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Pseudo-Hamiltonian-connected graphs

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Abstract

Given a graph G and a positive integer k, denote by $G[k]$ the graph obtained from G by replacing each vertex of G with an independent set of size k. A graph G is called pseudo-k Hamiltonian-connected if $G[k]$ is Hamiltonian-connected, i.e., every two distinct vertices of $G[k]$ are connected by a Hamiltonian path. A graph G is called pseudo Hamiltonian-connected if it is pseudo-k Hamiltonian-connected for some positive integer k. This paper proves that a graph G is pseudo-Hamiltonian-connected if and only if for every non-empty proper subset X of $V(G)$, $|N(X)| > |X|$. The proof of the characterization also provides a polynomial-time algorithm that decides whether or not a given graph is pseudo-Hamiltonian-connected. The characterization of pseudo-Hamiltonian-connected graphs also answers a question of Richard Nowakowski, which motivated this paper.

MSC: 05C15

Keywords: Pseudo-Hamiltonian-connected; Regular Hamiltonian walk; Pseudo-edge; Vertex packing; Regularizable

1. Introduction

Graphs in this paper are finite, undirected, loopless and without parallel edges. The term multigraph is used for that with parallel edges. For standard terminology and notation, see [2].

The following question was asked by Nowakowski [6]: Given a graph G, with each vertex assigned an integer. A beetle crawls from vertex to vertex along its edges. As it arrives at a vertex, it increases the integer assigned to that vertex by 1. A graph G is called a *beetle graph* if for any initial position and any initial assignment of integers, the beetle, by crawling along the edges, can change it to an assignment in which the

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integers assigned to all the vertices are the same. The question is to characterize beetle graphs.

It turns out that such graphs have other interesting properties, which are also related to the vertex packing problem [5,7,9]. To describe these properties, we first introduce the notion of pseudo-Hamiltonian-connected graphs, regular Hamiltonian walks and pseudo-edges.

Given a graph G and a positive integer k, denote by $G[k]$ the graph obtained from G by replacing each vertex of G with an independent set of size k. To be precise, $G[k]$ has vertex set $\{v_i : v \in V(G),\ i = 1, 2, \ldots, k\}$, two vertices v_i and u_j are adjacent if and only if vu is an edge of G. A graph G is called *pseudo-k Hamiltonian-connected* if $G[k]$ is Hamiltonian-connected, i.e., every two distinct vertices of $G[k]$ are connected by a Hamiltonian path. A graph G is called *pseudo-Hamiltonian-connected* if it is pseudo-k Hamiltonian-connected for some positive integer k.

Suppose G is a graph and x and y are vertices of G. An x–y walk W of G is called a *regular Hamiltonian walk* if there is a positive integer k such that each vertex of $V(G)$ occurs exactly k times in W. It is easy to see that if G is pseudo-Hamiltonian-connected, then for every pair of distinct vertices x and y of G there exists an x–y regular Hamiltonian walk.

An x–y walk W is called a *pseudo-edge* if there is an integer $k \geqslant 0$ such that each vertex of $V(G) - \{x, y\}$ occurs k times in W, and each of x and y occurs $(k+1)$ times in W. We are interested in graphs for which every pair of distinct vertices is connected by a pseudo-edge.

The following result shows that the classes of graphs defined above are indeed the same classes of graphs, for which there is a simple characterization and their membership can be determined in polynomial time. For a subset X of $V(G)$, we denote by $N_G(X)$ (or $N(X)$, if there is no confusion) the set of neighbors of vertices of X, i.e., $N_G(X) = \{y : xy \in E(G)$ for some $x \in X\}$. For any vertex x, $N_G(x)$ stands for $N_G(\{x\})$.

Theorem 1. *Given a graph G of at least three vertices, the following statements are equivalent*:
(1) *G is pseudo-Hamiltonian-connected.*
(2) *Every two distinct vertices of G are connected by a regular Hamiltonian walk.*
(3) *Every two distinct vertices of G are connected by a pseudo-edge.*
(4) *G is a beetle graph.*
(5) *G is connected and for every non-empty independent set I of G, $|N(I)| > |I|$.*
(6) *For every non-empty proper subset X of $V(G)$, $|N(X)| > |X|$.*

This paper proceeds as follows. Section 2 gives a proof of the main theorem and discusses the recognition problem for this class of graphs. Section 3 discusses the relation between these graphs and the vertex packing problem. Section 4 investigates pseudo-2 Hamiltonian-connected graphs and raises some open problems. The complexity issue is discussed in Section 5.

