poisson process, formulas for the Laplace transforms of the expectation and variance of the population size as a function of time as well as moments of the odds of extinction are derived (these odds are random since they depend on the intercatastrophe times). Finally, we study numerical techniques leading to plots of the density of the probability of extinction.

linear birth-and-death process * catastrophes * delay differential equations * edgeworth expansion * extinction probability * time-dependent killing

1. Introduction

In this paper we consider a population process, say $Z(t)$, subject to 'disasters' or 'catastrophes'. Between disasters $Z(t)$ will be assumed to be a linear birth and death process. Disasters are instantaneous events, each consisting of a binomial killing of members of the population alive at the time of disaster. The novelty of this paper

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lies in assuming that the killing probability in a disaster, say ε, is a function of the time τ that has elapsed since the last disaster.

The motivation for studying this type of process arises from many situations where $\varepsilon(\tau)$ tends to increase with τ, that is, when disasters become more serious as the time since the last disaster increases. This may be exemplified by earthquakes (see [9]) where previous quakes may relieve stress on a fault, by forest fires where previous fires remove flammable underbrush, or some types of epidemics. For the latter case, assume, for instance, that an epidemic leaves survivors with some immunity. As time passes, this immunity may be lost, or the immune persons die out, making the next epidemic more serious when it comes long after the preceding one.

The main objective of this paper will be to analyze the distribution and extinction of the process $Z(t)$. In the next section we shall introduce notation, specify the assumptions of our model and establish a method for studying the expectation and variance of $Z(t)$. Section 3 begins our study of the probability of extinction. In particular we obtain a necessary and sufficient condition for almost sure extinction. When this condition is not met, we investigate the distribution of the probability of extinction (note that this probability depends on the sequence of the times of disasters, so that it itself is a random variable). In Section 4 we find recursive formulas which allow us to compute the moments of the odds of extinction. Finally, in Section 5 we display plots of the density of the probability of extinction. When ε is constant we can draw these plots by using numerical techniques for solving differential-difference equations. In the general case, we plot the density using an Edgeworth expansion together with the results of Section 4.

2. The distribution of $Z(t)$

We shall consider a process $\{Z(t), t \geqslant 0\}$ representing the evolution of some population with initial size $Z(0)=1$, defined as follows. At times $\tau_1, \tau_1+\tau_2, \ldots$, there occur catastrophes. We assume that the inter-catastrophe times $\{\tau_i\}$ are positive i.i.d. random variables with finite expectation. We also assume that τ_n is independent of $Z(\tau_{n-1}^*)$ for $n=1,2,\ldots$, where $\tau_i^* = \tau_1+\tau_2+\cdots+\tau_i$ is the time of the ith catastrophe. With the occurrence of the ith catastrophe, each member of the population, independently of others, dies with probability $\varepsilon(\tau_i) = 1-\delta(\tau_i)$, where $\varepsilon(\tau)$ is some function satisfying the relation $0<\varepsilon(\tau)<1$. Thus, given τ_n, and $Z(\tau_n^*-0)=K$, $Z(\tau_n^*+0) \sim \mathrm{Bin}(K, \delta(\tau_n))$ for $n=1,2,\ldots$. Before the first catastrophe and in between the $(n-1)$st and the nth catastrophe, $Z(t)$ is a linear birth and death process, with parameters λ and μ.

The sample space of the process under consideration can be visualized, for instance, as follows. An individual sample path $\omega \in \Omega$ is a history of the process represented by a sequence of triplets, each consisting of (i) τ_i, (ii) the segment of the path of the linear birth and death process for times between τ_{i-1}^*+0 and

$\tau_{i-1}^* + \tau_i - 0$ and (iii) the number of members of the population killed at the ith catastrophe. From now on we shall make the convention that the process Z is continuous on the left and write $Z(t)$ for $Z(t-0)$ and $Z(t+)$ for $Z(t+0)$.

When δ is constant this model becomes a special case of a branching process subject to catastrophes as developed in [10], [1] and [15]. Other investigators have examined the occurrence of disasters as caused by population pressure (see for example [5], [6], and [13]). An excellent review of the area is given in [14] and an extensive bibliography covering population processes with disasters is given in [4].

To fix notation, we let

$$\delta_j = \delta(\tau_j(\omega)) \quad \text{and} \quad \psi_j = e^{\rho\tau_j(\omega)} \tag{1}$$

where $\rho = \lambda - \mu$. Next, following the techniques used in [7], we define $S_0 = 1$,

$$S_j = S_j(\tau_1 \cdots \tau_j) = \prod_{i=1}^{j} (\delta_i \psi_i)^{-1}, \tag{2}$$

and

$$X_n = X_n(\tau_1 \cdots \tau_j) = \prod_{j=1}^{n} (1 - \delta_j) S_j. \tag{3}$$

We start from the conditional probability generating function (p.g.f.) of the process $Z(t)$ at the time immediately following the nth catastrophe.

Theorem 1. *Given the first n inter-catastrophe times* $\tau_1, \ldots, \tau_n$,

$$\begin{aligned} H_n(s) &\equiv E(s^{Z(\tau_n^*+)} \mid \tau_1, \ldots, \tau_n)(\omega) \\ &= 1 - \frac{\rho(1-s)}{(\lambda + \lambda X_n - \mu S_n) - (\lambda + \lambda X_n - \lambda S_n)s} \end{aligned} \tag{4}$$

for $\omega \in \Omega - \mathcal{N}_n$ *where* $P(\mathcal{N}_n) = 0$.

