ables as well as actuator inputs and outputs directly undergo the (usually disastrous) effects of the generated bang-bang type of discontinuities. This fact makes possible the application of sliding mode control techniques to areas where they were not traditionally feasible, such as, chemical process control, bilogical systems control, and the regulation of mechanical and electromechanical systems (see also [7]).

In this article a nonlinear DC-motor example, dealing with smooth controlled transitions of nominal angular velocities to new constant operating values was presented along with encouraging simulation results. As topics for further research, the dynamical variable structure feedback controller here proposed could be implemented in an actual DC-motor using nonlinear analog electronics. Also, a robust controller that effectively handles the uncertainty of system parameters could be developed. Profitable connections could also be established with the work of Charlet *et al.* [25].

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Stabilization, Parameterization, and Decoupling Controller Design for Linear Multivariable Systems

Ching-An Lin and Tung-Fu Hsieh

Abstract—**We study linear multivariable systems under the unity-feedback configuration. For nonsquare plants with no coincidences of unstable poles and zeros, we prove a simplified condition for closed-loop stability. The simplification leads to a simple description of the set of all achievable I / O maps and a simple parameterization of all controllers achieving the same I / O map. The results are used to describe the set of all achievable decoupled I / O maps and prove a necessary and sufficient condition for the existence of *stable* decoupling controllers.**

I. INTRODUCTION

We study linear time-invariant MIMO plants under the unity-feedback configuration. For plants with no coincidences of unstable poles and zeros, we prove a simplified condition for

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closed-loop stability. The simplification leads to a simple characterization of all achievable I/O maps and a simple parameterization of all stabilizing controllers which achieve the same I/O map. For plants with more inputs than outputs, there are infinitely many controllers achieving an I/O map or a sensitivity map. We give an example to show how the parameterization can be used to quantitatively study the benefits that can be achieved by this extra degree of freedom in controller selection.

The results are used to study decoupling controller design for nonsquare plants. We give a simple description of the set of all achievable decoupled I/O maps and prove a necessary and sufficient condition for the existence of *stable* decoupling controllers. We note that descriptions of achievable decoupled I/O maps and decoupling controllers, based on coprime factorizations, for general nonsquare plants can be found in [9] and [6]. The descriptions that we obtain is simpler in that they only involve scalar polynomials satisfying certain interpolation conditions. Similar descriptions for square plants can be found in [7].

This note is organized as follows. Section II describes the feedback system and basic properties. In Section III we prove the simplified stability condition and propose the parameterization results. In Section IV we characterize the set of all achievable decoupled I/O maps, and prove a necessary and sufficient condition for the existence of a stable decoupling controller. Section V is a brief conclusion.

A. Notations and Definitions

The expression $a := b$ means a denotes b. $\mathbb{R} :=$ the field of real numbers; $\mathbb{C} :=$ the field of complex numbers. $\mathbb{C}_- := \{s \in \mathbb{C} | Re(s) < 0\}$; $\mathbb{C}_+ := \{s \in \mathbb{C} | Re(s) \geq 0\}$; $\mathbb{C}_{+e} := \mathbb{C}_+ \cup \{\infty\}$. $\mathbb{R}[s] :=$ the set of polynomials in s with real coefficients; $\mathbb{R}(s) :=$ the set of rational functions in s with real coefficients; $\mathbb{R}_p(s)(\mathbb{R}_{po}(s)$, respectively$) :=$ the set of proper (strictly proper, respectively) rational functions in s with real coefficients; $S := \{h \in \mathbb{R}_p(s) | h$ has no pole in $\mathbb{C}_+\}$. A rational matrix $H \in \mathbb{R}(s)^{m \times n}$ is said to be stable if and only if $H \in S^{m \times n}$. We say that $H \in S^{n \times n}$ is S-unimodular or a unit in $S^{n \times n}$ if and only if $H^{-1} \in S^{n \times n}$. For $p, q \in \mathbb{R}[s]$, $\mathscr{Z}[p] := \{s \in \mathbb{C} | p(s) = 0\}$, $\deg(p) :=$ degree of p, and $q|p$ means that q divides p, or equivalently, $p = qr$ for some $r \in \mathbb{R}[s]$.

II. Preliminary

Consider the unity-feedback system $S(P, C)$ shown in Fig. 1, where $P \in \mathbb{R}_{po}(s)^{n_0 \times n_i}$ is the plant, $C \in \mathbb{R}_p(s)^{n_i \times n_o}$ the controller, (u_1, u_2) the input, and (y_1, y_2) the output. It is assumed that the dynamical systems described by P and C contain no unstable hidden modes. Since P is strictly proper, $S(P, C)$ is well posed [2, p. 222]. Let $u := [u_1^T u_2^T]^T$ and $y := [y_1^T y_2^T]^T$. The closed-loop transfer matrix $H_{yu} \in \mathbb{R}_p(s)^{(n_i+n_o)\times(n_i+n_o)}$ and is given by

$$H_{yu} = \begin{bmatrix} H_{y_1u_1} & H_{y_1u_2} \\ H_{y_2u_1} & H_{y_2u_2} \end{bmatrix} = \begin{bmatrix} C(I+PC)^{-1} & -CP(I+CP)^{-1} \\ PC(I+PC)^{-1} & P(I+CP)^{-1} \end{bmatrix}. \quad (2.1)$$

Throughout this note, we refer to $H_{y_2u_1}$ as the I/O map of the system $S(P, C)$.