2. Proof of Theorem 1

(1) $\Rightarrow$ (2): This is trivial, because when $x \neq y$, an x–y Hamiltonian path of $G[k]$ corresponds to an x–y regular Hamiltonian walk of G.

(2) $\Rightarrow$ (3): Suppose every two distinct vertices of G are connected by a regular Hamiltonian walk. For any two distinct vertices x and y, let W be an x–y regular Hamiltonian walk. Replace each edge uv of W by a u–v regular Hamiltonian walk. Then it is straightforward to verify that the resulting walk is a pseudo-edge connecting x and y.

(3) $\Rightarrow$ (4): Suppose every two distinct vertices of G is connected by a pseudo-edge. If the beetle crawls along a pseudo-edge connecting x and y, the integers assigned to y is increased by $k+1$, while each of the other integers is increased by k. As we are only interested in the differences between the integers assigned to the vertices, when the beetle crawls along a pseudo-edge connecting x and y, it has the same effect as crawling along an edge connecting x and y. Therefore, G is a beetle graph if and only if K_n is a beetle graph, where $n = |V(G)|$. It is easy to verify that K_n is a beetle graph if and only if $n \geqslant 3$. Indeed, starting from an arbitrary vertex, the beetle may crawl along an edge towards a vertex with minimum integer, and the process is repeated. The integers assigned to the vertices will eventually become the same.

(4) $\Rightarrow$ (5): Assume that G is a beetle graph. Obviously G is connected. We prove that for each independent set I, $|N(I)| > |I|$. Assume to the contrary that for some independent set I, $|N(I)| \leqslant |I|$. Assign 1 to each vertex of $N(I)$ and assign 0 to each vertex of I. Integers assigned to the other vertices are arbitrary. Suppose the beetle is initially at a vertex of I. Before arriving at any vertex of I, the beetle must arrive at a vertex of $N(I)$. Thus, no matter how the beetle crawls along the edges, the total increase of the integers assigned to $N(I)$ is at least as large as the total increase of the integers assigned to I. As $|N(I)| \leqslant |I|$, the average increase of the integers assigned to $N(I)$ is at least as large as the average of the increase of the integers assigned to I. Thus, it is impossible that all the integers become the same.

(5) $\Rightarrow$ (6): Suppose X is an arbitrary non-empty proper subset of $V(G)$. Each non-isolated vertex of the induced subgraph $G[X]$ belongs to $N(X)$. Let I be the set of isolated vertices of $G[X]$. If $I \neq \emptyset$, then $|N(I)| > |I|$. As $N(I) \cap (X - I) = \emptyset$, it follows that $|N(X)| \geqslant |X - I| + |N(I)| > |X|$. If $I = \emptyset$, then $X \subset N(X)$. Therefore, either $|X| < |N(X)|$ or $X = N(X)$. Since G is connected, $X = N(X)$ would imply that $X = V(G)$.

(6) $\Rightarrow$ (1): Finally, assume that for every non-empty proper subset X of $V(G)$, $|N(X)| > |X|$. To see that there is an integer k such that $G[k]$ is Hamiltonian-connected, it suffices to prove that there is an integer k such that for any two vertices x and y of G, there is an x–y walk of G in which each vertex of G occurs exactly k times. Note that x and y need not be distinct.

We call a multigraph H a *spanning sub-multigraph* of G if H is obtained from G by "multiply" some edges, where "multiplying an edge" means replacing the edge by t parallel edges for some integer $t \geqslant 0$.

Suppose W is an x–y regular Hamiltonian walk in which each vertex occurs k times. Then the edges of W induce a spanning sub-multigraph of G, in which each vertex of $V(G)-\{x,y\}$ has degree $2k$, and each of x and y has degree $2k-1$. (In case $x=y$, the vertex x has degree $2k-2$.) Before proving the existence of such an x–y Hamiltonian walk for every pair of vertices x and y, we prove the existence of such sub-multigraphs of G.

