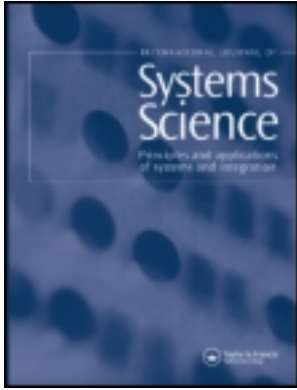


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# A new method for constructing sliding surfaces of linear time-invariant systems

YON-PING CHEN<sup>†</sup> and JEANG-LIN CHANG<sup>‡</sup>

*A new method for constructing the sliding surfaces is developed by using the pole-assignment method; unlike the usual cases, the pole-assignment method is applied to the overall closed-loop system and not the system in the sliding mode. Most importantly, the design procedure is simple since the sliding surfaces can be determined in an explicit form. Also, three significant properties listed in a theorem are proved and demonstrated by a numerical example.*

## 1. Introduction

In general, the sliding-mode controller design has two fundamental steps (Decarlo *et al.* 1988). The first step is to select the sliding surfaces such that the system exhibits the desired behaviour in the sliding mode. The second step is to determine control laws to guarantee the reaching and sliding condition. Recently, several methods available for the sliding surface design have been developed. Utkin and Young (1979) proposed three design procedures based on optimal control and pole placement. Zinober (Elghezawi *et al.* (1983) and Dorling and Zinober (1986, 1988) used a geometric approach and eigenstructure assignment to design sliding surfaces. Young and Özgüner (1993) incorporated the optimal approach with frequency shaping technique. Su *et al.* (1996) constructed sliding surfaces based on the Lyapunov equation. Here, this paper will introduce a distinct and simple method for constructing the sliding surfaces based on the the pole-assignment method.

In fact, the commonest way used in the sliding surface design is to assign the eigenvalues for the system in the sliding mode by using the pole-assignment method. Unfortunately, since the mathematical expression of the sliding motion is quite complicated, it is always an

effort by adopting the pole-assignment method. In order to ameliorate this drawback, this paper employs the pole-assignment method for the overall closed-loop system, but the system in the sliding mode. Because this is based on the linear system theories, many techniques have been well developed for the pole-assignment method applied to the overall closed-loop system and most importantly, many software packages are available for these techniques, such as MATLAB software by The Mathworks, Inc. This will make the pole-assignment method more useful and convenient.

In section 2, the problem statement is given. Section 3 shows the sliding surface design based on the pole assignment. Three conditions are considered for the pole-assignment method to determine the feedback gain matrix of the overall closed-loop system. One important theorem is also presented, which has three properties related to the sliding surface. This theorem will be proved in detail. A numerical example is shown in section 4 to verify the developed method. Finally, concluding remarks are given in section 5.

## 2. Problem statement

In general, a linear time-invariant system is described by

$$\dot{x} = \mathbf{A}x + \mathbf{B}(u + d), \quad (1)$$

where  $x \in R^n$  is the system state,  $u \in R^m$  is the control input and  $d \in R^m$  represents the matching-type disturbance. It is further assumed that the system matrices  $\mathbf{A} \in R^{n \times n}$  and  $\mathbf{B} \in R^{n \times m}$  are exactly known and the pair  $(\mathbf{A}, \mathbf{B})$  is controllable. Also, the matrix  $\mathbf{B}$  is of full rank, that is  $\text{rank}(\mathbf{B}) = m$ . Usually, in the design of the

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sliding-mode controller, the first step is to choose the sliding vector as

$$s = \mathbf{C}\mathbf{x}, \quad (2)$$

where  $s = [s_1 \ s_2 \ \cdots \ s_m]$ ,  $\mathbf{C} \in \mathbb{R}^{m \times n}$ , and  $\mathbf{CB}$  is non-singular. Here,  $s_i = 0$ ,  $i = 1, 2, \dots, m$ , are called the sliding surfaces. Once the sliding condition  $s^T \dot{s} < 0$  is guaranteed, the system will slide along the intersection of these  $m$  sliding surfaces  $s = \mathbf{0}$ , called the sliding mode. It is known that the system in the sliding mode is described by

