Note that compared to the H^∞ square-root formulas, the size of the pre-array in the H^∞ fast recursions has been reduced from $(p+q+n)\times(p+q+n+m)$ to $(p+q+n)\times(p+q+d)$ where m, p, and q are the dimensions of the driving disturbance, output, and states to be estimated, respectively, and where n is the number of the states. Thus the number of operations for each iteration has been reduced from $O(n^3)$ to $O(n^2d)$ with d typically much less than n.

As in the square-root case, the fast recursions do not require explicitly checking the positivity conditions of Theorem 1—if the recursions can be carried out then an H^∞ estimator of the desired level exists, and if not, such an estimator does not exist.

B. The Central Filters

We finally remark that fast array algorithms can also be developed for the central H^∞ filters (3.8) and (3.9). The resulting statements are straightforward and will be omitted for brevity.

VI. Conclusion

In this paper, we developed square-root and fast array algorithms for the H^∞ *a priori* and *a posteriori* and filtering problems. These algorithms involve propagating the indefinite square-roots of the quantities of interest and have the interesting property that the appropriate inertia of these quantities is preserved. Moreover, the conditions for the existence of the H^∞ filters are built into the algorithms, so that filter solutions will exist if, and only if, the algorithms can be executed.

The conventional square-root and fast array algorithms are preferred because of their better numerical behavior (in the case of square-root arrays) and their reduced computational complexity (in the case of the fast recursions). Since the H^∞ square-root and fast array algorithms are the direct analogs of their conventional counterparts, they may be more attractive for numerical implementations of H^∞ filters. However, since J-unitary rather than unitary operations are involved, further numerical investigation is needed.

Our derivation of the H^∞ square-root and fast array algorithms demonstrates a virtue of the Krein space approach to H^∞ estimation and control; the results appear to be more difficult to conceive and prove in the traditional H^∞ approaches. We should also mention that there are many variations of the conventional square-root and fast array algorithms, e.g. for control problems, and the methods given here are directly applicable to extending these variations to the H^∞ setting as well. Finally, the algorithms presented here are equally applicable to risk-sensitive estimation and control problems and to quadratic dynamic games.

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Reliable Control of Nonlinear Systems

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***Abstract*—In this paper, we extend Veillette's results (1995) to the study of reliable linear-quadratic regulator problem for nonlinear systems. This is achieved by employing the Hamilton–Jacobi inequality in the nonlinear case instead of algebraic Riccati equation in the linear one. The proposed state-feedback controllers are shown to be able to tolerate the outage of actuators within a prespecified subset of actuators. Both the gain margins of guaranteeing system stability and retaining a performance bound are estimated.**

***Index Terms*—Algebraic Riccati equation, Hamilton–Jacobi inequality, linear-quadratic regulator problem.**

I. Introduction

The study of the design of reliable control systems which can tolerate the failure of the control components and retain the desired system performance has recently attracted considerable attention (see e.g., [1] and [6]–[10]). Several approaches for the design of the reliable controllers have been proposed; however, most of those efforts are focused on linear control systems [1], [6]–[8] rather than nonlinear ones. For instance, Veillette employed the algebraic Riccati equation approach to develop a procedure for the design of a state-feedback controllers, which could tolerate the outage within a selected subset of actuators while retaining the stability and the known quadratic performance bound [7]. Both the gain margins for guaranteeing system's stability and preserving system performance were also estimated in [7]. Two recent papers employed the Hamilton–Jacobi inequality approach to investigate the nonlinear reliable control problem. One studied the design of controllers that could guarantee locally asymptotic stability and H_∞ performance even when some components failed within a prespecified subset of control components

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[9] and the other investigated the single contingency reliable controller design problem with strictly redundant control and sensing elements [10].