Lemma 2. *Suppose G is a graph such that for every non-empty proper subset X of $V(G)$, $|N_G(X)|>|X|$. Then for any distinct vertices x and y of G, there exists a sub-multigraph H of G in which each vertex of $V(G)-\{x,y\}$ has degree* 2, *and each of x and y has degree* 1; *for every vertex x of G, there exists a sub-multigraph H of G in which each vertex of $V(G)-\{x\}$ has degree* 2, *and x has degree* 0.

Proof. Let x and y be (not necessarily distinct) vertices of G. Construct a bipartite graph $Q_{x,y}=(A\cup B,E)$ as follows: $A=\{u_A: u\in V(G)$ and $u\neq x\}$, $B=\{v_B: v\in V(G)$ and $v\neq y\}$ and $E=\{(u_A,v_B):(u,v)\in E(G)\}$. Then for every subset X of A, $|N_{Q_{x,y}}(X)|\geqslant|N_G(X)|-1\geqslant|X|$. It follows from Hall's theorem that $Q_{x,y}$ has a perfect matching, say M. Let H be the sub-multigraph of G with edge multiset $\{(u,v):(u_A,v_B)\in M\}$. It is straightforward to verify that H satisfies the requirement of Lemma 2. This completes the proof of Lemma 2. Note that the proof is similar to a proof of a Petersen theorem on 2-factorization of a $2k$-regular graph. □

For each pair of distinct vertices x and y, let $H(x,y)$ be a spanning sub-multigraph of G such that $d_{H(x,y)}(x)=d_{H(x,y)}(y)=1$ and $d_{H(x,y)}(u)=2$ for all other vertices u. Since x and y are the only two vertices of H of odd degree, these two vertices are in the same connected component of $H(x,y)$. Let $H(x,x)$ be a sub-multigraph of G such that $d_{H(x,x)}(x)=0$ and $d_{H(x,x)}(u)=2$ for all other vertices u.

Let $x_1,x_2,\ldots,x_n$ be an arbitrary ordering of the vertices of G. Let H be the union of $H(x_1,x_2),H(x_2,x_3),\ldots,H(x_{n-1},x_n),H(x_n,x_1)$. (Here the union means add the edges together, thus $V(H)=V(G)$, and the multiplicity of an edge e in H is the sum of the multiplicities of e in the multigraphs $H(x_i,x_j)$.) Now H is a connected spanning sub-multigraph of G in which each vertex has the same degree $2n-2$. For any two vertices x and y, let $F(x,y)$ be the union of H and $H(x,y)$. If x and y are distinct, then $F=F(x,y)$ is a connected spanning sub-multigraph of G with $d_F(x)=d_F(y)=2n-1$ and $d_F(u)=2n$ for every other vertex u. If $x=y$, then $F=F(x,x)$ is a connected spanning sub-multigraph of G with $d_F(x)=2n-2$ and $d_F(u)=2n$ for every other vertex u. Thus F has an Eulerian trail P connecting x and y. Now P considered as a walk of G is a regular Hamiltonian walk in which each vertex of G occurs n times. (Note that when $x=y$, we count the initial occurrence and the terminal occurrence of x as different occurrences.) This completes the proof of Theorem 1. □

The proof of Theorem 1 also gives a polynomial-time algorithm that determines whether a connected graph G is pseudo-Hamiltonian-connected. To determine whether or not G is pseudo-Hamiltonian-connected, it amounts to determining whether or not for every non-empty proper subset X of $V(G)$, $|N(X)| > |X|$. By Hall's theorem, this is equivalent to determine whether or not for each pair of (not necessarily distinct) vertices x and y, the bipartite graph $Q_{x,y}$ as defined in the proof of Lemma 2 has a perfect matching, which can be determined in polynomial time.