$$\dot{\mathbf{x}} = [\mathbf{I}_n - \mathbf{B}(\mathbf{CB})^{-1}\mathbf{C}]\mathbf{A}\mathbf{x}, \quad (3)$$

which, as expected, is robust to the matching-type disturbance  $\mathbf{d}$ . Here,  $\mathbf{I}_n$  represents the  $n \times n$  identity matrix. Later, for convenience, we define  $\mathbf{I}_k$  as the  $k \times k$  identity matrix. Since  $[\mathbf{I}_n - \mathbf{B}(\mathbf{CB})^{-1}\mathbf{C}]\mathbf{A}$  has  $m$  zero eigenvalues, let its eigenvalues be denoted as

$$\left\{ \lambda_1, \lambda_2, \dots, \lambda_{n-m}, \underbrace{0, 0, \dots, 0}_{m \text{ zeros}} \right\}. \quad (4)$$

It is known that, if the  $n - m$  eigenvalues  $\{\lambda_1, \lambda_2, \dots, \lambda_{n-m}\}$  are stable, then the system stability is guaranteed. However, owing to the complexity of  $[\mathbf{I}_n - \mathbf{B}(\mathbf{CB})^{-1}\mathbf{C}]\mathbf{A}$ , it is often not easy to determine the matrix  $\mathbf{C}$ . The aim of this paper is to propose a simpler design method for the sliding surfaces.

### 3. Design of the sliding surfaces

The design method introduced here is mainly based on the pole-assignment method for the linear time-invariant system (1) without any disturbance  $\mathbf{d}$ , expressed by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}. \quad (5)$$

Since  $(\mathbf{A}, \mathbf{B})$  is controllable, by using the pole-assignment method a feedback gain  $\mathbf{K}$  can be obtained by assigning  $n$  eigenvalues  $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$  for  $\mathbf{A} - \mathbf{BK}$  (Chen 1984). To design the matrix  $\mathbf{C}$  in (2) simply and appropriately, the following conditions are considered.

**Condition C1:** The matrix  $\mathbf{A} - \mathbf{BK}$  has no eigenvalues in common with those of  $\mathbf{A}$ .

**Condition C2:** The eigenvalues of  $\mathbf{A} - \mathbf{BK}$  are chosen to be stable and assigned as

$$\left\{ \lambda_1, \lambda_2, \dots, \lambda_{n-m}, \underbrace{\lambda, \lambda, \dots, \lambda}_{m \lambda s} \right\}, \quad (6)$$

with  $\lambda \neq \lambda_i, i = 1, 2, \dots, n - m$ .

**Condition C3:** The matrix  $\mathbf{A} - \mathbf{BK}$  is diagonalizable although it possesses  $m$  repeated eigenvalues  $\lambda$ .

Note that condition C1 is usually required for the techniques to determine the feedback gain  $\mathbf{K}$  via the pole-placement method. Also, since  $\lambda$  is not in the spec-

trum of  $\mathbf{A}$ , the matrix  $\mathbf{A} - \lambda\mathbf{I}_n$  is non-singular. In other words,  $(\mathbf{A} - \lambda\mathbf{I}_n)^{-1}$  exists and later it will be included in the expression for the matrix  $\mathbf{C}$ . As for condition C2, it will be used to prove theorem 1 given in this section. For condition C3, in fact, it has been proved by Sinswat and Fallside (1977) that  $\mathbf{A} - \mathbf{BK}$  is diagonalizable with the eigenvalues given as (6), where the number of the repeated eigenvalue  $\lambda$  is not greater than  $m$ ; this means that

$$(\mathbf{A} - \mathbf{BK})\mathbf{W} = \mathbf{W}\mathbf{J} \quad (7)$$

where  $\mathbf{J} = \text{diag}[\lambda_1 \ \lambda_2 \ \cdots \ \lambda_{n-m} \ \lambda \ \cdots \ \lambda]$  and  $\mathbf{W}$  is non-singular. Significantly, it can be further decomposed into

$$(\mathbf{A} - \mathbf{BK})\mathbf{W}_{n-m} = \mathbf{W}_{n-m}\mathbf{J}_{n-m}, \quad (8)$$

