Forest Leaves and Four-Cycles

H. L. Fu[1] **and C. A. Rodger**[2]

[1] *DEPARTMENT OF APPLIED MATHEMATICS*
NATIONAL CHIAO TUNG UNIVERSITY
HSIN CHU, TAIWAN
REPUBLIC OF CHINA

[2] *DEPARTMENT OF DISCRETE AND STATISTICAL SCIENCES*
120 MATH ANNEX
AUBURN UNIVERSITY, ALABAMA 36849-5307

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Abstract: In this article, we find necessary and sufficient conditions for the existence of a 4-cycle system of $K_n - E(F)$ for any forest F of K_n. © 2000 John Wiley & Sons, Inc. J Graph Theory 33: 161–166, 2000

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1. INTRODUCTION

The *m-cycle* $(v_0, v_1, \ldots, v_{m-1})$ is the graph with vertex set $\{v_i | i \in \mathbb{Z}_m\}$ and edge set $\{\{v_i, v_{i+1}\}, \{v_0, v_{m-1}\} | i \in \mathbb{Z}_{m-1}\}$. An *$m$-cycle system* of G is an ordered pair $(V(G), C)$, where C is a set of m-cycles whose edges partition the edge set of G. There have been many results found on the existence of m-cycle systems of K_n and of $K_{m,n}$. Most recently, the set of integers n for which there exists an m-cycle system of K_n, where m is odd, has been completely settled [1]. It has also been common in the situation where no m-cycle system of K_n exists, to find the smallest set of edges E such that $K_n - E$ does have an m-cycle system. See [3, 6] for results on these problems.

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Another particularly interesting problem studied is to let H be one of a family of spanning subgraphs of K_n and find m-cycle system of $K_n - E(H)$; we say that H is the leave of the m-cycle system. This type of problem is somewhat different, because H grows with n. By using difference methods, inductive methods, and amalgamations of graphs, this problem has been solved if H is an 1-factor [1], and if $m = 3, 4$ or n (of course, n-cycles are Hamilton cycles) when H is any 2-factor of K_n [2, 4, 5]. In this article, we completely solve another of these problems, finding necessary and sufficient conditions for the existence of a 4-cycle system of $K_n - E(F)$ for any spanning forest F of K_n. We say that a forest F is *odd*, if all vertices in F have odd degree. To avoid confusion with cycles, let $\langle v_0, v_1, \ldots, v_{m-1}\rangle$ denote the *path* with vertex set $\{v_i | i \in \mathbb{Z}_m\}$ and edge set $\{\{v_i, v_{i+1}\} | i \in \mathbb{Z}_{m-1}\}$. Let $(A \cup B, K(A, B))$ denote a 4-cycle system of $K_{|A|,|B|}$ with bipartition A and B of the vertex set. The following is well known, and has been proved in more generality by Sotteau [7], but a proof is included here to keep the article self-contained.

Lemma 1.1. *There exists a* 4-*cycle system of* $K_{a,b}$ *if and only if* a *and* b *are even.*

Proof. The necessity is clear, and $((\mathbb{Z}_a \times \{0\}) \cup (\mathbb{Z}_b \times \{1\}), \{((2i, 0), (2j, 1), (2i + 1, 0), (2j + 1, 1)) | i \in \mathbb{Z}_{a/2}, j \in \mathbb{Z}_{b/2}\})$ proves the sufficiency. ∎

2. RESULT

We begin with a small partial 4-cycle system that is vital in the proof of Theorem 2.1.

Lemma 2.1. *There exists a* 4-*cycle system* $(\{v_i | i \in \mathbb{Z}_8\}, C^*)$ *of* $K_8 - P$ *with leave* P *consisting of the union of the four* 2-*paths* $p_0 = \langle v_0, v_2, v_1\rangle$, $p_1 = \langle v_2, v_6, v_3\rangle$, $p_2 = \langle v_4, v_1, v_5\rangle$, *and* $p_3 = \langle v_6, v_5, v_7\rangle$.

