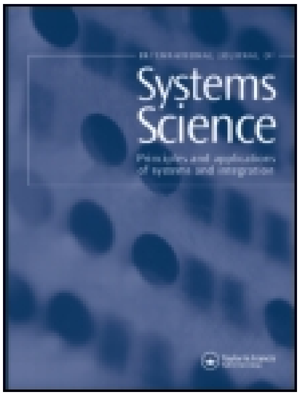

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Minimization of the life cycle cost for a multistate system under periodic maintenance

Chao-Ton Su†‡ and Cheng-Chang Chang

Although preventive maintenance policies have received extensive interest, limited attention has been paid to implementing multiple maintenance actions for a multistate system over a finite horizon in continuous time. Therefore, this study closely examines such maintenance actions by viewing a coherent multicomponent system as a multistate system and assuming that the necessary maintenance action and cost which move the current state to an extremely better state depends on the current state. The above problem is also formulated as a periodic maintenance model. In addition, the model's characteristics are elucidated to obtain the optimal cycle time of maintenance actions, thereby minimizing the life cycle cost over a specific finite horizon.

1. Introduction

Owing to the age-dependent deterioration of a system, the extremely complex acquisition cost of that system is normally less than the ownership cost with respect to life cycle years. Depending on the system type, the ownership cost over the life cycle span may vary from ten to 100 times the acquisition cost (Dhillon 1989). Therefore, developing an optimal maintenance policy to reduce the life cycle cost is a critical task.

Viewing the system as of a multistate nature is an effective means of accounting for the age-dependent failure. Restated, many states must be defined to account for the different operational conditions. Considerable attempts have been made to model the equipment maintenance–replacement problem based on the multistate and the Markov theory. Earlier studies including those by Derman (1963) and Pierskalla and Voelker (1976) attempted to replace the equipment when the operational and maintenance costs in net present value terms exceeded a certain level to justify a replacement. This approach has found extensive applications (see for example Hatoyama (1984), Kusaka (1985), Hopp (1988), Bylka *et al.* (1992), Hontelez (1996) and Wu and Chang (1996)). Other notable surveys have been given by Cho and Parlar (1991) and Pintelon and Gelders (1992). Moreover, for extremely complex systems, such as airplanes, boats, building and bridges, the planning life cycle years are extremely long and non-replacement actions must be considered to reduce the life cycle cost. Hopp and Wu (1990) considered the Markov maintenance problem with non-replacement actions. Despite the above models involving the Markov maintenance theory, theoretical results for application purposes are somewhat lacking. Difficulties encountered in applying the models in the real world include the complexity of the theories, to the extent that most practitioners cannot read them; measuring the discounted rates is also extremely difficult.

Another feature of maintenance–replacement models is their relatively simple and satisfactory implementation. A notable example consists of the periodic or block maintenance–replacement models based on the renewal theory. Earlier investigations include those of Beichelt (1981) and Cl'erous *et al.* (1979). These models have been extended in recent years (see for example Block *et al.* (1988), Sheu (1992, 1996), Gu (1993) and Sheu and Jhang (1996)). Such models, although relatively easy to understand and implement, generally tend to describe inadequately the process of system deterioration and maintenance which is attributed to the loss of multistate and preventive non-replacement actions. Another more practical concept is imperfect preventive maintenance (PM). Most investigations

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have treated imperfect preventive maintenance as a circumstance in which either the system is not restored like new but younger (see for example Nakagawa 1988, Jayabaian and Chaudhuri 1992, 1995 and Jack and Dagpunar 1994) or the system stochastically moves to the best condition after PM (see for example Brown and Proschan 1983, Nakagawa and Yasui 1987, Wang and Pham 1997 and Yun and Bai 1987). Several imperfect PM models capable of integrating both imperfect PM concepts can be found in the paper by Wang and Pham (1996). Pham and Wang (1996) carefully reviewed various treatment methods and optimal policies on the imperfect PM. However, the assumption that the failure rate of a younger system is less than that of the older system may be invalid. Direct use of the characteristics to categorize a system as finite states to account for the system's failure rate may be preferred merely to using the concept of usage. Combining the multistate, the usage and the number of PMs is a more general approach to explain a system's failure rate. Therefore, whether or not the minimal PM cost reverses the entire system to the best achievable state depends on the current state. The system failure rate after a PM depends on the parameters of state, the length of periods used, and the number of PMs.