$$(\mathbf{A} - \mathbf{BK})\mathbf{W}_{\sim m} = \lambda\mathbf{W}_{\sim m}, \quad (9)$$

where  $\mathbf{W}_{n-m}$  and  $\mathbf{W}_{\sim m}$  contain the first  $n - m$  columns and the last  $m$  columns of  $\mathbf{W}$ . Also,  $\mathbf{J}_{n-m} = \text{diag}[\lambda_1 \ \lambda_2 \ \cdots \ \lambda_{n-m}]$ , and  $\mathbf{W}_{\sim m}$  is related to the repeated eigenvalue  $\lambda$ . Now, according to these conditions, an explicit form of the matrix  $\mathbf{C}$  is introduced as

$$\mathbf{C} = \mathbf{K}(\mathbf{A} - \lambda\mathbf{I}_n)^{-1}, \quad (10)$$

which will generate three important properties listed in the following theorem.

**Theorem 1:** Let  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times m}$ , and the pair  $(\mathbf{A}, \mathbf{B})$  is controllable. If  $\mathbf{B}$  is of full rank and  $\mathbf{C} = \mathbf{K}(\mathbf{A} - \lambda\mathbf{I}_n)^{-1}$  with  $\mathbf{K}$  satisfying conditions C1, C2 and C3, then

- (a)  $\mathbf{CB} = \mathbf{I}_m$ ,
- (b)  $\mathbf{CW}_{n-m} = \mathbf{0}$ ,
- (c)  $[\mathbf{I}_n - \mathbf{B}(\mathbf{CB})^{-1}\mathbf{C}]\mathbf{A}$  has the eigenvalues  $\lambda_1, \dots, \lambda_{n-m}$  and  $m$  zeros.

**Proof:**

- (a) Let us show that  $\mathbf{CB} = \mathbf{I}_m$ . Rearranging (9) yields

$$(\mathbf{A} - \lambda\mathbf{I}_n)\mathbf{W}_{\sim m} = \mathbf{BKW}_{\sim m} \quad (11)$$

and then

$$\mathbf{W}_{\sim m} = (\mathbf{A} - \lambda\mathbf{I}_n)^{-1}\mathbf{BKW}_{\sim m} \quad (12)$$

Pre-multiplying  $\mathbf{K}$  by (12) yields

$$\mathbf{KW}_{\sim m} = \mathbf{K}(\mathbf{A} - \lambda\mathbf{I}_n)^{-1}\mathbf{BKW}_{\sim m} = \mathbf{CBKW}_{\sim m} \quad (13)$$

or

$$(\mathbf{I}_m - \mathbf{CB})\mathbf{KW}_{\sim m} = \mathbf{0}. \quad (14)$$

From (11), it can be declared that  $\mathbf{KW}_{\sim m}$  is of full rank because the other matrices  $\mathbf{A} - \lambda\mathbf{I}_n$ ,  $\mathbf{W}_{\sim m}$ , and

$\mathbf{B}$  are all of full rank; as a consequence of (14),  $\mathbf{I}_m - \mathbf{CB} = \mathbf{0}$ . This proves that  $\mathbf{CB} = \mathbf{I}_m$ .

- (b) From (10), we have  $\mathbf{CA} = \mathbf{K} + \lambda\mathbf{C}$ . Pre-multiplying  $\mathbf{C}$  by (8) yields

$$(\mathbf{CA} - \mathbf{CBK})\mathbf{W}_{n-m} = \mathbf{CW}_{n-m}\mathbf{J}_{n-m} \quad (15)$$

By using  $\mathbf{CB} = \mathbf{I}_m$  and  $\mathbf{CA} = \mathbf{K} + \lambda\mathbf{C}$ , (15) is rewritten as

$$\mathbf{CW}_{n-m}(\mathbf{J}_{n-m} - \lambda\mathbf{I}_{n-m}) = \mathbf{0}. \quad (16)$$

According to the condition  $\lambda \neq \lambda_i$ ,  $i = 1, 2, \dots, n - m$  in condition C2, the diagonal components of  $\mathbf{J}_{n-m} - \lambda\mathbf{I}_{n-m}$  are all non-zero. Hence,  $\mathbf{J}_{n-m} - \lambda\mathbf{I}_{n-m}$  is non-singular and then  $\mathbf{CW}_{n-m} = \mathbf{0}$  is proved.