Proof. Let $C^* = \{(v_0, v_6, v_1, v_7), (v_0, v_4, v_7, v_3), (v_0, v_1, v_3, v_5), (v_4, v_5, v_2, v_3), (v_4, v_6, v_7, v_2)\}$. ∎

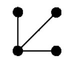
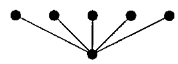
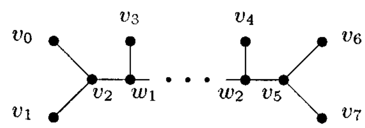
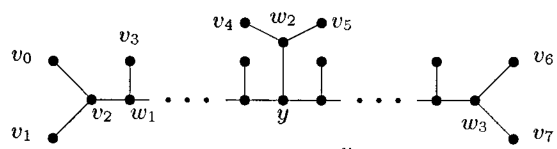
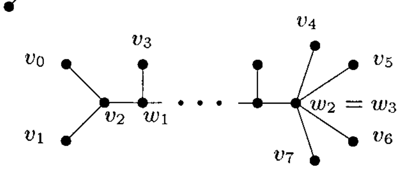
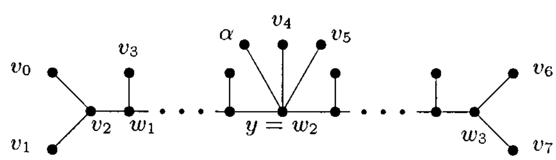
A *leaf pairing* of *size* x in a forest F is a set L of x sets, each containing two leaves (vertices of degree 1) of F, such that each vertex appears in at most one set in L, and such that for each pair $\{v, w\} \in L$ there exists a vertex z in F that is adjacent to both v and w (so $d_F(v, w) = 2$). It is easy to check that an odd *tree* T has a leaf pairing of size at least 4 unless T is K_2 or is one of the trees in Table I. Some of the vertices are labeled; this helps in the proof of Theorem 2.1.

Theorem 2.1. *Let* F *be a forest of* K_n *that has at least one edge. There exists a* 4-*cycle system of* $K_n - E(F)$ *if and only if* 4 *divides* $|E(K_n)| - |E(F)|$, F *is odd and spanning*, *and* n *is even.*

Proof. Since each such forest has a vertex v of degree 1, and since the other edges incident with v are partitioned into pairs by 4-cycles, n is even. Therefore, F is odd and spanning, and so the necessity is clear.

We begin proving the sufficiency by noting that, if $n \in \{2, 4, 6, 8\}$, then the necessary conditions require that F be a 1-factor; in each such case, it is trivial

TABLE I. All odd trees with at most 3 leaf pairs, excluding K_2.

Leaf Pair Size	Tree
1	$K_{1,3} =$ 
2	$K_{1,5} =$ 
	T_1 : 
3	T_2 : 
	T_3 : 
	T_4 : 

to find a 4-cycle system of $K_n - E(F)$. The remainder of the proof follows by induction, so suppose that for some even $n \geq 10$ and for all even $n' < n$ there exists a 4-cycle system of $K_{n'} - E(F')$ for any odd spanning forest F' of $K_{n'}$ for which 4 divides $|E(K_{n'})| - |E(F')|$. Let F be any odd spanning forest of K_n for which 4 divides $|E(K_n)| - |E(F)|$. We consider several cases in turn. Let K_n have vertex set $\mathbb{Z}_n$.

Case 1. Suppose that F contains a leaf pairing of size 4, say $\{\{v_{2i}, v_{2i+1}\} | i \in \mathbb{Z}_4\}$. Let $V = \{v_i | i \in \mathbb{Z}_8\}$ and $V' = \mathbb{Z}_n \backslash V$. For each $i \in \mathbb{Z}_4$, let w_i be adjacent to v_{2i} and v_{2i+1} in F; then, since $d_F(w_i) \geq 2$, we know $w_i \in V'$. For each $i \in \mathbb{Z}_4$, let $z_i \in V' \backslash \{w_i\}$ be a vertex "paired" with w_i; z_i exists because both n and $|V|$ are even, so $|V'|$ is even. Note that $w_1, w_2, w_3, w_4, z_1, z_2, z_3$, and z_4 need not all be distinct.