In this work, we present a model for multistate systems with state-dependent cost to determine the optimal cycle time of preventive non-replacement actions over a finite horizon. The proposed model combines the multistate system of Christer and Goodbody (1980), multiple non-replacement actions introduced by Hopp and Wu (1988), periodic maintenance–replacement described by Boland and Proschan (1982), and the imperfect maintenance concepts of Pham and Wang (1996). By doing so, the maintenance model proposed herein can be viewed as being of an imperfect maintenance nature while the number of PMs significantly impact the system's failure rate and can be viewed as perfect maintenance modelling if the number of PMs only slightly impacts the system's failure. Consequently, the proposed model can replace expensive and extremely complex systems such as those found in military aircraft and satellites. Also, the proposed model is more general than the conventional periodic replacement–maintenance models since the former considers multiple states and multiple non-replacement actions. The rest of this paper is organized as follows. Sections 2 and 3 describe the problem and the proposed model. Section 4 presents the model's structural characteristics and section 5 provides a numerical example which demonstrates the proposed model's feasibility. Section 6 provides a comparison and a real example of application of this work. Concluding remarks are finally made in section 7.

2. Problem description

A situation is considered in which an enterprise owns some coherent multicomponent systems which have a planning horizon K (life cycle years). Each system can be categorized as two state spaces, S_1 and S_2. Allow $S_1 = \{1, 2, \ldots, n\}$ to be the state space of normal operations, where 1 denotes the optimal state and n refers to the worst state. Moreover, allow $S_2 = \{\text{malfunction}\}$ to be the state space of system failure. In order to reduce the life cycle cost and to increase the productivity, the system must be maintained at an extremely better state than current as possibly for every T unit times. Herein, state 2 is assumed to be achievable under the available maintenance actions. We further define two action spaces, A_1 and A_2, where A_1 contains $n-1$ kinds of action, that is $A_1 = \{a_0, a_1, \ldots, a_{n-2}\}$. The maintenance action a_0 costs c_0 (including test and inspection cost) which represents the state test or inspection. The maintenance action a_1 costs c_1 which deterministically moves the system to state 2 from state $j, j \in \{2, 3\}$ and action a_2 deterministically moves the system to state 2 from state $j, j \in \{2, 3, 4\}$. Similarly, maintenance a_{n-2} deterministically moves the system to state 2 from state $j, j \in \{2, 3, 4, \ldots, n\}$. Consequently, after performing action a_0, the operational state is known and a further maintenance action $a \in A_1 - \{a_0\}$ must be chosen to move the system to state 2. Moreover, action space A_2 contains only one maintenance action, minimal repair, that is $A_2 = \{\text{minimal repair}\}$. After performing the action 'minimal repair', the system is repaired to the previous operational state before failure. After that, three maintenance policies are considered as follows.

Policy 1 (unplanned maintenance actions): The maintenance action 'minimal repair' is performed when it deteriorates into a failed situation. Restated, if the deteriorating process was 'state $j \rightarrow$ failure', then the repair process was 'failure $\rightarrow$ state j'. Policy 1 attempts to reduce the unavailability. □

Policy 2 (planned–preventive maintenance actions, PM): The system is maintained at state 2 when the age reaches $T, \quad 2T + \tau, \ldots, hT + (h-1)\tau, \quad hT + (h-1)\tau \leq K$, where K is the planning horizon. Policy 2 attempts to promote an enhanced productivity by the means of a large preventive maintenance. □

Policy 3 (planned replacement actions): The system is replaced by an advanced system when the age reaches the planning horizon K. Policy 3 attempts to pertain the competition. □

This study attempts to model the above problem, thereby facilitating the analysis of even more complex and realistic maintenance problems. The other relevant assumptions are as follows.