The main goal of this paper is to extend Veillette's results [7] to the study of reliable linear-quadratic regulator problem for nonlinear systems. Rather than solving the algebraic Riccati equation in the linear case [7], we will apply the Hamilton–Jacobi inequality to the study of the nonlinear one. Although our approach of using Hamilton–Jacobi inequality approach is similar to those of [9] and [10], there are four main differences between this paper and other two. Firstly, this study adopts quadratic performance index while those of [9] and [10] are concerned with H_∞ performance. Secondly, we consider static state feedback while those of [9] and [10] employed output feedback control structure. Thirdly, we will estimate both the gain margins of guaranteeing system stability and retaining a performance bound, which were not discussed in [9] and [10]. Finally, we seek possible positive semidefinite solutions of the Hamilton–Jacobi inequality instead of positive definite ones as in [9] and [10].

The organization of this paper is as follows. Problem formulation and required assumptions for the study are given in Section II. It is followed by the derivation of the reliable state-feedback control laws and the corresponding properties of the closed-loop system. Finally, an illustrative example is given in Section IV to demonstrate the application of the proposed design.

II. Problem Formulation

Consider the control systems as given by

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $g(x) = (g_1(x), \cdots, g_m(x))$ and $u = (u_1, \cdots, u_m)^T$. Here, $x \in \mathbb{R}^n, u_i \in \mathbb{R}$ for $i = 1, \cdots, m$ and $f(x)$ as well as $g_i(x)$ for $i = 1, \cdots, m$ are all assumed to be smooth vector fields. For the interest of study, we assume $f(0) = 0$. The objective of this paper is to design a state-feedback controller that can tolerate the outage of certain actuators and simultaneously minimizes the cost function as given by

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt \tag{2}$$

where $Q \geq 0$, and $R > 0$ are constant matrices.

In the following, we adopt the notation Ω from [6], [7] as the selected subset of actuators, within which the outages must be tolerated. The notation Ω' is defined as the complementary subset of Ω. Then we can decompose the corresponding matrix function $g(x)$, the control u, and the weighting matrix R as follows:

$$g(x) = (g_{\Omega'}(x), g_\Omega(x)) \tag{3}$$

$$u = \begin{pmatrix} u_{\Omega'} \\ u_\Omega \end{pmatrix} \tag{4}$$

and

$$R = \begin{pmatrix} R_{\Omega'} & 0 \\ 0 & R_\Omega \end{pmatrix}. \tag{5}$$

In addition, denote Ω an arbitrary subset of Ω and Ω' the complementary subset of Ω, respectively. The corresponding decompositions of g, u, and R with respective to Ω and Ω' can hence be defined in the same way as those in (3)–(5).

For the case of which system (1) is linear, that is, system (1) is replaced by $\dot{x} = Ax + Bu$, Veillette [7] proposed a reliable state-feedback controller to tolerate the outage of specified actuators while retaining stability and quadratic performance. In the study of [7], three assumptions are required for the reliable design. They are: i) the linear control system is stabilizable for the worst case; ii) (A, Q) is a detectable pair; and iii) R is a diagonal matrix. For the nonlinear case to be studied in this paper, besides the three assumptions above need to be modified, two more assumptions are required to provide the existence of controllers. Details are discussed as follows.

Assumption 1: $(f, g_{\Omega'})$ is a stabilizable pair. That is, there exists a function $\alpha(x)$, which is defined around a neighborhood of the origin, such that the origin of the system $\dot{x} = f(x) + g_{\Omega'}(x)\alpha(x)$ is locally asymptotically stable.

In order to introduce the detectable condition, we define the virtual output of (1) as

$$y = h(x) = Q^{1/2}x. \tag{6}$$

The detectability condition for (1) is then given in the next assumption.

Assumption 2: (f, h) is locally detectable. To employ the definition of detectability as in [2], it implies that there exists a neighborhood U of the origin such that for any state trajectory $x(t)$ of $\dot{x} = f(x)$ with initial $x(0) \in U$, we have $\lim_{t\to\infty} x(t) = 0$ if $h(x(t)) = 0$ for all $t \geq 0$.

Assumption 3: $R > 0$ is a diagonal matrix.