Definition 2.1: The system $S(P, C)$ is stable if H_{yu} is stable, that is, $H_{yu} \in S^{(n_i+n_o)\times(n_i+n_o)}$. We say that C stabilizes P or C is a stabilizing controller for P if $S(P, C)$ is stable. □

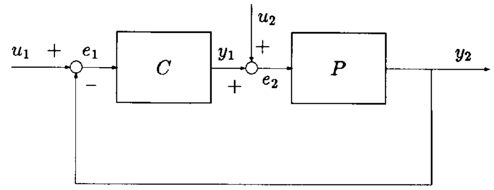

Fig. 1. The unity-feedback system $S(P, C)$.

Definition 2.2: The system $S(P, C)$ is *decoupled* if C stabilizes P and the resulting I/O map $H_{y_2u_1}$ is nonsingular and diagonal. We say that C is a decoupling controller for P if $S(P, C)$ is decoupled. □

Definition 2.3: A stable rational matrix $M \in S^{n_o \times n_o}$ is an *achievable* I/O *map* if there exists a controller C such that $S(P,C)$ is stable and $H_{y_2u_1} = M$; An achievable I/O map M is *strongly achievable* if it can be achieved by a stable controller. □

Since P is strictly proper, there is a *one-to-one correspondence* between the controller C and the transfer matrix $H_{y_1u_1} =: Q$. More precisely, $Q = C(I + PC)^{-1} \in \mathbb{R}_p(s)^{n_i \times n_o}$ if and only if $C = Q(I - PQ)^{-1} \in \mathbb{R}_p(s)^{n_i \times n_o}$ [2, ch. 8]. In terms of Q, the closed-loop transfer matrix H_{yu} in (2.1) becomes

$$H_{yu} = \begin{bmatrix} Q & -QP \\ PQ & (I-PQ)P \end{bmatrix}. \quad (2.2)$$

Thus we have the following result.

Proposition 2.4 [2, ch. 8] Let $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$.

i) If $C \in \mathbb{R}_p(s)^{n_i \times n_o}$ stabilizes P, then, with $Q := C(I + PC)^{-1}$, the transfer matrix H_{yu} in (2.2) is stable;
ii) Conversely, if H_{yu} in (2.2) is stable for some $Q \in \mathbb{R}_p(s)^{n_i \times n_o}$, then $C := Q(I - PQ)^{-1}$ stabilizes P. □

Let (D, N) be a left coprime factorization (l.c.f.[1]) over S of P, that is, $D \in S^{n_o \times n_o}$ and $N \in S^{n_o \times n_i}$ are left coprime and $D^{-1}N = P$. We shall need the following assumptions:

P1) The plant P has full normal rank and $n_o \leq n_i$, and
P2) $D(s)$ and $N(s)$ do not lose rank at the same point in $\mathbb{C}_+$, that is, for all $s_0 \in \mathbb{C}_+$, rank $D(s_0) < n_o$ implies rank $N(s_0) = n_o$ and rank $N(s_0) < n_o$ implies rank $D(s_0) = n_o$.

Comment: i) Assumption P1) is necessary for decoupling [5, p. 64], and assumption P2) means that the $\mathbb{C}_+$-poles and $\mathbb{C}_+$-zeros of P do not coincide. ii) Such assumptions were originally used in [3, theorem 3.2, 4.2] to establish simplified stability conditions for *simple unstable plants*, and also appeared in [9] as sufficient conditions for the existence of decoupling controllers for P.

We shall need the following lemma.

Lemma 2.5 [11, lemma 7.5.5]: Suppose that $H \in S^{m \times n_o}$, $P \in \mathbb{R}_p(s)^{n_o \times n_i}$, and (D, N) is an l.c.f. of P. Under these conditions, $HP \in S^{m \times n_i}$ if and only if $HD^{-1} \in S^{m \times n_o}$. □

III. Stability Conditions and Parameterizations

A general stability condition for $S(P, C)$ is given in [1], which is a set of conditions on $Q := H_{y_1u_1}$ to ensure the stability of the four submatrices in (2.2). Under the assumptions P1) and P2), we prove a simplified stability condition (Theorem 3.1). Based on such a simplified stability condition, we describe the set of all

[1] Throughout, l.c.f. denotes left coprime factorization over S.

achievable I/O maps and the set of all strongly achievable I/O maps, and parameterize the set of all controllers which achieve the same I/O map. An example is given to show how the controller parameterization can be used to improve controller designs involving linear plants which have more inputs than outputs.

A. Stability Conditions

Let (D, N) be an l.c.f. of P and $U \in S^{n_i \times n_i}$ be any S-unimodular matrix such that

$$NU = [\bar{N} \quad 0] \text{ with } \bar{N} \in S^{n_o \times n_o}. \tag{3.1}$$

Such a U exists and can be obtained by performing elementary column operations on N [8, ch. 3]. We now state the simplified stability condition.