Proof. The probability generating function of a linear birth and death process without catastrophes, starting with one individual, is (see [2]):

$$F_0(s, t) = \frac{\mu - \mu\, e^{\rho t} - (\lambda - \mu\, e^{\rho t})s}{\mu - \lambda\, e^{\rho t} - (\lambda - \lambda\, e^{\rho t})s}. \tag{5}$$

Thus, immediately following the first catastrophe the conditional generating function of $Z(\tau_1+)$ is

$$H_1(s) = E(s^{Z(\tau_1+)} \mid \tau_1) = F_0(\delta(\tau_1)s + \varepsilon(\tau_1), \tau_1), \tag{6}$$

for $\omega \in \Omega - \mathcal{N}_1$ with $P(\mathcal{N}_1) = 0$. Indeed $F_0(s, \tau_1)$ is the conditional p.g.f. of $Z(\tau_1)$, while $Z(\tau_1+) \sim \text{Bin}(Z(\tau_1), \delta(\tau_1))$. Proceeding by induction and using the branching property of the linear birth and death process, we have,

$$H_n(s) = H_{n-1}(F_0(\delta(\tau_n)s + \varepsilon(\tau_n), \tau_n)) \tag{7}$$

for all $\omega \in \Omega - \mathcal{N}_n$ where $P(\mathcal{N}_n) = 0$. It follows from (5), (6) and (7) by induction that each H_n is of the form

$$H_n(s) = \frac{\mathcal{A}_n s + \mathcal{B}_n}{\mathcal{C}_n s + \mathcal{D}_n}, \tag{8}$$

where

$$\begin{bmatrix} \mathcal{A}_n & \mathcal{B}_n \\ \mathcal{C}_n & \mathcal{D}_n \end{bmatrix} = \prod_{i=1}^{n} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}$$

with

$$A_i = -(\lambda - \mu\psi_i)\delta_i, \quad B_i = (\lambda - \mu\psi_i)\delta_i - \rho,$$
$$C_i = -(\lambda - \lambda\psi_i)\delta_i, \quad D_i = (\lambda - \lambda\psi_i)\delta_i - \rho.$$

The conditional probability of extinction, which we shall study in Section 3, equals $H_n(0) = \mathcal{B}_n / \mathcal{D}_n$ almost surely. To find the values $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{C}_n$, and $\mathcal{D}_n$ we use a matrix decomposition. We write

$$\prod_{i=1}^{n} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = (-\rho)^n \prod_{i=1}^{n} P_i \begin{bmatrix} 1 & 0 \\ 0 & \psi_i\delta_i \end{bmatrix} P_i^{-1} \tag{9}$$

where we define the matrix

$$P_i = \begin{bmatrix} 1 & B_i \\ 1 & -C_i \end{bmatrix}$$

so that

$$P_i^{-1} = \frac{1}{\rho(1 - \psi_i\delta_i)} \begin{bmatrix} -C_i & -B_i \\ -1 & 1 \end{bmatrix}.$$

We write the product of the right-hand side of (9) as

$$P_1 \left\{ \begin{bmatrix} 1 & 0 \\ 0 & \psi_1\delta_1 \end{bmatrix} P_1^{-1} P_2 \begin{bmatrix} 1 & 0 \\ 0 & \psi_2\delta_2 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ 0 & \psi_n\delta_n \end{bmatrix} \right\} P_n^{-1}. \tag{10}$$

Finally, by noticing that $P_i^{-1} P_{i+1}$ has the simple form

$$P_i^{-1} P_{i+1} = \begin{bmatrix} 1 & -\lambda\left((1 - \delta_{i+1}) - (1 - \delta_i) \dfrac{1 - \psi_{i+1}\delta_{i+1}}{1 - \psi_i\delta_i} \right) \\ 0 & \dfrac{1 - \psi_{i+1}\delta_{i+1}}{1 - \psi_i\delta_i} \end{bmatrix},$$

we can evaluate (10), and after considerable algebra we obtain

$$\begin{aligned} \mathcal{A}_n &= (-\rho)^{n-1} \frac{1}{S_n} (\mu + \lambda X_n - \lambda S_n), & \mathcal{B}_n &= -(-\rho)^{n-1} \frac{1}{S_n} (\mu + \lambda X_n - \mu S_n), \\ \mathcal{C}_n &= (-\rho)^{n-1} \frac{1}{S_n} (\lambda + \lambda X_n - \lambda S_n), & \mathcal{D}_n &= -(-\rho)^{n-1} \frac{1}{S_n} (\lambda + \lambda X_n - \mu S_n). \end{aligned} \tag{11}$$

Substituting these formulas into (8) completes the proof of Theorem 1. □

For the remainder of this section, we will examine the special case when the τ_i's are exponential with mean $1/\beta$, so that catastrophes follow the Poisson process.