Assumption 4: There exists a smooth positive semidefinite function $V(x)$, which is locally defined in a neighborhood of the origin in $\mathbb{R}^n$, which satisfies the Hamilton–Jocobi inequality

$$\nabla_x V(x) f(x) + h(x)^T h(x) - \tfrac{1}{4} \cdot \nabla_x V(x) g_{\Omega'}(x) R_{\Omega'}^{-1} g_{\Omega'}^T(x) \nabla_x^T V(x) \leq 0. \tag{7}$$

Here, $\nabla_x V(x)$ denotes the gradient of $V(x)$.

Assumption 5: The origin of the uncontrolled version of (1) is locally Lyapunov stable in the set $S_1 = \{x \mid V(x) = 0\}$. For the definition of Lyapunov stability in a set, please refer to [4].

Note that, if $V(x)$ in Assumption 4 is taken to be a positive definite function as considered in [9] and [10] instead of a positive semidefinite one, then the set S_1 in Assumption 5 contains the origin only. Assumption 5 can then be removed.

III. Main Results

In the following, we will study the reliable control laws for (1) under Assumptions 1–5. Both the stability gain margin and the performance gain margin of the closed-loop system will also be discussed. Details are given as follows.

Suppose $V(x)$ is a smooth positive semidefinite function that satisfies the Hamilton–Jacobi inequality (7), which is associated with the worst fault condition for the reliable design. From optimal control theory and (7), the state-feedback controls for actuators in Ω' are obtained as

$$u_{\Omega'}^* = -\tfrac{1}{2} \cdot R_{\Omega'}^{-1} g_{\Omega'}^T(x) \nabla_x^T V(x). \tag{8}$$

Let the remaining controls associated with the actuators in Ω be

$$u_\Omega^* = -\tfrac{1}{2} \cdot R_\Omega^{-1} g_\Omega^T(x) \nabla_x^T V(x). \tag{9}$$

Thus, the overall state-feedback control becomes

$$u^* = \begin{pmatrix} u_{\Omega'}^* \\ u_\Omega^* \end{pmatrix} = -\tfrac{1}{2} \cdot R^{-1} g^T(x) \nabla_x^T V(x). \tag{10}$$

The closed-loop system can then be rewritten as

$$\dot{x} = f(x) - \tfrac{1}{2} g_{\Omega'}(x) R_{\Omega'}^{-1} g_{\Omega'}^T(x) \nabla_x^T V(x) - \tfrac{1}{2} g_\Omega(x) R_\Omega^{-1} g_\Omega^T(x) \nabla_x^T V(x). \tag{11}$$

From (7) and the time derivative of $V(x)$ along the state trajectory of (11), we have

$$\begin{aligned}\dot{V} \le& -h(x)^T h(x) - \tfrac{1}{4}\nabla_x V(x) g_{\Omega'}(x) R_{\Omega'}^{-1} g_{\Omega'}^T(x) \nabla_x^T V(x)\\ &- \tfrac{1}{2}\nabla_x V(x) g_{\Omega}(x) R_{\Omega}^{-1} g_{\Omega}^T(x) \nabla_x^T V(x). \end{aligned} \tag{12}$$

First, we check the stability of (11) without any actuator outage. It is observed from (12) that we have $\dot{V} \le 0$ for all x in a neighborhood of the origin. Denote $x(t)$ the timing trajectory of (11) with initial state $x(0)$. We then have

$$V(x(t)) - V(x(0)) = \int_0^t \frac{d}{d\tau} V(x(\tau))\, d\tau \le 0 \tag{13}$$

for any $t > 0$. Let $S_1 := \{x \mid V(x) = 0\}$. It is not difficult to check from (13) that S_1 is an invariant set of (11) and a subset of the set $S := \{x \mid \dot{V}(x) = 0\}$. From (12), for any $x \in S$, we have $h(x) = 0, g_{\Omega'}^T(x)\nabla_x^T V(x) = 0$ and $g_{\Omega}^T(x)\nabla_x^T V(x) = 0$. These imply that $u_{\Omega'}^* = 0$ and $u_{\Omega}^* = 0$ for all $x \in S$. System (11) then becomes $\dot{x} = f(x)$ and $\lim_{t\to\infty} x(t) = 0$ for all $x \in S$ if Assumption 2 holds. This leads to the conclusion that the origin is locally asymptotically stable in the set S_1 if both Assumptions 2 and 5 hold. By employing LaSalle's Invariant Set Theorem (e.g., [3]) and [4, Lemma 2], we then have the next stabilization result.