- (c) The truth of  $\mathbf{CB} = \mathbf{I}_m$  implies that  $\text{rank}(\mathbf{C}) = m$ . Therefore,  $\mathbf{C}[\mathbf{I}_n - \mathbf{B}(\mathbf{CB})^{-1}\mathbf{C}]\mathbf{A} = \mathbf{0}$  implies that the eigenvalues of  $[\mathbf{I}_n - \mathbf{B}(\mathbf{CB})^{-1}\mathbf{C}]\mathbf{A}$  should contain  $m$  zeros. Further, by using  $\mathbf{CB} = \mathbf{I}_m$  and  $\mathbf{CA} = \mathbf{K} + \lambda\mathbf{C}$ , we have

$$[\mathbf{I}_n - \mathbf{B}(\mathbf{CB})^{-1}\mathbf{C}]\mathbf{A} = \mathbf{A} - \mathbf{BK} - \lambda\mathbf{BC}. \quad (17)$$

Then, from (8) and  $\mathbf{CW}_{n-m} = \mathbf{0}$ , after post-multiplying  $\mathbf{W}_{n-m}$  by (17), it can be shown that

$$[\mathbf{I}_n - \mathbf{B}(\mathbf{CB})^{-1}\mathbf{C}]\mathbf{A}\mathbf{W}_{n-m} = \mathbf{W}_{n-m}\mathbf{J}_{n-m}. \quad (18)$$

Obviously, the diagonal components of  $\mathbf{J}_{n-m}$ , that is  $\lambda_1, \lambda_2, \dots, \lambda_{n-m}$ , are the eigenvalues of  $[\mathbf{I}_n - \mathbf{B}(\mathbf{CB})^{-1}\mathbf{C}]\mathbf{A}$  and the matrix  $\mathbf{W}_{n-m}$  contains the eigenvectors corresponding to these eigenvalues. This completes the proof.  $\square$

Based on this theorem, the matrix  $\mathbf{C}$  can be directly calculated from the explicit form  $\mathbf{C} = \mathbf{K}(\mathbf{A} - \lambda\mathbf{I}_n)^{-1}$  after the feedback gain  $\mathbf{K}$  is obtained for the overall closed-loop system (1) via the pole-assignment method. In addition, this theorem also implies that the eigenvalues of the system in the sliding mode can be arbitrarily assigned under the restriction of  $\mathbf{CB} = \mathbf{I}_m$ . In the next section, an example is used as a demonstration.

#### 4. Numerical example

This section adopts the example given by Su *et al.* (1996). In their work, the design of  $\mathbf{C}$  is mainly based on the Lyapunov equation. Although the matrix  $\mathbf{C}$  can be obtained rather straightforwardly, the system eigenvalues in the sliding mode cannot be pre-assigned and thus the sliding behaviour is hard to predict. The system matrices  $\mathbf{A}$  and  $\mathbf{B}$  of (1) are given as

$$\mathbf{A} = \begin{bmatrix} 0.2325 & -0.9285 & 0.0154 & 0.1222 \\ -0.7274 & 1.01116 & -0.0224 & 0.1576 \\ -1.6883 & 0.2214 & 0.6534 & 1.6278 \\ -0.5310 & -0.2603 & -0.0052 & 1.1025 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 3.0 & 2.0 \\ 0 & 1.0 \\ 0.5 & -2.0 \\ 1.3 & 0 \end{bmatrix}.$$