Let $N(t)$ be the number of catastrophes occurring before time t. In order to find the expected value and variance of $Z(t)$, observe first that the p.g.f. of $Z(t)$, conditioned on $N(t)=n$ and on $\tau_1, \ldots, \tau_n$ is

$$F_0(H_n(s), t-\tau_n^*), \quad \text{or equivalently,} \quad H_n(F_0(s, t-\tau_n^*)) \tag{12}$$

with F_0 and H_n given by (5) and (4). Differentiating (12) and unconditioning with respect to $N(t)$ and $\{\tau_j\}$, we arrive at expressions for the unconditional mean and variance of $Z(t)$:

Corollary 1. *If catastrophes form a Poisson process with parameter β, then*

$$E(Z(t)) = \sum_{n=0}^{\infty} \frac{(\beta t)^n}{n!} \mathrm{e}^{(\rho-\beta)t} E_n\left[\prod_{i=1}^{n} \delta_i\right] \tag{13}$$

and

$$\begin{aligned}\operatorname{Var}(Z(t)) &= \mathrm{e}^{2\rho t} \sum_{n=0}^{\infty} \frac{(\beta t)^n}{n!} \mathrm{e}^{-\beta t}\left\{E_n\left[\frac{2\lambda}{\rho}(X_n+1)\prod_{i=1}^{n}\delta_i^2\right] - E_n\left(\prod_{i=1}^{n}\delta_i\right)\left[\frac{\lambda+\mu}{\rho}\mathrm{e}^{-\rho t}\right]\right\} \\ &\quad - E(Z(t))^2,\end{aligned} \tag{14}$$

where E_n stands for expectation of functions of τ_i's conditional on $N(t)=n$. □

Next, we simplify (13) by noting that given $N(t)=n$, the time τ_j^* of the jth catastrophe has the same distribution as the jth order statistic in a sample of size n from the distribution uniform on $[0, t]$. This gives us after some algebra the Laplace Transform

$$\mathscr{L}_{EZ(t)}(\xi) = \int_0^{\infty} \mathrm{e}^{-\xi t} EZ(t)\,\mathrm{d}t = \frac{1}{(\xi+\beta-\rho)[1-\beta\mathscr{L}_\delta(\xi+\beta-\rho)]}, \tag{15}$$

which certainly exists for $\xi > \rho$. In expression (15) $\mathscr{L}_\delta$ is the Laplace Transform of $\delta(\tau_1)$:

$$\mathscr{L}_\delta(v) = \int_0^{\infty} \mathrm{e}^{-v\tau}\delta(\tau)\,\mathrm{d}\tau.$$

Three special cases are of interest. First, we will examine the case when $\delta(\tau)$ is constant (as covered in [3] and [7]). Next, we will study a case of accelerating killing power, $\delta(\tau)=\mathrm{e}^{-k\tau}$, and finally, that of decelerating killing power, $\delta(\tau) = 1-\mathrm{e}^{-k\tau}$, for $k>0$.

In the first case, $\mathscr{L}_\delta(v) = \delta/v$, so that inverting (15) gives

$$E(Z(t)) = \mathrm{e}^{(\rho-\beta(1-\delta))t}. \tag{16}$$

Note that (16) gives a sufficient condition for a.s. extinction of the process, namely $\rho < \beta(1-\delta)$. However, this is weaker than the necessary and sufficient condition (in this constant δ case)

$$\delta \leq e^{-\rho/\beta} \tag{17}$$

given in [7].

On the other hand, if $\delta(t) = e^{-k\tau}$, then $\mathscr{L}_\delta(v) = 1/(k+v)$, so that

$$E(Z(t)) = e^{\rho t}\left[\frac{k}{k-\beta}e^{-\beta t} - \frac{\beta}{k-\beta}e^{-kt}\right]. \tag{18}$$

Now (18) shows that a sufficient condition for a.s. extinction in the accelerating killing case is $\rho < \min(k, \beta)$. Again, a stronger condition will be given in the next section. When $\delta(\tau) = 1 - e^{-k\tau}$ (decelerating killing power), we have $\mathscr{L}_\delta(v) = k/(v(v+k))$, hence after some algebra

$$\mathscr{L}_{E(Z(t))}(\xi) = \frac{\xi+\beta-\rho+k}{(\xi+\beta-\rho)(\xi+\beta-\rho+k)-k\beta}.$$

The denominator is easily seen to have two distinct real roots, and we obtain, using a partial fractions expansion

$$E(Z(t)) = \frac{1}{2}\left(1+\frac{k}{\sqrt{\Delta}}\right)\exp\{-(\beta-\rho)t - \tfrac{1}{2}(k-\sqrt{\Delta})t\}$$
$$+\frac{1}{2}\left(1-\frac{k}{\sqrt{\Delta}}\right)\exp\{-(\beta-\rho)t - \tfrac{1}{2}(k+\sqrt{\Delta})t\}$$

where $\Delta = k^2 + 4k\beta$.

Note that when $k \to 0$ (killing rate becomes high) the expected size of the population at t is about $e^{(\rho-\beta)t}$, which is the product of the probability, $e^{-\beta t}$, of no disaster until t and the expected population size under no disasters $e^{\rho t}$.

As another check, taking $k \to \infty$ (killing rate becomes low), we obtain the limiting expected population size at t equals $e^{\rho t}$.

In the case of decelerating killing probabilities, a sufficient condition for a.s. extinction of the process is $2(\beta-\rho) > \sqrt{k^2+4k\beta} - k$.

Now, when δ is constant, evaluation of (14) gives

$$\mathrm{Var}(Z(t)) = \frac{\lambda+\mu+\beta\delta(1-\delta)}{\lambda-\mu-\beta\delta(1-\delta)} e^{(\rho-\beta(1-\delta))t}(e^{(\rho-\beta\delta(1-\delta))t}-1). \tag{19}$$

Observe that if $\beta = 0$ (no catastrophes) or $\delta = 1$ (no killings), formula (19) reduces to the known formula of the variance of a linear birth and death process (see e.g., [2]).