Theorem 3.1: Let $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$. Suppose that P1) and P2) are satisfied and let U be as defined in (3.1). Under these conditions, we have the following:

i) If $C \in \mathbb{R}_p(s)^{n_i \times n_o}$ stabilizes P, then $Q := C(I + PC)^{-1}$ satisfies that

Q1) $Q = U\begin{bmatrix} Q_1 \\ Q_2 D \end{bmatrix}$ for some $Q_1 \in S^{n_o \times n_o}$ and $Q_2 \in S^{(n_i - n_o) \times n_0}$, and

Q2) $(I - PQ)D^{-1} \in S^{n_o \times n_o}$.

ii) If $Q \in \mathbb{R}_p(s)^{n_i \times n_o}$ satisfies Q1) and Q2), then $C := Q(I - PQ)^{-1}$ stabilizes P.

Comment: i) Note that Q2) is exactly the same as the condition iv') in [1]. ii) It follows from Theorem 3.1 and the one-to-one correspondence between C and Q that the set of all stabilizing controllers for P is given by

$$C = \{Q(I - PQ)^{-1} | Q \text{ satisfies Q1) and Q2)}\}. \tag{3.2}$$

iii) Since, by Lemma 2.5, Q2) holds if and only if $H_{y_2 u_2} = (I - PQ)P \in S^{n_o \times n_i}$, it is easy to see that, for $n_o = n_i$ case, $S(P, C)$ is stable if and only if $H_{y_1 u_1} \in S^{n_i \times n_o}$ and $H_{y_2 u_2} \in S^{n_o \times n_i}$. iv) Note that for every Q which satisfies Q1) and Q2), the resulting I/O map $H_{y_2 u_1} = PQ = D^{-1}\bar{N}Q_1$ is independent of Q_2.

Proof: By Proposition 2.4, it suffices to show that $H_{\underline{yu}}$ in (2.2) is stable if and only if Q1) and Q2) hold. Note that $\bar{N}$ is nonsingular since P, and hence N, has full normal rank.

($\Leftarrow$) Suppose that Q1) and Q2) hold, then $Q \in S^{n_i \times n_o}$ and $(I - PQ)P \in S^{n_o \times n_i}$ by Lemma 2.5. It remains to show that a) $PQ \in S^{n_o \times n_o}$ and b) $QP \in S^{n_i \times n_i}$.

To establish a), let

$$G := (I - PQ)P \tag{3.3}$$

then $G \in S^{n_o \times n_i}$ and

$$G = (I - PQ)D^{-1}[\bar{N} \quad 0]U^{-1}. \tag{3.4}$$

Postmultiply (3.4) by $U\begin{bmatrix} \bar{N}^{-1} \\ 0 \end{bmatrix}D$ to get

$$(I - PQ) = GU\begin{bmatrix} \bar{N}^{-1} \\ 0 \end{bmatrix}D. \tag{3.5}$$

Since $P = D^{-1}N$, $N \in S^{n_o \times n_i}$, and $Q \in S^{n_i \times n_o}$, $I - PQ$ is analytic in $\mathbb{C}_+ - \mathscr{D}$, where

$$\mathscr{D} := \{s \in \mathbb{C}_+ | \det D(s) = 0\} = \{s \in \mathbb{C}_+ | \text{rank } D(s) < n_o\}. \tag{3.6}$$

From (3.5), it follows that $I - PQ$ is analytic in $\mathbb{C}_+ - \mathscr{N}$, where

$$\mathscr{N} := \{s \in \mathbb{C}_+ | \det \bar{N}(s) = 0\}. \tag{3.7}$$

Thus $I - PQ$ is analytic in $(\mathbb{C}_+ - \mathscr{D}) \cup (\mathbb{C}_+ - \mathscr{N}) = \mathbb{C}_+ - (\mathscr{D} \cap \mathscr{N})$. It is easy to see from (3.1) that $\mathscr{N} = \{s \in \mathbb{C}_+ | \text{rank } N(s) < n_o\}$ since U is S-unimodular. Since $\mathscr{D} \cap \mathscr{N} = \varnothing$ by P2), it follows that $I - PQ$ is analytic in $\mathbb{C}_+$ and hence $PQ \in S^{n_o \times n_o}$.

To establish b), note that since $QP = U\begin{bmatrix} Q_1 P \\ Q_2 N \end{bmatrix}$, we only have to show that $Q_1 P \in S^{n_o \times n_i}$. By (3.3) we have

$$G = P(I - QP) = D^{-1}[\bar{N} \quad 0]U^{-1}\left(I - U\begin{bmatrix} Q_1 P \\ Q_2 N \end{bmatrix}\right). \tag{3.8}$$

Premultiply (3.8) ny $\bar{N}^{-1}D$ to get

$$\bar{N}^{-1}DG = [I \quad 0]U^{-1} - Q_1 P. \tag{3.9}$$

Similar arguments as those following (3.5) show that $Q_1 P \in S^{n_o \times n_i}$.