Let $F' = F - V$. Then $|E(F')| = |E(F')| - 8$, so, since 4 divides $\binom{n}{2} - |E(F)|$, it can easily be seen that 4 divides $\binom{n-8}{2} - |E(F')|$. Also, $|V'| = n - 8$ is even and F' is an odd spanning forest of K_{n-8}, so by induction there exists a 4-cycle system (V', C_1) of $K_{n-8} - E(F')$. For each $i \in \mathbb{Z}_4$, let c_i be the 4-cycle formed by adding the edges $\{v_{2i}, z_i\}$ and $\{v_{2i+1}, z_i\}$ to the 2-path p_i defined in Lemma 2.1, let $C_2 = \{c_i | i \in \mathbb{Z}_4\}$, and let C^* be as defined in Lemma 2.1. Finally, let $C_3 = \cup_{i \in \mathbb{Z}_4} K(\{v_{2i}, v_{2i+1}\}, V' \backslash \{z_i, w_i\})$. Then the edges $e_{2i} = \{v_{2i}, w_i\}$ and $e_{2i+1} = \{v_{2i+1}, w_i\}$ for $i \in \mathbb{Z}_4$ occur in no 4-cycle defined so far, which is good, because $E(F) = E(F') \cup \{e_i | i \in \mathbb{Z}_8\}$. So $(\mathbb{Z}_n, C_1 \cup C_2 \cup C_3 \cup C^*)$ is a 4-cycle system of $K_n - E(F)$, as required.

Case 2. Suppose one component of F is K_2 with vertex set $\{n-2, n-1\}$. Since 4 divides $\binom{n}{2} - |E(F)|$, it is easy to check that 4 also divides $\binom{n-2}{2} - (|E(F)| - 1)$. Therefore, by induction there exists a 4-cycle system $(\mathbb{Z}_{n-2}, C_1)$ of $K_{n-2} - E(F')$, where $F' = F - \{n-2, n-1\}$. Then $(\mathbb{Z}_n, C_1 \cup K(\mathbb{Z}_{n-2}, \{n-2, n-1\}))$ is a 4-cycle system of $K_n - E(F)$.

In view of Cases 1 and 2, it remains to consider the cases where F contains at most 3 leaf pairs and no component of F is K_2. Trees with this property are listed in Table I. Notice that under this restriction that we just placed on F, if one component of F were to be $K_{1,5}$, then either $F = K_{1,5}$ or the other component of F would have to be $K_{1,3}$; since F is spanning, both these cases are excluded, since 4 divides neither $\binom{6}{2} - 5$ nor $\binom{10}{2} - 8$, respectively.

Case 3. Suppose that either $F \in T_1$ (see Table I), or $n \neq 12$ and F consists of two components, one of which is in T_1 and the other is $K_{1,3}$ (the necessary conditions preclude the possibility of $n = 10$ in this case). Then since F is spanning, the vertices in $V = \{v_i | i \in \mathbb{Z}_8\}$ marked on the tree in T_1 in Table I are all distinct. Let $\{w_1, v_2\}, \{w_1, v_3\}, \{w_2, v_4\}$ and $\{w_2, v_5\}$ be edges in F, as is shown in Table I. Let $V' = \mathbb{Z}_n \backslash V$, and for each $i \in \{1, 2\}$ let $z_i \in V' \backslash \{w_i\}$.

Let $F' = F - V$. Then $|E(F')| = |E(F)| - 8, |V'| = n - 8$, and F' is an odd spanning forest of K_{n-8}, so as in Case 1 by induction there exists a 4-cycle system (V', C_1) of $K_{n-8} - E(F')$. Using Lemma 2.1, let c_1 and c_2 be formed from the 2-paths p_1 and p_2 by adding the edges in $\{\{v_2, z_1\}, \{v_3, z_1\}\}$ and $\{\{v_4, z_2\}, \{v_5, z_2\}\}$, respectively, and let $C_2 = \{c_1, c_2\}$. Finally, let $C_3 = (\cup_{i \in \{0,3\}} K(\{v_{2i}, v_{2i+1}\}, V')) \cup (\cup_{i \in \{1,2\}} K(\{v_{2i}, v_{2i+1}\}, V' \backslash \{w_i, z_i\}))$. Then the edges $\{v_{2i}, w_i\}$ and $\{v_{2i+1}, w_i\}$) for $i \in \{1, 2\}$ and the edges in the 2-paths p_0 and p_3 occur in no 4-cycle defined so far; these are precisely the edges in $E(F) \backslash E(F')$. So, with C^* as defined in Lemma 2.1, $(\mathbb{Z}_n, C_1 \cup C_2 \cup C_3 \cup C^*)$ is a 4-cycle system of $K_n - E(F)$.