It is possible to obtain a formula for the variance of $Z(t)$ in the case of accelerating killing power. However, we shall not reproduce it here because of its length.

In the case of decelerating killing power, the formula for the variance of $Z(t)$ involves roots of a seventh degree polynomial. After these roots are found numerically, one can write an explicit expression for the variance of $Z(t)$. Again, we omit the details.

3. The probability of extinction

We shall now study the properties of the extinction probability. Throughout this section we will exclude some extreme situations by assuming that $E[\log((1-\delta_1)/\delta_1)]$ is finite.

Observe that X_n, defined in formula (3), is a non-negative random variable, being a function of the inter-catastrophe times $\tau_1, \ldots, \tau_n$. Since the series (3) is always convergent (to a finite or infinite limit), we let

$$X = \lim_{n\to\infty} X_n = \sum_{n=1}^{\infty} (1-\delta_n)S_n \tag{20}$$

where now $X \geqslant 0$ is a random variable depending on the infinite sequence $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots)$.

From (20) we see that,

$$E(X) \leqslant \sum_{n=1}^{\infty} Q_1^n$$

where $Q_1 = E(1/(\psi\delta))$, so that $Q_1 < 1$ implies that $E(X) < \infty$. Similarly, by induction it can be shown that $Q_k = E(1/(\psi\delta)^k) < 1$ implies $E(X^k) < \infty$.

Now observe that extinction, that is the event $\mathscr{E} = \{Z(t) = 0 \text{ for some } t\}$, depends on the sequence of inter-catastrophe times $\boldsymbol{\tau}$, on the events pertaining to the development of the process Z, and on the random killings in the catastrophes. However,

$$P\{\text{extinction}\} = P[Z(t) = 0 \text{ for some } t] = E_{\tau}\{P[Z(t) = 0 \text{ for some } t \,|\, \boldsymbol{\tau}]\};$$

on the other hand,

$$P[Z(t) = 0 \text{ for some } t \,|\, \boldsymbol{\tau}] = \lim_{n\to\infty} P[Z(\tau_n^* +) = 0 \,|\, \boldsymbol{\tau}] = \lim_{n\to\infty} H_n(0 \,|\, \boldsymbol{\tau}).$$

Thus the probability of extinction is also a random variable depending on $\boldsymbol{\tau}$. Its value for a particular $\boldsymbol{\tau}$ is denoted by $P(\mathscr{E} \,|\, \boldsymbol{\tau})$. Next, we will show that this dependence is only through the value of $X = X(\boldsymbol{\tau})$.

We first claim that $P(X < \infty)$ is either 0 or 1. Indeed, the shift transformation consisting of omitting the history of the process up until the time of the first catastrophe is a mapping $\Omega \to \Omega$ which leaves the set $\{X < \infty\}$ invariant (provided $\tau_1 < \infty$). The assertion follows by the independence of the τ_i's and the Kolmogorov 0–1 Law (see [8]).

Define $\Omega_1 = \{\omega : X(\omega) < \infty\}$ and $\Omega_2 = \Omega - \Omega_1$. Notice that if the underlying birth and death process is critical or subcritical ($\rho \leqslant 0$), then $X = \infty$ a.s. and $P(\mathscr{E} \mid \boldsymbol{\tau}) = 1$. For the supercritical case, we can prove the following theorem.

Theorem 2. *Suppose that $\lambda > \mu$. Then there exist sets $\mathcal{M}_1 \subset \Omega_1$ and $\mathcal{M}_2 \subset \Omega_2$ with $P(\mathcal{M}_1) = P(\mathcal{M}_2) = 0$ such that either*

$$\text{(a)} \qquad P(\mathscr{E} \mid \boldsymbol{\tau})(\omega) = \frac{\mu + \lambda X(\omega)}{\lambda + \lambda X(\omega)} \quad \textit{if } \omega \in \Omega_1 - \mathcal{M}_1 \tag{21}$$

or

$$\text{(b)} \qquad P(\mathscr{E} \mid \boldsymbol{\tau})(\omega) = 1 \quad \textit{if } \omega \in \Omega_2 - \mathcal{M}_2.$$

Moreover, X has the same distribution as the random variable

$$\frac{1 - \delta_1 + X}{\psi_1 \delta_1}. \tag{22}$$

Before proving this theorem, it is useful to recall that X depends on ω only through the τ's, so that $\boldsymbol{\tau}(\omega) = \boldsymbol{\tau}(\omega') \Rightarrow X(\boldsymbol{\tau}(\omega)) = X(\boldsymbol{\tau}(\omega'))$. On the other hand, this is not true for the occurrence of extinction, so that we may have $\boldsymbol{\tau}(\omega) = \boldsymbol{\tau}(\omega')$ but $\omega \in \mathscr{E}$ and $\omega' \notin \mathscr{E}$.

Proof. Substituting $s = 0$ in (4) we obtain

$$H_n(0) = \frac{\mu + \lambda X_n - \mu S_n}{\lambda + \lambda X_n - \mu S_n} \quad \text{except possibly on the set } \mathcal{N}_n. \tag{23}$$

Now,

$$S_n = \prod_{j=1}^{n} (\psi_j \delta_j)^{-1} = \exp\left\{ -\sum_{j=1}^{n} (\rho \tau_j + \log \delta_j) \right\}. \tag{24}$$

Since the τ_i's are i.i.d. and $E[\log(\delta_1)] > -\infty$ (in view of $E[\log((1-\delta_1)/\delta_1)]$ being finite), by the law of large numbers, if

$$E\{\rho \tau_1 + \log \delta_1\} > 0, \tag{25}$$

then $S_n \to 0$ except possibly on a set $\mathcal{M}_3$ with $P(\mathcal{M}_3) = 0$.