Assumption A1: *The deteriorating time from state* 1 *into state* 2 *is negligible. This represents that state* 2 *is an approximated best condition..*

Assumption A2: *The planning horizon is known in advance.*

Assumption A3: *A worse operational state implies a quicker system failure, that is*

$$P\{\text{failure in } (t, t+\Delta t) \| Z_t = i\}$$
$$\leq P\{\text{failure in } (t, t+\Delta t) | Z_t = j\}, \quad \forall i \leq j, i, j \in S,$$

where Z_t denotes the operational state at time t.

Assumption A4: *A better operational state implies an enhanced productivity..*

Assumption A5: *A worse operational state implies a faster state transition to another worse state. Restated, if $i \leq j \leq l, i, j, l \in S$, then*

$$P\{Z_{t+s} = l \| Z_t = i\} \leq P\{.Z_{t+s} = l \| Z_t = j\}$$

Remark 1:

(i) For policy 3, age-dependent reliability and technological improvement normally determine the planning horizon K. For instance, for complex and expensive systems such as military aircraft and satellites, some alternative advanced systems are generally available over the planning horizon K (owing to a technological improvement or breakthrough). By doing this, the probability of an extremely poor condition, implying a perfect maintenance action (replacing the entire system), is usually extremely low over this planning horizon. Restated, the system can be recovered to an extremely better state from any operational state by an approximated replacement action regardless of the high maintenance cost.

(ii) Assumptions A3 and A4 verify that policy 2, which maintains the system at state 2 for every T unit times, attempts not only to reduce the unplanned maintenance cost regarding policy 1 but also to promote productivity.

(iii) Assumption A5 states that the transition way is in order, that is $1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n$.

3. Model formulation

To construct the model, the relevant notation is initially stated as follows:

- $\hat{c}(j, 2)$ minimal maintenance cost of reversing state j to state 2
- $E\{B(2, T)\}$ expected total planned maintenance cost in time interval $(0, K)$ when performing the preventive maintenance policy $(2, T)$
- $E\{\hat{B}(2, T)\}$ expected total unplanned maintenance cost in time interval $(0, K)$ when performing the preventive maintenance policy $(2, T)$
- $E_k[B(2, T)]$ kth expected planned maintenance cost regarding policy 2, $k = 1, 2, \ldots$
- $E_k[\hat{B}(2, T)]$ expected unplanned maintenance cost regarding policy 1 which takes in time interval $((k-1)(T+\tau), k(T+\tau)-\tau)$, $k = 1, 2, \ldots$, under given policy 2
- $LCC_K(2, T)$ system's life cycle cost over finite horizon K under the given system that was maintained to state 2 for every T unit times
- $N_i(t)$ number of system failures occurring in time interval $(0, t)$ in the situation in which the state at time 0 is in i and only policy 1 is performed in $(0, t)$
- Q_t number of planned maintenance actions regarding policy 2 over time interval $(0, t)$
- w number of planned maintenance actions regarding policy 2 over time interval $(0, K)$
- X_{ij} a random variable which represents the deterioration time from state i to state j in a situation in which no planned maintenance action regarding policy 2 has been performed
- $X_{ij}(k)$ random variable which represents the deterioration time from state i to j in a situation in which the kth planned maintenance action regarding policy 2 has been performed, $k \geq 1$
- Z_t operational state at time t
- γ expected repair time for each unplanned maintenance regarding policy 1
- $\delta_i(t)$ total expected repair times regarding policy 1 over time interval $(0, t)$ in the situation in which the state at time 0 is in i, that is $\delta_i(t) = \gamma\phi_i(t)$
- τ PM times regarding policy 2 for each PM (owing to the fixed contract, the PM times regarding policy 2 for each PM is a constant)
- $\phi_i(t)$ cumulative hazard rate function of $N_i(t)$, that is $E\{N_i(t)\} = \phi_i(t)$.