($\Rightarrow$) Since $(I - PQ)P \in S^{n_o \times n_i}$, Q2) follows from Lemma 2.5. Since U is S-unimodular and $QP \in S^{n_i \times n_i}$, we have $U^{-1}QD^{-1} \in S^{n_i \times n_o}$ by Lemma 2.5. Let $\begin{bmatrix} \bar{Q}_1 \\ Q_2 \end{bmatrix} := U^{-1}QD^{-1}$, where $\bar{Q}_1 \in S^{n_o \times n_o}$ and $Q_2 \in S^{(n_i - n_o) \times n_o}$. Then Q1) follows with $Q_1 := \bar{Q}_1 D \in S^{n_o \times n_o}$. □

Under the condition that $S(P, C)$ is stable, the following proposition is a necessary and sufficient condition for the stability of the controller C.

Proposition 3.2: Suppose that $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$ and that $C \in \mathbb{R}_p(s)^{n_i \times n_o}$ stabilizes P. Let $Q := C(I + PC)^{-1}$ and (D, N) be an l.c.f. of P. Under these conditions, C is stable if and only if QS) $D(I - PQ)^{-1} \in S^{n_o \times n_o}$.

Comment: i) The proposition is useful in the characterization of all strongly achievable I/O maps (Corollary 3.4), which will be used to establish a necessary and sufficient condition for the existence of a *stable* decoupling controller for P (Theorem 4.2). ii) By Theorem 3.1 and Proposition 3.2, the set of all stable controllers which stabilize P is given by

$$\mathbf{C}_s = \{Q(I - PQ)^{-1} | Q \text{ satisfies Q1) and } (I - PQ)D^{-1} \text{ is a unit}\}. \tag{3.10}$$

iii) It is shown in [12, p. 119] that the stabilizing controller C is stable if and only if $H_{y_2 u_2}$ contains exactly the same $\mathbb{C}_{+e}$-zeros as P does. Since $H_{y_2 u_2} = (I - PQ)D^{-1}N$, and QS) holds if and only if $(I - PQ)D^{-1}$ has no $\mathbb{C}_{+e}$-zeros, the assertion also follows from Proposition 3.2.

Proof:

($\Leftarrow$) Since C stabilizes P and $Q = C(I + PC)^{-1}$, we have $QP \in S^{n_i \times n_i}$ by Proposition 2.4 and thus $QD^{-1} \in S^{n_i \times n_o}$ by Lemma 2.5. Since $C = Q(I - PQ)^{-1}$, it follows from QS) that

$$C = (QD^{-1})(D(I - PQ)^{-1}) \in S^{n_i \times n_o}. \tag{3.11}$$

($\Rightarrow$) Since $D \in S^{n_o \times n_o}$, $N \in S^{n_o \times n_i}$, and $C \in S^{n_i \times n_o}$, it follows that

$$\begin{aligned} D(I - PQ)^{-1} &= D(I - PC(I + PC)^{-1})^{-1} = D(I + PC) \\ &= D + NC \in S^{n_o \times n_o}. \end{aligned} \tag{3.12}$$

B. Parameterizations

In this subsection, we describe the set of all achievable I/O maps and all strongly achievable I/O maps, and then parameterize the set of all stabilizing controllers which achieve the same I/O map.

Proposition 3.3: Let $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$. Suppose that P1) and P2) are satisfied and $M \in S^{n_o \times n_o}$. Let $\bar{N}$ be as defined in (3.1).

Under these conditions, M is an achievable I/O map if and only if

M1) $\bar{N}^{-1}DM \in S^{n_o \times n_o}$, and

M2) $(I - M)D^{-1} \in S^{n_o \times n_o}$.

Comment: i) The condition M2) also appeared in [9, remark 2]. ii) The set of all achievable I/O maps is then given by

$$\mathbf{M} = \{M \in S^{n_o \times n_o} | \bar{N}^{-1}DM \in S^{n_o \times n_o} \text{ and } (I - M)D^{-1} \in S^{n_o \times n_o}\}. \quad (3.13)$$

The description in (3.13) is useful in the parameterization of all controllers which achieve the same I/O map (Theorem 3.5).

Proof:

($\Leftarrow$) Let $Q_2 \in S^{(n_i - n_o) \times n_o}$ and define

$$Q = U\begin{bmatrix} \bar{N}^{-1}DM \\ Q_2 D \end{bmatrix}. \quad (3.14)$$

By M1) and (3.14), the condition Q1) in Theorem 3.1 is satisfied. Also by (3.14), we have $PQ = M$ and thus $(I - PQ)D^{-1} = (I - M)D^{-1} \in S^{n_o \times n_o}$ by M2). It then follows from Theorem 3.1 that $C := Q(I - PQ)^{-1}$ stabilizes P and $H_{y_2u_1} = PQ = M$.

($\Rightarrow$) Suppose that C stabilizes P and $H_{y_2u_1} = M$. Then, with $Q := C(I + PC)^{-1}$, the four submatrices in (2.2) are all stable by Proposition 2.4 and $PQ = M$. It follows from (31) that

$$M = PQ = D^{-1}[\bar{N} \quad 0]U^{-1}Q. \quad (3.15)$$

Premultiply (3.15) by $\bar{N}^{-1}D$ to get $\bar{N}^{-1}DM = [I \ 0]U^{-1}Q \in S^{n_o \times n_o}$, and thus M1) is satisfied. Since $(I - M)P = (I - PQ)P \in S^{n_o \times n_i}$, it follows from Lemma 2.5 that M2) is satisfied. □

Now suppose that M is achieved by C and let $Q := C(I + PC)^{-1}$. Then $PQ = M$ and it follows from Proposition 3.2 that $C \in S^{n_i \times n_o}$ if and only if

MS) $D(I - M)^{-1} \in S^{n_o \times n_o}$.