Now, we shall show that $X < \infty$ implies the inequality in (25). Firstly, $X < \infty$ implies that for any $\Delta > 0$ we have for all k greater than some $K(\omega)$,

$$(1 - \delta_k) S_k < \Delta.$$

Therefore, using (24),

$$\exp\left\{ -\sum_{j=1}^{k} (\rho \tau_j + \log \delta_j) \right\} < \frac{\Delta}{1 - \delta_k}$$

which implies

$$\frac{1}{k} \sum_{j=1}^{k} (\rho \tau_j + \log \delta_j) > \frac{\log[(1 - \delta_k)]}{k} - \frac{\log(\Delta)}{k}.$$

The random variables $\log[(1-\delta_k)]$ are i.i.d. with finite (negative) expectation. Thus, the Borel–Cantelli Lemma implies that the right-hand side converges to 0 a.s. The left-hand side converges to $E\{\rho\tau_1+\log\delta_1\}$ by the law of large numbers, so that we have shown that $X<\infty$ a.s. implies

$$E\{\rho\tau_1+\log\delta_1\}\geqslant 0. \tag{26}$$

It remains to eliminate the possibility of achieving equality in (26).

Let us write

$$X=\sum_{i=1}^{\infty}(1-\delta_i)S_i=\sum_{n=1}^{\infty}\frac{1-\delta_n}{\psi_n\delta_n}\exp\left\{-\sum_{j=1}^{n-1}(\rho\tau_j+\log\delta_j)\right\}$$

which is of the form $\sum_{n=1}^{\infty}V_n\exp\{-W_{n-1}\}$. Here $W_{n-1}=-\sum_{j=1}^{n-1}U_j$ is a random walk generated by the i.i.d. variables $U_j=\rho\tau_j+\log\delta_j$ and $V_n=(1-\delta_n)/(\psi_n\delta_n)$ is independent of W_{n-1}. Now wee need to show that if $E(U_j)=0$, then $X=\infty$ a.s. To this end, it will suffice to show that the event $W_{n-1}<0$ and $V_n>\eta$ for some $\eta>0$ occurs infinitely often with probability one. We let $T_0=0$ and, for $k\geqslant 1$ define recursively,

$$T_k=T_{k+1}+\min\{n\geqslant 2;\ W_{T_{k-1}+n}-W_{T_{k-1}+1}<0,\ V_{T_{k-1}+n+1}>\eta>0\}.$$

In particular, to show that $X=\infty$ a.s. in this case, it is enough to show that each T_i is a stopping variable. The fact that $P(T_i<\infty)=1$ follows from the Chung–Fuchs Theorem because $\{W_{T_{k-1}+n}-W_{T_{k-1}+1}$ for $n\geqslant 2\}$ is a symmetric random walk starting at the point $W_{T_{k-1}+1}$.

Finally, if $X_n\to\infty$, then the relation

$$P(\mathscr{E}\mid\boldsymbol{\tau})=\lim_{n\to\infty}\frac{\mu/X_n+\lambda-\mu S_n/X_n}{\lambda/X_n+\lambda-\mu S_n/X_n}\quad\text{a.s.}$$

implies $P(\mathscr{E}\mid\boldsymbol{\tau})=1$ a.s., unless $S_n/X_n\to\lambda/\mu$. After some algebra we may write

$$\frac{S_n}{X_n}=\frac{S_n}{S_n+\sum_{i=1}^{n}(1-1/\psi_i)S_{i-1}-1}. \tag{27}$$

If $S_n/X_n\to\lambda/\mu$, then $S_n\to\infty$. This implies that $\limsup S_n/X_n\leqslant 1$, which is a contradiction, since $\lambda>\mu$.

To prove that X has the same distribution as (22), let use write ($\overset{\mathrm{d}}{=}$ stands for equality in distribution)

$$X_{n-1}=\sum_{i=1}^{n-1}(1-\delta_i)\prod_{j=1}^{i}(\psi_j\delta_j)^{-1}\overset{\mathrm{d}}{=}\sum_{i=2}^{n}(1-\delta_i)\prod_{j=2}^{i}(\psi_j\delta_j)^{-1}=X_{n-1}^{*}.$$

Therefore

$$X_n\overset{\mathrm{d}}{=}\frac{1-\delta_1+X_{n-1}^{*}}{\psi_1\delta_1},$$

which proves the result by passing to the limit. □

Since Theorem 2 gives us the probability of extinction in terms of X, we may find the conditions under which $X=\infty$, leading to a.s. extinction.

Theorem 3. *The necessary and sufficient condition for $P(\mathscr{E}\,|\,\boldsymbol{\tau})=1$ a.s. is*

$$E\{\rho\tau_1+\log\delta_1\}\leq 0. \tag{28}$$

When δ is constant and τ_1 is exponential with mean $1/\beta$, this condition gives (17) *which agrees with the result in* [7].