The expected life cycle cost over time interval $(0, K)$ can be written as

$$LCC_K(2,T) = E\{\hat{B}(2,T)\} + E\{B(2,T)\}$$
$$= \begin{cases} \sum_{k=1}^{w+1} E_k\{\hat{B}(2,T)\}, & \text{if } w = 0, \\ \sum_{k=1}^{w+1} E_k\{\hat{B}(2,T)\} + \sum_{k=1}^{w} E_k\{B(2,T)\}, & \text{if } w \geq 1, \end{cases} \quad (1)$$

for all $T \in (0, K]$, where

$$w = \begin{cases} \left[\frac{K}{T}\right] - 1, & \text{if } \frac{K}{T} \text{ is an integer,} \\ \left[\frac{K}{T}\right], & \text{if } \frac{K}{T} \text{ is not an integer.} \end{cases}$$

(Note that $[t]$ represents the largest integer that is less than or equal to t.)

The purpose of this paper is to find an optimal cycle time T^* such that

$$LCC_K(2,T^*) = \min_{T\in(0,K]}\{LCC_K(2,T)\}. \quad (2)$$

(Note that, if $T^* = K$, then this implies that the optimal strategy is one of no PM in policy 2.)

According to policy 1, the unplanned maintenance will move the system's failure to its original operational state before failing, then the following equations are derived.

$$E\{P\{Z_t = j|Z_0 = i; N_i(t);\ \text{policy } 1; Q_t = 0\}\}$$
$$= \begin{cases} P\{X_{ij} \leq t - \delta_i(t), \min\{X_{ik}, k = j+1,\ldots,n\} > t - \delta_i(t)\}, & \text{if } i < j \leq n-1 \\ P\{X_{in} \leq t - \delta_i(t)\}, & \text{if } j = n \\ 1 - P\{\min\{X_{ik}, k = i+1,\ldots,n\} \leq t - \delta_i(t)\}, & \text{if } j = i \end{cases}$$
$$= \begin{cases} P\{X_{ij} \leq t - \delta_i(t), X_{i,j+1} > t - \delta_i(t)\}, & \text{if } i < j \leq n-1 \\ P\{X_{in} \leq t - \delta_i(t)\}, & \text{if } j = n \\ 1 - P\{X_{i,i+1} \leq t - \delta_i(t)\}, & \text{if } j = i \end{cases}$$
$$= \begin{cases} F_{ij}(t - \delta_i(t)) - F_{i,j+1}(t - \delta_i(t)), & \text{if } i < j \leq n-1, \\ F_{in}(t - \delta_i(t)), & \text{if } j = n, \\ 1 - F_{i,i+1}(t - \delta_i(t)), & \text{if } j = i, \end{cases} \quad (3)$$

where $F_{ij}(x)$ denotes the cumulative distribution function of X_{ij}.

Similarly, the following equation can be obtained by policy 1.

$$E\{P\{Z_{t+s} = j|Z_t = i; N_i(t+s) - N_i(t);\ \text{policy } 1; Q_t = k\}\}$$
$$= \begin{cases} F_{ij|k}(s - \delta_i(s)) - F_{i,j+1|k}(s - \delta_i(s)), & \text{if } i < j \leq n-1, \\ F_{in|k}(s - \delta_i(s)), & \text{if } j = n, \\ 1 - F_{i,i+1|k}(s - \delta_i(s)), & \text{if } j = i, \end{cases} \quad (4)$$

where $F_{ij|k}(x)$ represents the cumulative distribution function of $X_{ij}(k)$.