Thus, we have the following characterization.

Corollary 3.4: Let $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$. Suppose that P1) and P2) are satisfied and let $\bar{N}$ be as defined in (3.1). Then the set of all strongly achievable I/O maps is given by

$$\mathbf{M}_s = \{M \in S^{n_o \times n_o} | \bar{N}^{-1}DM \in S^{n_o \times n_o} \text{ and } (I - M)D^{-1} \text{ is a unit}\}. \quad (3.16)$$

□

We now parameterize the set of all stabilizing controllers which achieve the same I/O map.

Theorem 3.5: Let $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$. Suppose that P1) and P2) are satisfied, and let U and $\bar{N}$ be as defined in (3.1). Under these conditions, for each achievable I/O map M, the set of all stabilizing controllers which achieve M is given by

$$\mathbf{C}_M = \left\{ U\begin{bmatrix} \bar{N}^{-1}DM \\ Q_2 D \end{bmatrix}(I - M)^{-1} \middle| Q_2 \in S^{(n_i - n_o) \times n_o} \right\}. \quad (3.17)$$

Proof:

a) Suppose that $C \in \mathbf{C}_M$, that is, $C = U\begin{bmatrix} \bar{N}^{-1}DM \\ Q_2 D \end{bmatrix}(I - M)^{-1}$ for some $Q_2 \in S^{(n_i - n_o) \times n_o}$. Let $Q := U\begin{bmatrix} \bar{N}^{-1}DM \\ Q_2 D \end{bmatrix}$, then $PQ = M$ and $C = Q(I - PQ)^{-1}$. Since M is achievable, we have $\bar{N}^{-1}DM \in S^{n_o \times n_o}$ and $(I - PQ)D^{-1} = (I - M)D^{-1} \in S^{n_o \times n_o}$ by Proposition 3.3. It follows from Theorem 3.1 that C stabilizes P. Also, $H_{y_2u_1} = PC(I + PC)^{-1} = PQ = M$.

b) Suppose that C achieves M and let $Q := C(I + PC)^{-1}$. Then, by Theorem 3.1, $Q = U\begin{bmatrix} Q_1 \\ Q_2 D \end{bmatrix}$ for some $Q_1 \in S^{n_o \times n_o}$ and $Q_2 \in S^{(n_i - n_o) \times n_o}$, and $(I - PQ)D^{-1} \in S^{n_o \times n_o}$. Since $M = H_{y_2u_1} = PQ = D^{-1}\bar{N}Q_1$, we have $Q_1 = \bar{N}^{-1}DM$. Hence, $C = Q(I - PQ)^{-1} = U\begin{bmatrix} \bar{N}^{-1}DM \\ Q_2 D \end{bmatrix}(I - M)^{-1}$ and $C \in \mathbf{C}_M$. □

If the plant has more inputs than outputs, then there is an extra degree of freedom in controller selection in addition to achieving a desirable I/O map or sensitivity map. The parameterization in (3.17) makes it easier to quantitatively answer the question: what can be achieved by such an extra degree of freedom? For example, suppose that the design objective is to achieve a prespecified (achievable) decoupled I/O map M while maintaining the optimal robustness with respect to additive plant uncertainty. Based on (3.17), the problem can be formulated as follows [4, p. 19]: find $Q_2 \in S^{(n_i - n_o) \times n_o}$ such that the H_∞-norm of $Q := H_{y_1u_1} = U\begin{bmatrix} \bar{N}^{-1}DM \\ Q_2 D \end{bmatrix}$ is minimized. Write $U = [U_1 \ U_2]$, where $U_1 \in S^{n_i \times n_o}$ and $U_2 \in S^{n_i \times (n_i - n_o)}$, then the problem becomes

$$\min_{Q_2} \|Q\|_\infty = \min_{Q_2} \|U_1 \bar{N}^{-1}DM + U_2 Q_2 D\|_\infty. \quad (3.18)$$

Note that (3.18) is a standard *model-matching problem*. Since U is S-unimodular, we have rank $U_2(j\omega) = n_i - n_o$ for all $0 \le \omega \le \infty$, and it follows that if P has no poles on the $j\omega$ axis, then (3.18) has an optimal solution [4, p. 62]. Suppose that $\hat{Q}_2$ solves (3.18), then $C := U\begin{bmatrix} \bar{N}^{-1}DM \\ \hat{Q}_2 D \end{bmatrix}(I - M)^{-1}$ solves the design problem.

IV. Decoupling Controller Design

In this section, we first describe the set of all achievable decoupled I/O maps. Then we establish a necessary and sufficient condition for the existence of a stable decoupling controller for P, and characterize the set of all strongly achievable decoupled I/O maps.