Theorem 1: Suppose Assumptions 1–5 hold. Then the origin of (1) without any actuator outage is locally asymptotically stabilizable by the control u^* as in (10).

Proof: Suppose Assumptions 1–5 hold. By [4, Lemma 2] and the discussions above, we deduce the origin is locally Lyapunov stable. This results in every trajectory being locally bounded. Since each trajectory with $\dot{V} = 0$ has the property $\lim_{t\to\infty} x(t) = 0$, we thus have the invariant set in $\dot{V} = 0$ being the origin only. The conclusion of theorem is then implied by LaSalle's Invariant Set Theorem [3]. ■

Next, we consider the case of which actuators fail to operate or have change in gain magnitude. Denote $N_{\Omega'}$ and N_{Ω} the diagonal gain matrices corresponding to the control inputs $u_{\Omega'}$ and u_{Ω}, respectively. The effective control input of (1) then becomes

$$u = \begin{pmatrix} u_{\Omega'} \\ u_{\Omega} \end{pmatrix} \tag{14}$$

with $u_{\Omega'} = N_{\Omega'} u_{\Omega'}^*$ and $u_{\Omega} = N_{\Omega} u_{\Omega}^*$. Rewriting (1), we have

$$\begin{aligned}\dot{x} =& f(x) - \tfrac{1}{2} g_{\Omega'}(x) N_{\Omega'} R_{\Omega'}^{-1} g_{\Omega'}^T(x) \nabla_x^T V(x)\\ &- \tfrac{1}{2} g_{\Omega}(x) N_{\Omega} R_{\Omega}^{-1} g_{\Omega}^T(x) \nabla_x^T V(x). \end{aligned} \tag{15}$$

Since $V(x)$ satisfies (7), the time derivative of $V(x)$ along the state trajectory of (15) is hence calculated as

$$\begin{aligned}\dot{V} \le& -h(x)^T h(x)\\ &- \tfrac{1}{4}\nabla_x V(x) g_{\Omega'}(x)(2N_{\Omega'} - I) R_{\Omega'}^{-1} g_{\Omega'}^T(x) \nabla_x^T V(x)\\ &- \tfrac{1}{2}\nabla_x V(x) g_{\Omega}(x) N_{\Omega} R_{\Omega}^{-1} g_{\Omega}^T(x) \nabla_x^T V(x). \end{aligned} \tag{16}$$

It is easy to check from (16) that $\dot{V} \le 0$ for $N_{\Omega'} > 0.5 \cdot I$ and $N_{\Omega} \ge 0$, where I denotes an identity matrix. Note that the definition of I will be in effect throughout the remaining of this paper. We have the next result, which provides an estimation of the gain margin of $N_{\Omega'}$ and N_{Ω} to provide the reliable stability of (1).

Theorem 2: Suppose Assumptions 1–5 hold and let the control input be in the form of (14). Then the origin of (1) will be locally asymptotically stable for $N_{\Omega'} > 0.5 \cdot I$ and $N_{\Omega} \ge 0$. That is, the gains associated with the control input u_{Ω} can be within $[0, \infty)$ and those gains associated with $u_{\Omega'}$ can be within $(0.5, \infty)$.

The proof of Theorem 2 is similar to that of Theorem 1. Details are omitted.