Proof. In the proof of Theorem 2 we showed that $X<\infty$ implies (25), so that sufficiency has been demonstrated. For necessity we note that

$$X_n=\sum_{j=1}^{n}\frac{(1-\delta_j)}{\delta_j\psi_j}\prod_{i=1}^{j-1}(\delta_i\psi_i)^{-1}$$

is of the form considered by Vervaat [16] and the result follows by his Lemma 1.7. □

4. Moments of the odds of extinction

From now on, we will examine the important case where the intercatastrophe times τ_i are assumed to be exponential with mean $1/\beta$. In [3] we derived an integral equation for the distribution function and density of the probability of extinction under the above assumption when δ is constant. We found an explicit expression for this density over part of its range, and conducted an extensive simulation study to display the whole density in a special case. In this section, we instead study the moments of the odds of extinction. Firstly, we derive recursive formulas for the moments of these odds, and then in Section 5 we use numerical methods to plot the densities of the odds in two special cases.

From (21), the odds R of extinction equal

$$R=\frac{P(\mathscr{E}\,|\,\boldsymbol{\tau})}{1-P(\mathscr{E}\,|\,\boldsymbol{\tau})}=\frac{\mu}{\rho}+\frac{\lambda}{\rho}X.$$

Consequently, the moments of R satisfy

$$E(R^N)=\sum_{n=0}^{N}\binom{N}{n}\left(\frac{\mu}{\rho}\right)^{N-n}\left(\frac{\lambda}{\rho}\right)^{n}M_n$$

where $M_n=E(X^n)$.

Let now $\mathscr{L}(u)=E(\exp(-uX))$ be the Laplace transform of X, which exists for all $u>0$. From the proof of (22), it follows that

$$\mathscr{L}(u)=\beta\int_0^\infty e^{-\beta\tau}\exp\left[-u\frac{1-\delta(\tau)}{\psi(\tau)\delta(\tau)}\right]\mathscr{L}\left(\frac{u}{\psi(\tau)\delta(\tau)}\right)d\tau.$$

Taking repeated derivatives gives for $n=0,1,\ldots,$

$$\mathscr{L}^{(n)}(u)=\beta\int_0^\infty e^{-\beta\tau}\exp\left[-u\frac{1-\delta}{\psi\delta}\right]\frac{1}{(\psi\delta)^n}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(1-\delta)^{n-j}\mathscr{L}^{(j)}\left(\frac{u}{\psi\delta}\right)d\tau.$$

Evaluating the derivatives at $u=0$ we obtain for moments

$$M_n=\beta\sum_{j=0}^{n}\binom{n}{j}M_j\int_0^\infty e^{-\beta\tau}\frac{1}{(\psi\delta)^n}(1-\delta)^{n-j}\,d\tau. \tag{29}$$

Notice that M_n appears at both sides of (29), so that this relation is valid even if $M_n = \infty$. If $M_n < \infty$ then $M_j < \infty$ for $j < n$ and we can solve (29) obtaining

$$M_n = \frac{\beta \sum_{j=0}^{n-1} \binom{n}{j} M_j \int_0^\infty e^{-\beta\tau}[(1-\delta)^{n-j}/(\psi\delta)^n]\,d\tau}{1 - \beta \int_0^\infty e^{-\beta\tau}\,d\tau/(\psi\delta)^n} \quad \text{for } n \geq 1. \tag{30}$$

This recursive relationship is initiated by noting that $M_0 = 1$. When this equation yields a negative value then M_n must be infinite for all subsequent n. Also observe that the numerator of (30) is always positive, while the denominator equals $1 - Q_n$, where Q_n is defined following (20) in Section 3. Thus, $E(X^n)$ exists and is given by (30) if and only if the denominator of (30) is positive.

For particular functions $\delta(\tau)$, the integral in (30) can be evaluated, giving exact expressions for the moments of X, hence also for the moments of the odds of extinction. For instance, taking δ constant we obtain

$$E(R) = \frac{\lambda\delta}{\rho\delta - \beta(1-\delta)} - 1$$

and

$$\operatorname{Var}(R) = \frac{[\lambda(1-\delta)\delta]^2\beta}{[\rho\delta - \beta(1-\delta)]^2[2\rho\delta^2 - \beta(1-\delta^2)]}, \tag{31}$$

provided the condition eliminating a.s. extinction, $\delta > e^{-\rho/\beta}$, is satisfied.

The quantity (31) is maximized when δ is the (unique) real solution of the cubic equation

$$-\frac{\rho}{\beta}\left(1 + \frac{2\rho}{\beta}\right)\delta^3 + \left(1 + \frac{\rho}{\beta}\right)\delta^2 - 2\delta + 1 = 0. \tag{32}$$

This solution of (32) occurs when

$$\frac{1-\delta}{\delta}\left[\frac{1+\sqrt{1+8/\delta}}{4}\right] = \frac{\rho}{\beta}.$$

For a given inter-catastrophe growth rate (ρ/β) the resulting bound on the variance might be useful in problems of estimating $E(R)$.

In our previous paper [3] we investigated the special case $\lambda = 4$, $\mu = 2$ and $\beta = 2$. Along with those values we now take $\delta(\tau) = e^{-\tau}$, so that disasters become more severe with longer inter-catastrophe times. In this case, (30) becomes

$$M_n = 2(n-1)!(n+2)! \sum_{j=0}^{n-1} \frac{M_j}{j!(2n+2-j)!}$$

and

$$E(R^N) = \sum_{n=0}^{N} \binom{N}{n} 2^n M_n.$$

The first 20 values of M_n and $E(R^N)$ are given in Table 1.