First the eigenvalues  $\mathbf{A}$  are obtained as  $\{-0.2, 0.5, 0.7, 1.0\}$ . According to conditions C1 and C2, we choose the eigenvalues for the matrix  $\mathbf{A} - \mathbf{BK}$  as  $\{-0.612, -1.22, -0.5, -0.5\}$ . Clearly, the matrix  $\mathbf{A} - \mathbf{BK}$  has no eigenvalues in common with those of  $\mathbf{A}$ . Also, the eigenvalues of  $\mathbf{A} - \mathbf{BK}$  are stable and  $\lambda_1 \neq \lambda_2 \neq \lambda$  where  $\lambda_1 = -0.612$ ,  $\lambda_2 = -1.22$ , and  $\lambda = -0.5$ . Note that  $\lambda_1$  and  $\lambda_2$  are purposely assigned to be the same as those in the work of Su *et al.* (1996). As for the repeated eigenvalue  $\lambda$ , it is arbitrarily assigned. By using the MATLAB software, the feedback gain  $\mathbf{K}$  is processed by the command ' $\mathbf{K} = \text{place}(\mathbf{A}, \mathbf{B}, \mathbf{p})$ ' for pole-placement with  $\mathbf{p} = [-0.612, -1.22, -0.5, -0.5]^T$ . The command ' $\text{place}$ ' allows the user to assign the eigenvalues with multiplicity equal to the number of inputs, as the case introduced in this paper. The numerical result of  $\mathbf{K}$  is then obtained as

$$\mathbf{K} = \begin{bmatrix} 3.92 & -3.66 & 2.11 & -7.19 \\ -2.81 & 2.51 & -2.73 & 7.14 \end{bmatrix}.$$

Condition C3 can be checked by the command ' $[\mathbf{W}, \mathbf{J}] = \text{eig}(\mathbf{A} - \mathbf{BK})$ ' where  $\mathbf{W}$  and  $\mathbf{J}$  are obtained as

$$\mathbf{W} = [\mathbf{W}_2 \quad \mathbf{W}_{\sim 2}] = \begin{bmatrix} -0.72 & 0.54 & 0.35 & -0.77 \\ -0.14 & 0.66 & 0.62 & -0.44 \\ -0.54 & -0.51 & -0.66 & -0.35 \\ -0.42 & -0.12 & -0.24 & -0.30 \end{bmatrix},$$

$$\mathbf{J} = \text{diag}[-1.22 \quad -0.612 \quad -0.5 \quad -0.5].$$

Clearly, since  $\mathbf{J}$  is diagonal, the matrix  $\mathbf{A} - \mathbf{BK}$  is diagonalizable. Based on the explicit form  $\mathbf{C} = \mathbf{K}(\mathbf{A} - \lambda\mathbf{I}_4)^{-1}$ , the matrix  $\mathbf{C}$  is then simply attained as

$$\mathbf{C} = \begin{bmatrix} 2.75 & -2.06 & 1.72 & -6.24 \\ -2.57 & 1.59 & -2.27 & 6.80 \end{bmatrix}.$$

By direct calculation, it is easy to verify the three properties in theorem 1, namely  $\mathbf{CB} = \mathbf{I}_2$ ,  $\mathbf{CW}_{\sim 2} = \mathbf{0}$  and  $[\mathbf{I}_4 - \mathbf{B}(\mathbf{CB})^{-1}\mathbf{C}]\mathbf{A}$  has eigenvalues  $\{-0.612, -1.22, 0, 0\}$ . Hence, it can be declared that, by choosing the

sliding vector  $\mathbf{s} = \mathbf{C}\mathbf{x}$ , the system stability in the sliding mode is guaranteed.

## 5. Conclusions

This paper presents a new design method for sliding surfaces and via the pole-assignment method. Most significantly, the pole-assignment method is applied to the overall closed-loop system with the system in the sliding mode. This simplifies the design work because many numerical tools have been well developed for such a pole-assignment method. Three conditions are considered for the pole-assignment method of the overall closed-loop system. After the feedback gain is obtained, the sliding surfaces can be determined in an explicit form. The three properties listed in theorem 1 have been proved and demonstrated by a numerical example. Based on this theorem, the system stability in the sliding mode can be guaranteed. Most significantly, this theorem also implies that the eigenvalues of the system in the sliding mode can be arbitrarily assigned under the restriction of  $\mathbf{CB} = \mathbf{I}_m$ . This restriction has been employed in many sliding-mode controller designs.

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