In general, we have $N_{\Omega'} = I$ in the practical application. The time derivative of $V(x)$ along the trajectory of (1) with the control input $u = u_{\Omega'}^* + N_{\Omega} u_{\Omega}^*$ is calculated as

$$\begin{aligned}\dot{V} =& \nabla_x V(x) f(x) - \tfrac{1}{2}\nabla_x V(x) g_{\Omega'}(x) R_{\Omega'}^{-1} g_{\Omega'}^T(x) \nabla_x^T V(x)\\ &- \tfrac{1}{2}\nabla_x V(x) g_{\Omega}(x) N_{\Omega} R_{\Omega}^{-1} g_{\Omega}^T(x) \nabla_x^T V(x). \end{aligned} \tag{17}$$

By employing (7), we then have

$$\begin{aligned}\dot{V} \le& -h(x)^T h(x) - \tfrac{1}{4}\nabla_x V(x) g_{\Omega'}(x) R_{\Omega'}^{-1} g_{\Omega'}^T(x) \nabla_x^T V(x)\\ &- \tfrac{1}{2}\nabla_x V(x) g_{\Omega}(x) N_{\Omega} R_{\Omega}^{-1} g_{\Omega}^T(x) \nabla_x^T V(x)\\ =& -h(x)^T h(x) - (u_{\Omega'}^*)^T R_{\Omega'} u_{\Omega'}^*\\ &- \tfrac{1}{2}\nabla_x V(x) g_{\Omega}(x) N_{\Omega} R_{\Omega}^{-1} g_{\Omega}^T(x) \nabla_x^T V(x). \end{aligned} \tag{18}$$

Assume $0 \le N_{\Omega} \le 2I$; (18) can then become

$$\begin{aligned}\dot{V} \le& -h(x)^T h(x) - (u_{\Omega'}^*)^T R_{\Omega'} u_{\Omega'}^*\\ &- \tfrac{1}{4}\nabla_x V(x) g_{\Omega}(x) R_{\Omega}^{-1} N_{\Omega} R_{\Omega} N_{\Omega} R_{\Omega}^{-1} g_{\Omega}^T(x) \nabla_x^T V(x)\\ =& -h(x)^T h(x) - (u_{\Omega'}^*)^T R_{\Omega'} u_{\Omega'}^* - (N_{\Omega} u_{\Omega}^*)^T R_{\Omega} N_{\Omega} u_{\Omega}^*\\ =& -x^T Q x - u^T R u. \end{aligned} \tag{19}$$

Taking time integration on both sides of (19) from 0 to ∞, we have

$$V(x(\infty)) - V(x(0)) \le -\int_0^\infty (x^T Q x + u^T R u)\, dt. \tag{20}$$

Based on the discussion above, an estimation of the upper bound of the cost function for (1) is obtained in the next theorem.

Theorem 3: Suppose Assumptions 1–5 hold and let the control input be in the form of (14). Then the closed-loop system satisfies the following performance bound:

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt \le V(x_0) \tag{21}$$

for $N_{\Omega'} = I$ and $0 \le N_{\Omega} \le 2I$, where x_0 denotes the initial state of the system.

Proof: For $N_{\Omega'} = I$ and $0 \le N_{\Omega} \le 2I$, by Theorem 2, the origin of (1) is guaranteed to be locally asymptotically stable. Thus, we have $\lim_{t\to\infty} x(t) = 0$. From (20) and $V(0) = 0$, the conclusion of theorem is hence implied. ■

Remark 1: In Assumption 4, assume the Hamilton–Jacobi inequality as in (7) is replaced by the Hamilton–Jacobi equality

$$\begin{aligned}&\nabla_x V(x) f(x) + h(x)^T h(x)\\ &\quad - \tfrac{1}{4} \cdot \nabla_x V(x) g_{\Omega'}(x) R_{\Omega'}^{-1} g_{\Omega'}^T(x) \nabla_x^T V(x) = 0. \end{aligned} \tag{22}$$

Then it can be shown by a slight modification of the proof of Theorem 3 that the worst fault performance (that is, $N_{\Omega} = 0$) for (1) becomes

$$J = \int_0^\infty \left(x^T Q x + u_{\Omega'}^T R_{\Omega'} u_{\Omega'}\right) dt = V(x_0). \tag{23}$$

Here, x_0 denotes the given initial state.

For the case of which actuators in a subset Ω of Ω fail to operate, the next result follows readily from Theorem 3.