3. Other characterizations

This section discusses other interesting properties of pseudo-Hamiltonian-connected graphs. The class of pseudo-Hamiltonian-connected graphs turns out to be related to the *weighted vertex packing problem*, i.e., finding a maximum weight independent set. Assign to each vertex v of G a weight $c(v) \geqslant 0$. The weighted vertex packing problem is to find an independent set I such that $c(I) = \Sigma_{v \in I} c(v)$ is maximum. This can be formulated as an integer linear programming problem (which is referred as (VP)):

$$\text{(VP)} \quad \begin{array}{ll} \text{maximize} & cx, \\ \text{subject to} & Ax \leqslant 1_m, \ x \text{ binary}, \end{array}$$

where $m = |E(G)|, n = |V(G)|, 1_m = (1, 1, \ldots, 1)$ is an m-vector of ones, and A is the $m \times n$ edge–vertex incidence matrix of G ($a_{ij} = 1$ if v_j is an end vertex of edge e_i, and $a_{ij} = 0$ otherwise). Relaxing the binary constraints to $x \geqslant 0$ gives the *linear programming relaxation* (VLP) of (VP).

How an optimum solution to (VLP) (which is polynomial) can be helpful in finding an optimum solution to (VP) (which is NP-complete) was discussed in [5,7,9]. Many interesting results concerning this problem were obtained. It was proved by Nemhauser and Trotter [5] that for an optimum solution $\boldsymbol{x}$ to (VLP), the integral components of $\boldsymbol{x}$ can be extended to an optimum solution to (VP). Thus, we are interested in finding an optimum solution $\boldsymbol{x}$ to (VLP) which has as many integral components as possible. The results obtained in [5,7,9] concerning such an approach are negative. The following result proved by Nemhauser and Trotter [5] relates this problem to pseudo-Hamiltonian-connected graphs:

Theorem 3 (Nemhauser and Trotter [5]). *The solution $\boldsymbol{x}$ given by $x_i = \frac{1}{2}$ for all $v_i \in V$ is the unique optimum solution to (VLP) if and only if $c(I) < c(N(I))$ for all independent sets I.*

In particular, if $c(v_i) = 1$ for all v_i and G is connected, then the solution given by $x_i = \frac{1}{2}$ for all i is a unique solution to (VLP) if and only if $|I| < |N(I)|$ for all independent sets I, i.e., G is pseudo-Hamiltonian-connected.

Berge [1] has given another characterization of those graphs G for which (VLP) has a unique solution. Define a graph G as being *regularizable* if it is possible to replace each edge e with $n_e \geqslant 1$ multiple edges so that the resulting multigraph is regular.

Theorem 4 (Berge [1]). *The solution* $\mathbf{x}$ *given by* $x_i = \frac{1}{2}$ *for all* $v_i \in V$ *is the unique optimum solution to* (*VLP*) (*with* $c(v_i)=1$ *for all* v_i) *if and only if G is regularizable and that each component of G is non-bipartite.*

Combining Theorem 1 with Theorems 3 and 4, we have the following characterization of pseudo-Hamiltonian-connected graphs:

Corollary 5. *A graph G is pseudo-Hamiltonian-connected if and only if G is connected, regularizable and non-bipartite.*

A 2-matching of a graph G is sub-multigraph of G in which each vertex has degree 2. A graph G is called 2-*bicritical* [9] if $G-v$ has a 2-matching for every vertex v of G. The following result was proved by Pulleyblank:

Theorem 6 (Pulleyblank [9]). *A graph G is* 2-*bicritical if and only if for every independent set I, we have* $|N(I)| > |I|$.

Thus, a connected graph G is 2-bicritical if and only if it is pseudo-Hamiltonian-connected.

Pulleyblank also proved in [9] that almost all graphs are 2-bicritical. Hence almost all connected graphs are pseudo-Hamiltonian-connected.

4. Pseudo-2 Hamiltonian-connected graphs

Given a pseudo-Hamiltonian-connected graph G, we denote by $p(G)$ the minimum number k for which $G[k]$ is Hamiltonian connected. It follows from the proof of Theorem 1 that if a graph G of order n is pseudo-Hamiltonian-connected, then $p(G) \leqslant n$. This is not a sharp bound.