A. Achievable Decoupled I/O Maps

Consider the system $S(P, C)$ where P satisfies P1) and P2). Every achievable diagonal I/O map has the form

$$H_{y_2u_1} = \text{diag}\left[\frac{\tilde{\beta}_i}{\alpha_i}\right] =: M \quad (4.1)$$

where $\alpha_i, \tilde{\beta}_i \in \mathbb{R}[s]$ with α_i monic and Hurwitz, for $i = 1, \cdots, n_o$. In the following we derive necessary and sufficient conditions on α_i and $\tilde{\beta}_i$ so that M1) and M2) in Proposition 3.3 are satisfied and hence the M in (4.1) is achievable.

Write $\bar{N}^{-1}D$ as

$$\bar{N}^{-1}D = \left[\frac{N_{ij}}{D_{ij+}D_{ij-}}\right] \quad (4.2)$$

where $N_{ij}, D_{ij-}, D_{ij+} \in \mathbb{R}[s]$ are mutually coprime, $\mathscr{Z}[D_{ij-}] \subset \mathbb{C}_-$, and D_{ij+} is monic with $\mathscr{Z}[D_{ij+}] \subset \mathbb{C}_+$. It follows that M1) holds if and only if

$$\bar{N}^{-1}DM = \left[\frac{N_{ij}\tilde{\beta}_i}{D_{ij+}D_{ij-}\alpha_j}\right] \in S^{n_o \times n_o}. \quad (4.3)$$

To check M2), we note that, by Lemma 2.5, M2) holds if and only if $(I - M)P \in S^{n_o \times n_i}$. Write P as

$$P = \left[\frac{Z_{ij}}{P_{ij+}P_{ij-}}\right] \quad (4.4)$$

where $Z_{ij}, P_{ij-}, P_{ij+} \in \mathbb{R}[s]$ are mutually coprime, $\mathscr{Z}[P_{ij-}] \subset \mathbb{C}_-$, and P_{ij+} is monic with $\mathscr{Z}[P_{ij+}] \subset \mathbb{C}_+$. Thus, M2) holds if and only if

$$(I - M)P = \left[\frac{(\alpha_i - \tilde{\beta}_i)Z_{ij}}{\alpha_i P_{ij+} P_{ij-}} \right] \in S^{n_o \times n_i}. \tag{4.5}$$

Let D_{i+} be the monic least common multiple (l.c.m.) of $\{D_{ji+}\}_{j=1}^{n_o}$, and P_{i+} the monic l.c.m. of $\{P_{ij+}\}_{j=1}^{n_o}$, it then follows from [7] that (4.3) and (4.5) hold if and only if, for $i = 1, \cdots, n_o$

Md1) α_i is Hurwitz and $\tilde{\beta}_i = D_{i+}\beta_i$ for some $\beta_i \in \mathbb{R}[s]$,
Md2) $\deg(\alpha_i) - \deg(\beta_i) \geq \max_j [\deg(N_{ji}) - \deg(D_{ji-}) - \deg(D_{ji+})] + \deg(D_{i+})$, and
Md3) $P_{i+} | (\alpha_i - D_{i+}\beta_i)$.

We thus have the following theorem.

Theorem 4.1: Let $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$. Suppose that P1) and P2) are satisfied. Then the set of all achievable decoupled I/O maps is given by

$$\mathbf{M}_d = \left\{ \begin{bmatrix} \frac{\tilde{\beta}_1}{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \frac{\tilde{\beta}_{n_o}}{\alpha_{n_o}} \end{bmatrix} \middle| \begin{matrix} \alpha_i \text{ and } \tilde{\beta}_i \text{ satisfy} \\ \text{Md1), Md2), and Md3)} \end{matrix} \right\}. \tag{4.6}$$

Comment: Since all decoupling controllers which achieve a given decoupled I/O map are parameterized in (3.17), the construction of a decoupling controller is straightforward after an achievable decoupled I/O map is chosen. A design algorithm to construct achievable decoupled I/O maps can be found in [7].

B. Decoupling by Stable Controllers

Let P_{i+}, D_{i+}, N_{ji}, D_{ji-}, and D_{ji+} be as defined in Section IV-A. For $i = 1, \cdots, n_o$, define

$$s_i := \deg(P_{i+}) \tag{4.7}$$

$$t_i := \deg(D_{i+}), \text{ and} \tag{4.8}$$

$$w_i := \max_j [\deg(N_{ji}) - \deg(D_{ji-}) - \deg(D_{ji+})]. \tag{4.9}$$

Let $0 < \gamma \in \mathbb{R}$ and define the diagonal matrix

$$P_{\gamma+} := \text{diag}\left[\frac{P_{i+}}{(s+\gamma)^{s_i}} \right] \in S^{n_o \times n_o}. \tag{4.10}$$

The following theorem gives a necessary and sufficient condition for the existence of a stable decoupling controller for P.

Theorem 4.2: Let $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$. Suppose that P1) and P2) are satisfied, and let s_i, t_i, w_i, and $P_{\gamma+}$ be as defined in (4.7)–(4.10). Under these conditions, there exists a *stable* decoupling controller for P if and only if

Sd1) $\text{rank}[P_{\gamma+}(s) \;\; P_{\gamma+}(s)P(s)] = n_o$ for all $\mathbb{C}_+$-poles s of P, and
Sd2) $(D_{i+}/P_{i+}(s+\gamma)^{t_i - s_i + w_i})$ has the parity interlacing property [12, p. 54], for $i = 1, \cdots, n_o$.