Corollary 1: Suppose Assumptions 1–5 hold and let the control input be in the form of (14). Then the closed-loop system satisfies the following performance bound:

$$J = \int_0^\infty \left(x^T Q x + u_{\Omega'}^T R_{\Omega'} u_{\Omega'}\right) dt \le V(x_0) \tag{24}$$

for $N_{\Omega'} = I$ and $N_{\Omega} = 0$, where $\Omega \subseteq \Omega$ and x_0 denotes the given initial state of the system.

Note that, for the case of which (1) is a linear control system, Assumptions 1–3 as given in Section II are the same as those of Veillette's [7]. Moreover, Assumptions 4 and 5 automatically hold for linear systems since the algebraic Riccati equation is a special case of the Hamilton–Jacobi inequality as (7). The results for linear system obtained in [7] can be abstracted from this paper.

IV. Illustrative Example

This section presents an example to illustrate the use of the main results.

Example 1: Consider system (1) with

$$f(x)=\begin{pmatrix}-x_1^3\\ -x_2+x_3x_4\\ -x_3+x_4^2\\ x_4+x_3x_4\end{pmatrix} \quad \text{and} \quad g(x)=\begin{pmatrix}1 & 0\\ 0 & x_1\\ x_3 & 0\\ 0 & 1\end{pmatrix}. \tag{25}$$

Denote $g_1(x)$ and $g_2(x)$ the first and second columns of $g(x)$, respectively. It is not difficult to check that (f, g_1) is not a stabilizable pair since it preserves the unstable eigenvalue $\lambda = 1$ no matter what control is applied. However, (f, g_2) is a stabilizable pair. For instance, the system can be stabilizable by choosing $u_1 = 0$ and $u_2 = -\alpha x_4$ with $\alpha > 1$. This results in the second actuator cannot be taken as the susceptible input. Thus, in this example, we consider $\Omega' = \{u_2\}$ and $\Omega = \{u_1\}$. It follows that the condition of Assumption 1 is satisfied.

As noted in Remark 1, the performance index can be calculated if the Hamilton–Jacobi equation (22) is able to be solved. Otherwise, an upper bound of the performance index can be obtained from the solution of the associated Hamilton–Jacobi inequality. Thus, without solving the Hamilton–Jacobi equation, it is in general hard to judge which of the control input has better performance when all the actuators are taken to be the susceptible input. However, unlike the algebraic Riccati equation which can be explicitly solved in the linear case, there does not have to be a systematic way to solve the Hamilton–Jacobi equation so far because of its nonlinear nature (see e.g., [5] and [12]). A parallel study of linear systems obtained in [7] for the selection of susceptible inputs is generally hard to derive for the nonlinear system. In this example, we will only calculate an upper bound of the performance index from the solution of the Hamilton–Jacobi inequality by taking u_1 to be susceptible input.

Let the weighting matrices Q and R for the performance index as in (2) be chosen as

$$Q=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \quad \text{and} \quad R=\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{26}$$

The virtual output is hence obtained from (6) as $y = h(x) = (x_3, x_4)^T$. In the following, we will verify the satisfaction of Assumptions 2, 4, and 5 to guarantee the reliable stability and estimate the basin of attraction of the reliable closed-loop system.

To verify Assumption 2, it is noted that if $x(t)$ is any state trajectory of the uncontrolled version of (1) satisfying the condition $h(x(t)) = 0$ for all $t \ge 0$, then we have $x_3(t) = x_4(t) = 0$ for all $t \ge 0$. It follows that $x(t)$ satisfies the constrained dynamics

$$\dot{x_1} = -x_1^3 \quad \text{and} \quad \dot{x_2} = -x_2 \tag{27}$$

of (25) with $x_3(t) = x_4(t) = 0$ for all $t \ge 0$. It is not difficult to check that the origin of the reduced system (27) will be asymptotically stable. This results in $\lim_{t\to\infty} x(t) = 0$. Assumption 2 is hence satisfied.