In fact, a slight modification, as follows, of the last paragraph of the proof of Theorem 1 gives that $p(G) \leqslant \lceil n/2 \rceil$. For any two (not necessarily distinct) vertices x and y, the graph $H(x,y)$ has at most one isolated vertex and so has $r \leqslant \lceil n/2 \rceil$ components. As G is connected (otherwise, the vertex set X of a connected component is a non-empty proper subset such that $N(X)=X$), there exist $r-1$ edges $x_i y_i$ $(1 \leqslant i \leqslant r-1)$ connecting these components into a connected graph. Then the union of $H(x,y)$ and $H(x_i,y_i)+x_i y_i$ for $1 \leqslant i \leqslant r-1$ plays the same role as $F(x,y)$ in the proof. And the degree of any vertex in $V(G)-\{x,y\}$ is $2r \leqslant 2\lceil n/2 \rceil$. Therefore, $p(G) \leqslant \lceil n/2 \rceil$.

By definition, $p(G) \leqslant 1$ if and only if G is Hamiltonian-connected. It is difficult to determine whether a given graph is Hamiltonian-connected. In this section, we prove that any pseudo-Hamiltonian-connected graph with a Hamiltonian cycle is pseudo-2 Hamiltonian-connected.

Theorem 7. *Suppose G is pseudo-Hamiltonian-connected. If G has a Hamiltonian cycle then* $p(G) \leqslant 2$.

Proof. Suppose G is pseudo-Hamiltonian-connected and has a Hamiltonian cycle. By Theorem 1, for every proper subset X of $V(G)$, $|N_G(X)| > |X|$. For any (not necessarily distinct) vertices x and y of G, let $H(x,y)$ be the sub-multigraph of G defined as in the proof of Lemma 2. Namely, when $x \neq y$, each of x and y has degree 1 and every other vertex has degree 2 in $H(x,y)$; when $x = y$, then x has degree 0 and every other vertex has degree 2 in $H(x,x)$. Let C be a Hamiltonian cycle of G. Then $H = C \cup H(x,y)$ is a connected sub-multigraph of G such that when $x \neq y$ then $d_H(x) = d_H(y) = 3$ and $d_H(u) = 4$ for $u \neq x, y$; and when $x = y$ then $d_H(x) = 2$ and $d_H(u) = 4$ for $u \neq x$. Thus, H has a Eulerian trail connecting x and y, which is then an x–y regular Hamiltonian walk of G which traverses each vertex of G exactly twice. □

The toughness $t(G)$ of a graph G is defined as

$$t(G) = \min\{|S|/k(G-S) : S \text{ is a vertex cut set of } G\},$$

where $k(G-S)$ is the number of components of $G-S$. It was conjectured by Chvátal [3] that there is a real number r_0 such that any graph G with $t(G) \geqslant r_0$ is Hamiltonian. Chvátal also conjectured that letting $r_0 = 2$ would be enough. Note that non-Hamiltonian graphs exist with toughness at least t for each $t < 2$, see [4]. While the first conjecture remaining open, the second one is recently disproved by Bauer, Broersma and Veldman.

Theorem 8. *If $t(G) > 1$, then G is pseudo-Hamiltonian-connected.*

Proof. If G is not pseudo-Hamiltonian-connected, then there is a non-empty independent set I such that $|N(I)| \leqslant |I|$. Now $k(G - N(I)) \geqslant |I|$. Hence $t(G) \leqslant |I|/k(G - N(I)) \leqslant 1$. □

Note that even cycles are not pseudo-Hamiltonian-connected and they are 1 tough.

If Chvátal's conjectures are true, then $t(G) \geqslant r_0 \geqslant 2$ implies that G has a Hamiltonian cycle. Since $t(G) > 1$ implies that G is pseudo-Hamiltonian-connected by Theorem 8, it would follow from Theorem 7 that G is pseudo-2 Hamiltonian-connected. Therefore, the following conjecture is implied by Chvátal's conjecture:

Conjecture 1. *There is a real number $r_0 > 2$ such that for any graph G, if $t(G) \geqslant r_0$ then G is pseudo-2 Hamiltonian-connected.*

An even weaker conjecture is the following:

Conjecture 2. *There is a real number r_0 and an integer k such that for any graph G, if $t(G) \geqslant r_0$ then G is pseudo-k Hamiltonian-connected.*