Now, suppose that $X_{ij}(k) = \beta^{-k}X_{ij}$, $\beta \geq 1$, for all $k = 1, 2, \ldots$ and X_{ij} be a uniform random variable on $[0, a_{ij}]$, that is

$$F_{ij}(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{t}{a_{ij}} & \text{if } 0 < t < a_{ij}, \\ 1 & \text{if } t \geq a_{ij}. \end{cases} \quad (5)$$

Then,

$$P\{X_{ij}(k) \leq x\} = P\{\beta^{-k}X_{ij} \leq x\} = P\{X_{ij} \leq \beta^k x\} = \frac{\beta^k x}{a_{ij}}. \quad (6)$$

Therefore,

$$\mathrm{d}P\{X_{ij}(k) \leq x\} = \mathrm{d}\left(\frac{\beta^k x}{a_{ij}}\right) = \beta^k\,\mathrm{d}F_{ij}(x) \quad (7)$$

Let c_j (cost of action a_j) be non-decreasing in j, that is $c_m \geq c_l$, if $m \geq l$. Therefore, the minimal total maintenance cost $\hat{c}(j,2)$ of reversing operational state j to state 2 is

$$\hat{c}(j,2) = c_0 + \min_{j-2\leq l\leq n-2}\{c_l\} = c_0 + c_{j-2} \quad \text{for all } j > 2. \quad (8)$$

Moreover, this work also considers a situation in which at least an action a in A must be chosen while the system is sent to the maintenance shop (owing to contract). Hence we obtain the minimal total maintenance cost $\hat{c}(2,2)$ of reversing state 2 to state 2:

$$\hat{c}(2,2) = c_0 + \min_{1\leq j\leq n-2}\{c_j\} = c_0 + c_1. \quad (9)$$

Since the deteriorating time from state 1 into state 2 is negligible (assumption (A1)), hence it will surely deteriorate into state 2 instantaneously, that is

$$\lim_{\Delta_t\to 0}(P\{Z\Delta_t = 2|Z_0 = 1\}) = 1. \quad (10)$$

Equations (3) and (8)–(10) can be used to obtain the first planned maintenance cost regarding policy 2 as follows:

$$
\begin{aligned}
&E_1\{B(2,T)\}\\
&= \sum_{j=1}^{n} \hat{c}(j,2)\, E\{P\{Z_T = j | Z_0 = 1; Q_T = 0; N_2(T)\}\}\\
&= \sum_{j=2}^{n} \hat{c}(j,2)\, E\{P\{Z_T = j | Z_0 = 2; Q_T = 0; N_2(T)\}\}\\
&= \sum_{j=3}^{n-1} \hat{c}(j,2)[F_{2j}(T-\delta_2(T)) - F_{2,j+1}(T-\delta_2(T))]\\
&\quad + \hat{c}(n,2)F_{2n}(T-\delta_2(T)) + \hat{c}(2,2)[1 - F_{23}(T-\delta_2(T))]\\
&= c_0 + \sum_{j=3}^{n-1} c_{j-2}[F_{2j}(T-\delta_2(T)) - F_{2,j+1}(T-\delta_2(T))]\\
&\quad + c_{n-2}F_{2n}(T-\delta_2(T)) + c_1[1 - F_{23}(T-\delta_2(T))].
\end{aligned} \tag{11}
$$

Equations (4) and (7)–(9) can also be used to obtain the kth planned maintenance cost regarding policy 2 as follows:

$$
\begin{aligned}
E_k\{B(2,T)\} &= \sum_{j=2}^{n} \hat{c}(j,2)\\
&\quad \times E\{P\{Z_{kT+(k-1)\tau} = j | Z_{(k-1)(T+\tau)} = 2;\\
&\quad Q_{(k-1)(T+\tau)} = k-1;\\
&\quad N_2(k(T+\tau)-\tau) - N_2((k-1)(T+\tau))\}\}\\
&= \sum_{j=3}^{n-1} (c_0 + c_{j-2})[F_{2j|k-1}(T-\delta_2(T))\\
&\quad - F_{2,j+1|k-1}(T-\delta_2(T))]\\
&\quad + (c_0 + c_{n-2})F_{2n|k-1}(T-\delta_2(T))\\
&\quad + (c_0 + c_1)[1 - F_{23|k-1}(T-\delta_2(T))]\\
&= c_0 + \sum_{j=3}^{n-1} c_{j-2}\left(\int_0^{T-\delta_2(T)} \mathrm{d}F_{2j|k-1}(x)\right.\\
&\quad \left. - \int_0^{T-\delta_2(T)} dF_{2,j+1|k-1}(x)\right)\\
&\quad + c_{n-2}\int_0^{T-\delta_2(T)} dF_{2n|k-1}(x)\\
&\quad + c_1\left(1 - \int_0^{T-\delta_2(T)} dF_{23|k-1}(x)\right)\\
&= c_0 + \beta^{k-1}\left(\sum_{j=3}^{n-1} c_{j-2}[F_{2j}(T-\delta_2(T))\right.\\
&\quad \left. - F_{2,j+1}(T-\delta_2(T))] + c_{n-2}F_{2n}(T-\delta_2(T))\right)\\
&\quad + c_1[1 - \beta^{k-1}F_{23}(T-\delta_2(T))] \text{ for all } k = 2,3,\ldots.
\end{aligned} \tag{12}
$$