To verify Assumption 4, we choose the positive semidefinite function $V(x)$ as

$$V(x) = x_3^2 + kx_4^2. \tag{28}$$

For (25), we then have

$$\begin{aligned}&\nabla_x V(x) f(x) + h^T(x)h(x)\\ &\quad - \tfrac{1}{4}\nabla_x^T V(x) g_{\Omega'}(x) g_{\Omega'}^T(x) \nabla_x V(x)\\ &\quad = -x_3^2 + (2k+1-k^2)x_4^2 + (2+2k)x_3x_4^2.\end{aligned} \tag{29}$$

The Hamilton–Jacobi inequality is found from (7) to hold if $2k + 1 - k^2 < 0$. That means Assumption 4 holds if $k > \sqrt{2} + 1$. Moreover, it is not difficult to verify that Assumption 5 holds for (25) by the similar approach as those for the verification of Assumption 2. Details are omitted. Thus, as implied by Theorems 2 and 3, we can choose the control inputs as

$$u_1 = -x_3^2 \quad \text{and} \quad u_2 = -kx_4 \quad \text{with } k > \sqrt{2}+1 \tag{30}$$

to provide the reliable stability of (25).

It is noted that the closed-loop system with control in the form of (30) is a triangular system (for a definition, see e.g., [11]). In the following, we will employ the results of [11] to estimate the basin of attraction of the reliable closed-loop system. To this end, we first estimate the domain of attraction of the subsystem associated with the states x_3 and x_4. By defining the function

$$W_1(x_3, x_4) = x_3^2 + hx_4^2 \quad \text{with } h > 0 \tag{31}$$

from (25) and (30) we have

$$\dot{W}_1 = 2\{-x_3^2 - \delta x_3^4 + [(1+h)x_3 + h - hk]x_4^2\}. \tag{32}$$

Here, $\delta = 0$ if u_1 fails while $\delta = 1$ if u_1 is in normal operating condition. This implies that $\dot{W}_1 < 0$ if $x_3 < (h(k-1))/(h+1)$. Thus, the region

$$A = \left\{(x_3, x_4)^T \mid W_1(x_3, x_4) < \left(\frac{h(k-1)}{h+1}\right)^2\right\} \tag{33}$$

is an estimation of the domain of attraction of the subsystem associated with the states x_3 and x_4 regardless of whether the first actuator fails or gives normal operation. Next, it is not difficult to check that the closed-loop subsystem associated with the states x_1 and x_2 by setting $x_3 = x_4 = 0$ is globally asympaitically stable. Finally, to conclude that $\mathbb{R}^2 \times A$ is an estimation of the domain of attraction for the whole closed-loop system, it remains to show that each orbit of the example system with initial point in $\mathbb{R}^2 \times A$ is bounded for $t > 0$. To see this, let us define the function

$$W_2(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2. \tag{34}$$

Then the time-derivative of W_2 along the trajectories of (25) is calculated as

$$\begin{aligned}\dot{W}_2 &= 2\left[-x_1^4 - x_2^2 - \delta x_1 x_3^2 + x_2x_3x_4 - kx_1x_2x_4\right.\\ &\qquad \left.- x_3^2 + 2x_3x_4^2 - \delta x_3^4 + (1-k)x_4^2\right]\\ &\le 2\left[-x_1^4 - x_2^2 - x_3^2 - \delta x_3^4 + (1-k)x_4^2 + \epsilon k \cdot |x_1x_2|\right.\\ &\qquad \left.+ \epsilon^2(\delta\cdot|x_1| + |x_2|) + 2\epsilon^3\right]\end{aligned} \tag{35}$$

for $\|(x_3, x_4)^T\| < \epsilon < 1$. Here, $\|\cdot\|$ denotes the Euclidean norm of a vector. It follows that $\dot{W}_2 < 0$ on the set $\{x : \|(x_3, x_4)^T\| < \epsilon, \|(x_1, x_2)^T\| > \rho\}$ with ϵ sufficient small and ρ sufficient large. According to [11, Th. 4.5], every orbit with initial point in $\mathbb{R}^2 \times A$ is hence bounded for $t > 0$. The set $\mathbb{R}^2 \times A$ is then concluded by [11,