Table 1

Moments of X and R = the odds of extinction

n	$M_n = E(X^n)$	$E(R^n)/2n$
1	0.50000	1.00000
2	0.26667	1.01667
3	0.14841	1.04841
4	0.08526	1.09459
5	0.05023	1.15525
6	0.03021	1.23104
7	0.01850	1.32312
8	0.01150	1.43315
9	0.00725	1.56331
10	0.00462	1.71631
11	0.00298	1.89548
12	0.00194	2.10486
13	0.00127	2.34932
14	0.00084	2.63465
15	0.00056	2.96783
16	0.00038	3.35715
17	0.00026	3.81255
18	0.00017	4.34588
19	0.00012	4.97133
20	0.00008	5.70589

5. Some numerical results

To numerically evaluate the density of X (for explicit conditions under which X will have a density, see [12]) we first discuss the case when δ is constant. Here we let $Y = (1/\delta)\,\mathrm{e}^{-\rho\tau}$ so that the c.d.f. of Y is given by

$$P(Y \leqslant y) = \begin{cases} 0 & y < 0, \\ (y\delta)^{\alpha} & 0 \leqslant y < 1/\delta, \\ 1 & y \geqslant 1/\delta, \end{cases}$$

where $\alpha = \beta/(\lambda - \mu)$.

From (22) in Theorem 2 we have $X \overset{\mathrm{d}}{=} (X + 1 - \delta)Y$. Using this fact we may condition on the value of Y to get an integral equation for the density of X:

$$h(x) = \begin{cases} \displaystyle\int_0 \alpha y \quad \delta^{\alpha} h\left(\frac{\ }{y} - \varepsilon\right) \mathrm{d}y & \text{for } 0 < x < \varepsilon/\delta, \\ \displaystyle\int_0^{1/\delta} \alpha y^{\alpha-2} \delta^{\alpha} h\left(\frac{x}{y} - \varepsilon\right) \mathrm{d}y & \text{for } x \geqslant \varepsilon/\delta. \end{cases}$$

Substituting $z = x/y - \varepsilon$ and dividing by $c = \alpha\delta^{\alpha} \int_0^{\infty} [h(z)/(z+\varepsilon)^{\alpha}]\,\mathrm{d}z$ we obtain the delayed differential equation

$$h(x) = x^{\alpha-1}, \quad 0 < x < \varepsilon/\delta,$$

and

$$h'(x) = \frac{\alpha - 1}{x} h(x) - \frac{\delta\alpha}{x} h(\delta x - \varepsilon), \quad x > \varepsilon/\delta.$$

Now define $\varphi(t) = \delta^{-t} - 1$ for $0 < t < \infty$ so that $\delta\varphi(t) - \varepsilon = \varphi(t-1)$. Putting $w(t) = h(\varphi(t))$ we obtain a differential-difference equation with delayed argument of the form analyzed by Oberle and Pesch [11]:

$$w'(t) = [(\alpha - 1)w(t) - \alpha\delta w(t-1)][\log(\delta)/(\delta^t - 1)] \quad \text{for } t \geq 1.$$

Using their technique we may find a numerical approximation to the density of X to any degree of accuracy desired. Thus from (21) we can easily transform this approximate density into an approximation of the density of the probability of extinction.

Figure 1 is a graph of the density of the probability of extinction when $\beta = 2$, $\lambda = 4$, $\mu = 2$ and $\delta = 0.6$. This closely matches the corresponding figure in [3] which was obtained using extensive simulation combined with an analytical expression for the density over part of its range.

For the case of δ varying with time, we can plot the density by first determining the moments of X, then using the Edgeworth expansion to obtain a numerical

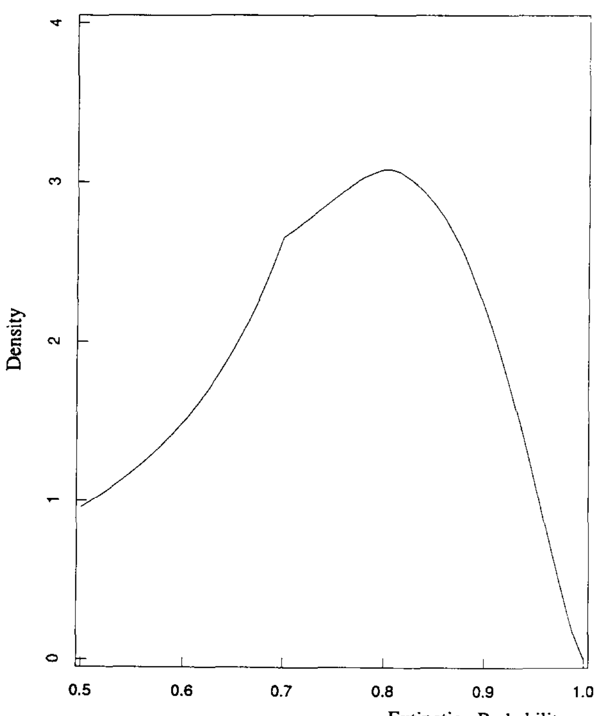

Fig. 1. The density of the probability of extinction when δ has constant value 0.6.

approximation of the density of X and finally, transforming it as before to the density of the probability of extinction.

For example, using the values from Table 1, the above procedure leads to the picture in Figure 2 when $\delta = e^{-\tau}$.