Since X_{ij} is a uniform random variable on $[0, a_{ij}]$, (11) and (12) can be rewritten as

$$
E_k\{B(2,T)\} = c_0 + c_1 + \beta^{k-1}\psi(T-\delta_2(T)), \quad \text{for all } k = 1,2,3,\ldots, \tag{13}
$$

where

$$
\psi(x) = \begin{cases}
\sum_{j=3}^{n-1} c_{j-2}\left(\frac{x}{a_{2j}} - \frac{x}{a_{2,j+1}}\right) + c_{n-2}\frac{x}{a_{2n}} + c_1\left(-\frac{x}{a_{23}}\right), & \text{if } x \le a_{23},\\
\left(c_1 - \frac{c_1 x}{a_{24}}\right) + \sum_{j=4}^{n-1} c_{j-2}\left(\frac{x}{a_{2j}} - \frac{x}{a_{2,j+1}}\right) + c_{n-2}\frac{x}{a_{2n}}, & \text{if } a_{23} \le x \le a_{24},\\
\vdots, & \\
c_{n-3} + (c_{n-2} - c_{n-3})\frac{x}{a_{2n}}, & \text{if } a_{2,n-1} \le x \le a_{2n},\\
c_{n-2}, & \text{if } a_{2n} \le x.
\end{cases}
$$

By combining (12) and (13), the expected total planned cost over finite horizon K is

$$
\begin{aligned}
E\{B(2,T)\} &= \begin{cases} 0, & \text{if } w = 0\\ \sum_{k=1}^{w} E_k\{B(2,T)\}, & \text{if } w \ge 1 \end{cases}\\
&= \begin{cases} 0, & \text{if } w = 0\\ wc_0 + \sum_{k=1}^{w} \beta^{k-1}\psi(T-\delta_2(T)), & \text{if } w \ge 1 \end{cases}\\
&= \begin{cases} 0, & \text{if } w = 0,\\ wc_0 + \frac{(\beta^{w-1}-1)}{\beta-1}\psi(T-\delta_2(T)), & \text{if } w \ge 1. \end{cases}
\end{aligned} \tag{14}
$$

The expected unplanned maintenance cost for the first time interval $(0, T)$ is

$$E_1\{\hat{B}(2,T)\} = E\{\text{cost of unplanned repairs in } (0,T)\}$$
$$= \theta E\{N_1(T)\}$$
$$= \lim_{\Delta t\to 0}\theta\{E\{N_1(\Delta t)\} + E\{N_2(T-\Delta t)\}\}$$
$$= \theta E\{N_2(T)\}$$
$$= \theta\phi_2(T). \quad (15)$$

Furthermore, a situation is considered in which the defined states cannot precisely account for the system failure. Assume that a real number $\lambda, \lambda \geq 1$, and a large M (threshold) exist such that

$$E\{N_2(t+s) - N_2(t)|Z_t = 2\} = \begin{cases} \lambda^{Q_t}E\{N_2(s)\}, & \text{if } Q_t \leq M, \\ E\{N_n(s)\}, & \text{if } Q_t > M. \end{cases} \quad (16)$$