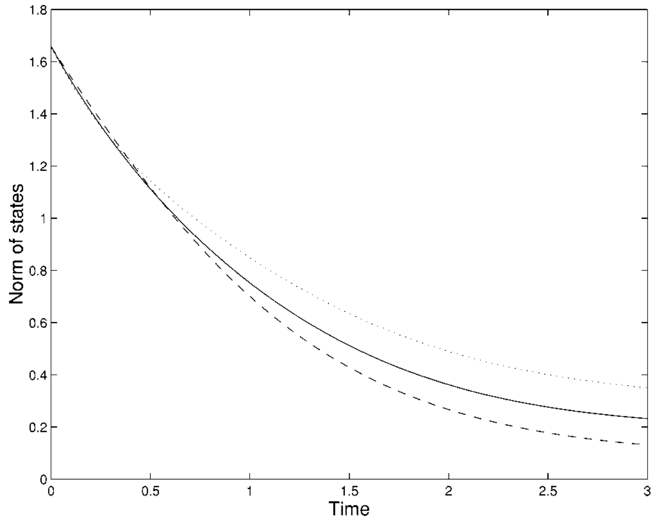

Fig. 1. Norm of states.

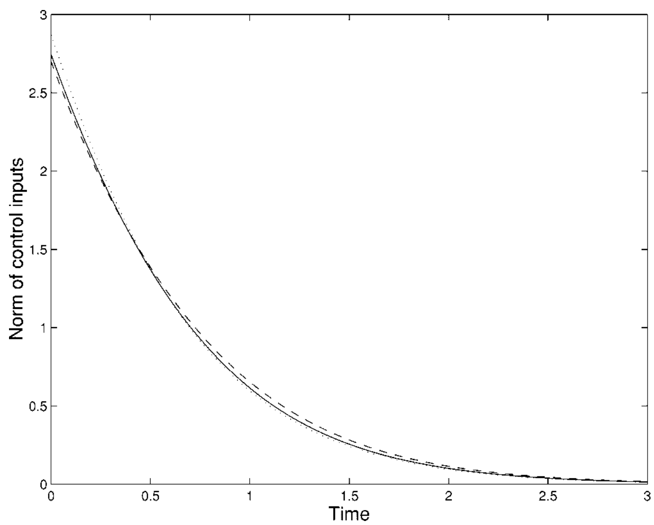

Fig. 2. Norm of control inputs.

Corollary 4.6] to be an estimation of the domain of attraction for the whole closed-loop system.

Numerical simulations for Example 1 are given in Figs. 1 and 2. In these simulations, the initial state and the positive semidefinite function $V(x)$ are chosen to be $x_0 = (0.1, 1.2, 0.7, 0.9)^T$ and $V(x) = x_3^2 + 3x_4^2$, respectively. Fig. 1 shows the time evolution of the norm of system state and Fig. 2 gives the norm of the applied control force. In these two figures, solid-line associates with the case of $(N_{\Omega'}, N_\Omega) = (1, 1)$, while the dashed line and dotted line correspond to those cases of which $(N_{\Omega'}, N_\Omega) = (1, 0)$ and $(N_{\Omega'}, N_\Omega) = (1, 2)$, respectively. That is, the dashed line shows the case in which the first actuator fails to operate, while the dotted line shows the case in which the first feedback-loop gain is amplified. In these three cases, all the states are observed to converge to the origin, which agrees with the conclusion of Theorem 2. Moreover, by Theorem 3, an upper bound $V(x_0)$ for the cost function J for all three cases is calculated to be 2.92.

V. Conclusion

In this paper, we have employed the Hamilton–Jacobi inequality approach to study the reliable linear-quadratic regulator problem for nonlinear systems. The proposed state-feedback controllers are shown to be able to tolerate the outage of actuators within a prespecified subset of actuators. Moreover, both the gain margins of guaranteeing system stability and retaining a performance are also estimated.

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