Each of the Cases 4 to 6 is similar to either Case 1 or Case 3, so the details of the proof there are omitted. Throughout Cases 4–6, we let $V = \{v_i | i \in \mathbb{Z}_8\}, V' = \mathbb{Z}_n \backslash V, z_i \in V' \backslash \{w_i\}$ whenever w_i is defined, and $F' = F - V$. In each case, $|E(F')| = |E(F)| - 8$ and F' is an odd spanning forest, so by induction there exists a 4-cycle system (V', C_1) of $K_{n-8} - E(F')$. With C^* defined in Lemma 2.1, form a 4-cycle system $(\mathbb{Z}_n, C_1 \cup C_2 \cup C_3 \cup C^*)$ by defining C_2 and C_3 as follows.

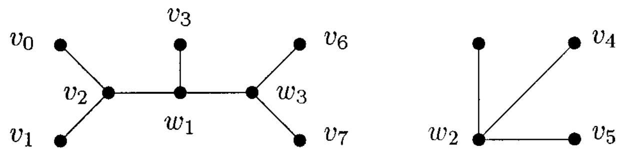


FIGURE 1. The forest F in Case 6.

Case 4. Suppose that $F \in T_2, T_3$, or T_4, and if $F \in T_2$ or T_4, then $n \neq 10$. Name the vertices as in Table I; if $F \in T_2$ or T_4, then, since $n > 10$, we can assume that vertex $y \neq w_1$. For each $i \in \{1,2,3\}$, let c_i be the 4-cycle formed by adding the edges $\{v_{2i}, z_i\}$ and $\{v_{2i+1}, z_i\}$ to the 2-path p_i, and let $C_2 = \{c_1, c_2, c_3\}$. Let $C_3 = (\cup_{i\in\{1,2,3\}} K(\{v_{2i}, v_{2i+1}\}, V' \backslash \{w_i, z_i\})) \cup K(\{v_0, v_1\}, V')$.

Case 5. If $F \in T_4$ with $n = 10$, then proceed as in Case 4 with $\alpha = v_3$ and $w_1 = w_2$.

Case 6. If $n = 12$ and F consists of two components, one of which is in T_1 and the other is $K_{1,3}$, then again proceed as in Case 4 with vertices labeled as in Fig. 1.

Finally, there remains one case to consider.

Case 7. If $F \in T_2$ with $n = 10$, then let

$$\begin{aligned} V &= \{v_i | i \in \mathbb{Z}_6\} \cup \{w_i | i \in \mathbb{Z}_3\} \cup \{\infty\}, \text{ and let} \\ C &= \{(v_{2i}, v_{2j}, v_{2i+1}, v_{2j+1}) | 0 \leq i < j \leq 2\} \\ &\cup \{(v_{2i}, v_{2i+1}, w_{i+1}, w_{i+2}), (v_{2i+1}, \infty, v_{2i+2}, w_{i+2}) | i \in \mathbb{Z}_3\}, \end{aligned}$$

reducing the subscript of w modulo 3. ∎

It may be of more use to state Theorem 2.1 in the following way.

Corollary 2.1. *Let F be an odd spanning forest of K_n. There exists a 4-cycle system of $K_n - E(F)$ if and only if $|E(F)| \in \{(n/2) + 4i | 0 \leq i \leq (n-2)/8\}$.*

Proof. Since F is spanning, clearly $|E(F)| \geq n/2$ (the size of a 1-factor of K_n), and, since F is a spanning forest, $|E(F)| \leq n - 1$ (the size of a spanning tree of K_n). If F_0 is a 1-factor, then $\binom{n}{2} - |E(F_0)| = n(n-1)/2 - n/2 = n(n-2)/2$, which is divisible by 4 for all even n; so F_0 satisfies the necessary conditions of Theorem 2.1. Since 4 must divide $|E(F_1)| - |E(F_2)|$ for any two forests F_1 and F_2 satisfying the conditions of Theorem 2.1, the corollary now follows. ∎

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