The following lemmas will be used in the proof of Theorem 4.2.

Lemma 4.3: Let $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$ and (D, N) be an l.c.f. of P. Let $P_{\gamma+}$ be as defined in (4.10). Under these conditions, Sd1) in Theorem 4.2 holds if and only if $P_{\gamma+}D^{-1}$ is a unit in $S^{n_o \times n_o}$.

Proof: The assertion follows immediately from the rank test [5, lemma 2.6.1] and [5, lemma 2.3.4]. □

Lemma 4.4: Let $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$ and suppose that P1) and P2) are satisfied. Let $\mathbf{M}_d$ be as defined in (4.6). Under these conditions, Sd2) in Theorem 4.2 holds if and only if there exist α_i and $\beta_i \in \mathbb{R}[s]$, for $i = 1, \cdots, n_o$, such that $M := \text{diag}[(D_{i+}\beta_i/\alpha_i)] \in \mathbf{M}_d$ and

$$\alpha_i = D_{i+}\beta_i + P_{i+}h_i \text{ for some Hurwitz } h_i \in \mathbb{R}[s]. \tag{4.11}$$

Proof:

($\Leftarrow$) Multiply (4.11) by $(1/(s+\gamma)^{s_i}h_i)$ to get

$$\frac{\alpha_i}{(s+\gamma)^{s_i}h_i} = \frac{D_{i+}}{(s+\gamma)^{t_i+w_i}} \frac{(s+\gamma)^{t_i - s_i + w_i}\beta_i}{h_i} + \frac{P_{i+}}{(s+\gamma)^{s_i}}. \tag{4.12}$$

Since $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$, we have $w_i \geq 1$ for $i = 1, \cdots, n_o$. Then by (4.11) and Md2), $\deg(P_{i+}h_i) = \deg(\alpha_i - D_{i+}\beta_i) = \deg(\alpha_i)$ and hence each term $((D_{i+}/(s+\gamma)^{t_i+w_i})$, etc.) in (4.12) is an element in S and $(\alpha_i/(s+\gamma)^{s_i}h_i)$ is a unit. It follows from (4.12) that $(s+\gamma)^{t_i - s_i + w_i}\beta_i/h_i \in S$ is a stabilizing controller for $g_i := (D_{i+}/(s+\gamma)^{t_i+w_i})/(P_{i+}/(s+\gamma)^{s_i})$, and thus g_i has the p.i.p. [11, corollary 5.3.2]

($\Rightarrow$) Suppose that $g_i := (D_{i+}/P_{i+}(s+\gamma)^{t_i - s_i + w_i})$ has the p.i.p. Since $((D_{i+}/(s+\gamma)^{t_i+w_i})/(P_{i+}/(s+\gamma)^{s_i}))$ is a coprime factorization of g_i, there exists an $(x_i/y_i) \in S$ with $y_i \in \mathbb{R}[s]$ and $x_i \in \mathbb{R}[s]$ coprime, and a unit (v_i/u_i) in S with $u_i \in \mathbb{R}[s]$ and $v_i \in \mathbb{R}[s]$ coprime, such that [11, lemma 3.1.4]

$$\frac{D_{i+}}{(s+\gamma)^{t_i+w_i}} \frac{x_i}{y_i} + \frac{P_{i+}}{(s+\gamma)^{s_i}} = \frac{v_i}{u_i}. \tag{4.13}$$

Multiply (4.13) by $(s+\gamma)^{s_i+t_i+w_i}y_iu_i$ to get

$$D_{i+}\underbrace{[(s+\gamma)^{s_i}x_iu_i]}_{=:\,\beta_i} + P_{i+}\underbrace{[(s+\gamma)^{t_i+w_i}y_iu_i]}_{=:\,h_i} = \underbrace{(s+\gamma)^{s_i+t_i+w_i}y_iv_i}_{=:\,\alpha_i}. \tag{4.14}$$

It can then be easily checked that $M := \text{diag}[D_{i+}\beta_i/\alpha_i] \in \mathbf{M}_d$ and h_i is Hurwitz. □

Proof of Theorem 4.2:

($\Leftarrow$) By Sd2) and Lemma 4.4, there exist α_i and $\beta_i \in \mathbb{R}[s]$ such that $M := \text{diag}[(D_{i+}\beta_i/\alpha_i)] \in \mathbf{M}_d$ and (4.11) is satisfied. Then $\alpha_i - D_{i+}\beta_i = P_{i+}h_i$ and thus

$$(I - M)D^{-1} = \text{diag}\left[\frac{P_{i+}h_i}{\alpha_i}\right]D^{-1} = \text{diag}\left[\frac{(s+\gamma)^{s_i}h_i}{\alpha_i}\right]P_{\gamma+}D^{-1} \tag{4.15}$$

is a unit since $\text{diag}[((s+\gamma)^{s_i}h_i/\alpha_i)]$ is a unit and $P_{\gamma+}D^{-1}$ is also a unit by Sd1) and Lemma 4.3. Thus M is strongly achievable by Corollary 3.4 and the assertion follows.