On the other hand, Conjecture 1 is strictly weaker than Chvátal's conjecture, in the sense that there are graphs G which do not contain a Hamiltonian cycle, and yet $p(G) = 2$. Petersen graph is such an example. The required x–y walks are as illustrated

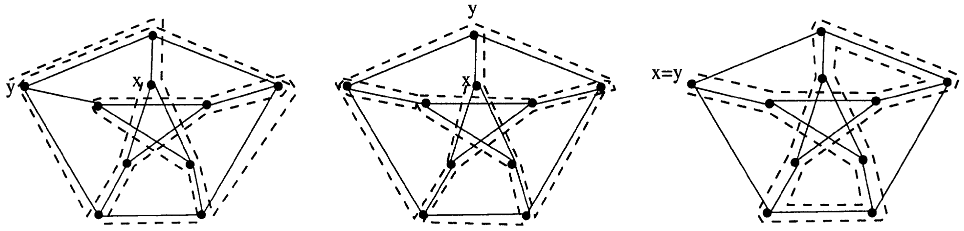

Fig. 1. Walks for Petersen graph.

in Fig. 1. Note that by symmetry, it suffices to consider three cases: $x=y$, x is adjacent to y, x is not adjacent to y.

Another conjecture, stronger than Conjecture 2, which seems to be not related to Chvátal's conjecture, is the following:

Conjecture 3. *For any real number $r>1$, there is an integer k_r such that for any graph G, if $t(G)\geqslant r$ then G is pseudo-k_r Hamiltonian-connected.*

We note that for any integer k, there are pseudo-Hamiltonian-connected graphs G such that $p(G)>k$. Indeed, take k disjoint copies of K_2, and then add a universal vertex u (i.e., add u and connect u to each of the $2k$ vertices of the k copies of K_2 by an edge). The resulting graph G is pseudo-Hamiltonian-connected and $p(G)=k+1$ if $k\geqslant 2$. We omit the verification which is quite straightforward.

5. Efficient beetle

Suppose G is pseudo-Hamiltonian-connected. Then for any initial assignment of integers to the vertices of G, the proof of Theorem 1 actually produces a route for the beetle so that by crawling along the route, all the integers will become the same. However, the route produced by the proof of Theorem 1 is usually not the shortest route to achieve the goal of changing all the integers to the same.

Given a graph G, an initial assignment f of integers to the vertices of G, and a vertex s of G which is the initial position of the beetle. We are interested in finding a shortest route for the beetle to crawl so that after finishing this route the integers assigned to the vertices will be changed to all being the same. Or equivalently, we need to find the minimum integer k, such that there is a walk W of G, starting from s, which arrives at each vertex v $k-f(v)$ times. Unfortunately, this problem is NP-complete, as one might have expected.

We now define the *efficient beetle problem* as the following minimization problem (G,f,s): Given a graph G, an assignment of each vertex v an integer $f(v)$, and a initial vertex s, the efficient beetle problem (G,f,s) is to find a minimum integer k such that there is a walk W of G, starting from s which arrives at each vertex v exactly $k-f(v)$ times.

Theorem 9. *The efficient beetle problem* (G, f, s) *is* NP-*hard even in the special case if* G *is a bipartite cubic planar graph,* $f(s) = 1$, *and* $f(v) = 0$ *for the other vertices* v.

Proof. The Hamiltonian path problem is NP-hard for such graphs, see [8]. Therefore the problem to find a Hamoltonian path with prescribed start vertex in such a graph is also NP-hard (because each vertex can be considered as a start vertex, one at a time). □

There are methods that produce better solutions to the beetle problem (i.e., find shorter route) than that given by the proof of Theorem 1. For example, one may first find a sub-multigraph Q of G such that there is an integer k and for every vertex v, $d_Q(v) = 2(k - f(v))$. If Q is connected, then of course, the Eulerian cycle gives the required walk. If Q is not connected, then we may add the sub-multigraph H as constructed in the proof of Theorem 1 to Q to obtain a connected sub-multigraph, which then produces the required walk. The sub-multigraph Q can be constructed similarly as in the proof of Lemma 2. There are other modification to the method of constructing the walk W, however, the ideas are similar, and the solutions seem to be far from being optimal.

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