One interesting feature of this figure is that the density is more concentrated than when δ is constant. This can be intuitively explained by observing that the exponentially decaying probability of survival $\delta = e^{-\tau}$ compensates for the increased variability of the population size at the time just before the catastrophe, for large inter-catastrophe times. To make a fair comparison we again used the Oberle and Pesch [11] technique to draw the density when $\delta = 1/\sqrt{2}$, which is the median of $e^{-\tau}$ for τ being exponential with $\beta = 2$ (see Figure 3).

Note also that $\delta = 0.6$ is close to the value which maximizes the variance of the odds of extinction among possible constant values of δ (32).

A second interesting feature of Figure 2 is that the density appears to drop to 0 for bigger values of the argument. In fact, the probability of extinction is bounded away from 1 for some functions $\delta(\tau)$. Indeed, suppose that for $X = x_0$, the random variable $(1 - \delta(\tau) + x_0)/(\delta(\tau)\, e^{\rho\tau})$ has maximum

$$u(x_0) = \sup_{0 \leq \tau < \infty} \frac{1 - \delta(\tau) + x_0}{\delta(\tau)\, e^{\rho\tau}}.$$

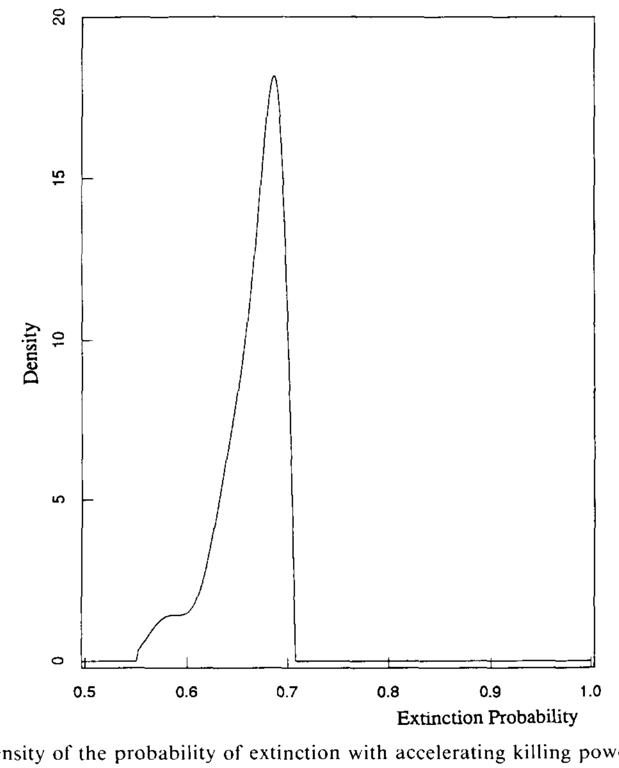

Fig. 2. The density of the probability of extinction with accelerating killing power, $\delta = \exp(-\tau)$.

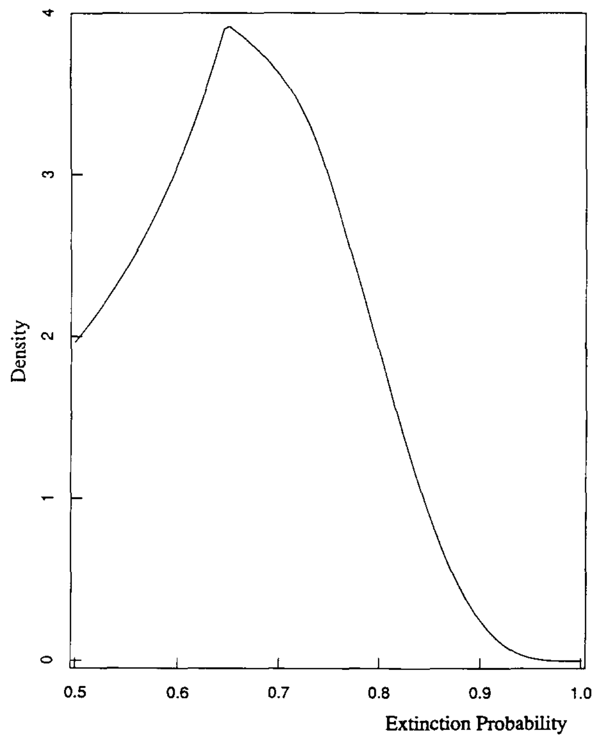

Fig. 3. The density of the probability of extinction when δ has constant value $1/\sqrt{2}$.

If $A=\inf\{y\colon P\{X<y\}=1\}$, then we must have $u(A)=A$, because X and $(1-\delta(\tau)+X)/(\delta(\tau)\,e^{\rho\tau})$ have the same distribution.

Since the probability of extinction is given by (21), it must be bounded from above by $(\mu+\lambda A)/(\lambda+\lambda A)$. For instance, in the case of acclerating killing power when $\delta(\tau)=e^{-k\tau}$, then

$$u(x)=\frac{(k/\rho)(x+1)}{[(x+1)(1-k/\rho)]^{1-\rho/k}}$$

and the equation $u(A)=A$ gives in this case $A=k/(\rho-k)$, provided $0\leqslant k\leqslant\rho$. Consequently, when $\delta(\tau)$ is exponential, the probability of extinction is less than $(\mu+k)/\lambda$ a.s. if $0\leqslant k\leqslant\rho$. By Theorem 3, it is equal to 1 a.s. if $k>\rho$, and, of course, is equal to μ/λ a.s. if $k=0$. In Figure 2, $k=1$, $\mu=2$ and $\lambda=4$, so that the probability of extinction is less than $\frac{3}{4}$.

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