($\Rightarrow$) Suppose that α_i and $\beta_i \in \mathbb{R}[s]$ are such that $M := \text{diag}[(D_{i+}\beta_i/\alpha_i)] \in \mathbf{M}_d$ and can be achieved by a stable controller. Then $(I - M)D^{-1}$ is a unit by Corollary 3.4. Denote $h_i := ((\alpha_i - D_{i+}\beta_i)/P_{i+})$, then $h_i \in \mathbb{R}[s]$ by Theorem 4.1. Simple manipulations lead to

$$(I - M)D^{-1} = \underbrace{\operatorname{diag}\left[\frac{(s+\gamma)^{s_i}h_i}{\alpha_i}\right]}_{=: W_i} P_{\gamma+}D^{-1}. \quad (4.16)$$

Since $(I - M)D^{-1}$ is a unit and $W_i \in S^{n_o \times n_o}$, it follows from (4.16) that $DP_{\gamma+}^{-1} = D(I - M)^{-1}W_i \in S^{n_o \times n_o}$. Also, since $P_{\gamma+}P \in S^{n_o \times n_i}$, we have $P_{\gamma+}D^{-1} \in S^{n_o \times n_o}$ by Lemma 2.5. Thus $P_{\gamma+}D^{-1}$ is a unit in $S^{n_o \times n_o}$ and Sd1) follows from Lemma 4.3. Since $(I - M)D^{-1}$ and $P_{\gamma+}D^{-1}$ are both units, W_i is also a unit by (4.16). Hence, h_i is Hurwitz since α_i is Hurwitz, and (4.11) is then satisfied. It follows from Lemma 4.4 that Sd2) holds. □

Corollary 4.5: Let $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$ and suppose that P1) is satisfied. Then there exists a stable decoupling controller for P if either

a) $P \in S^{n_o \times n_i}$, or
b) P satisfies Sd1) and has no $\mathbb{C}_+$-zeros. □

By the proof of Theorem 4.2, we see that an achievable decoupled I/O map $M := \operatorname{diag}[D_{i+}\beta_i/\alpha_i]$ can be achieved by a stable controller if and only if (4.11) holds. Thus we now have the following characterization.

Corollary 4.6: Let $P \in \mathbb{R}_{po}(s)^{n_o \times n_i}$. Suppose that P1), P2), Sd1), and Sd2) are satisfied. Then the set of all strongly achievable decoupled I/O maps is given by

$$\mathbf{M}_{sd} = \left\{ \begin{bmatrix} \frac{\tilde{\beta}_1}{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \frac{\tilde{\beta}_{n_o}}{\alpha_{n_o}} \end{bmatrix} \middle| \begin{array}{l} \alpha_i \text{ and } \tilde{\beta}_i \text{ satisfy Md1), Md2),} \\ \text{and } \alpha_i = \tilde{\beta}_i + P_{i+}h_i \\ \text{for some Hurwitz } h_i \in \mathbb{R}[s] \end{array} \right\}. \quad (4.17)$$

V. Concluding Remarks

We have shown that under a very mild assumption, that is, the plant has full normal rank and has no coincidences of unstable poles and zeros, the condition for closed-loop stability can be simplified. The simplification makes the description of all achievable I/O maps and the parameterization of all controller achieving the same I/O map direct and simple. It also makes the construction of achievable decoupled I/O maps straightforward.

Although only continuous-time systems with stability region designated as the open left-half plane are considered, the results remain true (with obvious modifications) if the stability region is changed to any subset of $\mathbb{C}$ symmetric with respect to the real axis, in particular, the open-unit-disk.

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Algebraic Versus Analytic Design Limitations Imposed by Ill-Conditioned Plants

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Abstract—A long-standing conjecture has been that ill-conditioned multivariable plants pose inherent design difficulty in that they limit the class of achievable robust performance specifications. The elusiveness of a proof of this conjecture has motivated study of an alternate design problem, involving only the nominal plant, in the hope that an inherent difficulty due to plant conditioning might emerge for this alternate problem. In this note we show, under mild assumptions, that such a difficulty, if it indeed exists, must take the form of a design tradeoff between system properties at _different_ frequencies, rather than between properties at the same frequency. (The terminology "analytic" and "algebraic" is motivated by the type of mathematics used to describe each class of tradeoff.) This analysis is also interpreted as implying the same conclusion for the original robust performance problem.

I. Introduction and Motivation

Consider a linear time-invariant feedback system whose plant and compensator have transfer functions denoted P, and C, respectively. We shall assume that the plant is square and invertible. Associated with this feedback system are several important transfer functions; namely, the open-loop transfer function, sensitivity function, and complementary sensitivity function defined at the plant output, $L_O = PC$, $S_O = (I + L_O)^{-1}$, and $T_O = L_O(I + L_O)^{-1}$, and their analogs defined at the plant input $L_I = CP$, $S_I = (I + L_I)^{-1}$, and $T_I = L_I(I + L_I)^{-1}$. It is well known (e.g., [1], [2]) that many design specifications may be stated in the form of frequency-dependent *bounds* upon the closed-loop transfer functions S_O, S_I, T_O, T_I, S_OP, and $P^{-1}T_O$. In particular, each of these six transfer functions

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