Suppose that $w < M$ (since M is a large number); then

$$E\{N_2(k(T+\tau)-\tau) - N_2((k-1)(T+\tau)) \times |Z_{(k-1)(T+\tau)} = 2; Q_{(k-1)(T+\tau)} = k-1\}$$
$$= \lambda^{k-1}E\{N_2(k(T+\tau)-\tau) - N_2((k-1)(T+\tau)) \times |Z_(k-1)(T+\tau) = 2; Q_{(k-1)(T+\tau)} = 0\}$$
$$= \lambda^{k-1}E\{N_2(T)\}$$
$$= \lambda^{k-1}\phi_2(T) \quad \text{for all } k = 1,2,\ldots \quad (17)$$

Hence,

$$E_k\{\hat{B}(2,T)\} = E\{\text{cost of unplanned actions in } ((k-1)(T+\tau), k(T+\tau)-\tau)\}$$
$$= \theta\lambda^{k-1}\phi_2(T) \quad \text{for all } k = 2,3,\ldots,w \quad (18)$$

and

$$E_{w+1}\{\hat{B}(2,T)\} = E\{\text{cost of unplanned actions in } (w(T+\tau),K)\}$$
$$= \theta\lambda^w\phi_2(K - w(T+\tau)). \quad (19)$$

The expected total unplanned maintenance cost regarding policy 1 over planning horizon K is

$$E\{\hat{B}(2,T)\} = \theta\left(\sum_{k=1}^{w}\lambda^{k-1}\phi_2(T) + \lambda^w\phi_2(K-w(T+\tau))\right)$$
$$= \theta\left(\frac{(\lambda^w-1)}{\lambda-1}\phi_2(T) + \lambda^w\phi_2(K-w(T+\tau))\right) \quad (20)$$

By combining (14) and (20), the life cycle cost (the sum of planned maintenance cost and unplanned maintenance cost) over the finite horizon K is

$$LCC_K(2,T) = E\{\hat{B}(2,T)\} + E\{B(2,T)\}$$
$$= \theta\left(\frac{\lambda^w-1}{\lambda-1}\phi_2(T) + \lambda^w\phi_2(K-w(T+\tau))\right) + wc_0 + \frac{\beta^w-1}{\beta-1}\psi(T-\delta_2(T)) \quad (21)$$

for all $T \in (0,K]$, where

$$w = \begin{cases} \left[\frac{K}{T}\right] - 1, & \text{if } \frac{K}{T} \text{ is an integer}, \\ \left[\frac{K}{T}\right], & \text{if } \frac{K}{T} \text{ is not an integer}. \end{cases}$$

(Note that $[t]$ represents the largest integer that is less than or equal to t.)

4. Structural Characteristics

In order to study the structural characteristics, we rewrite the equation (21) as

$$LCC_K(2,T) = E\{\hat{B}(2,T)\} + E\{B(2,T)\}$$
$$= \theta\left(\frac{(\lambda^w-1}{\lambda-1}\phi_2(T) + \lambda^w\phi_2(K-w(T+\tau))\right) + wc_0 + \frac{\beta^w-1}{\beta-1}\psi(T-\delta_2(T))$$

for all $T \in (0,K]$, where

$$w = 0,1,2,\ldots, \frac{K-w\tau}{w+1} \leq T < \frac{K-w\tau}{w}. \quad (22)$$

Theorem 1: *If the time of minimal repair is negligible, $r_2(T)$ is strictly increasing in T and $\lambda, \beta > 1$ then*

(*a*) $LCC_K(2,T)$ is differentiable on $(0,K]$ excepting possibly at the points
$$\{K,(K-\tau)/2,(K-2\tau)/3,\ldots\} \cup \{a_{23},a_{24},\ldots,a_{2n}\};$$
(*b*) $LCC_K(2,T)$ is continuous except at the points $\{K,(K-\tau)/2,(K-2\tau)/3,\ldots\}$, and
(*c*) $LCC_K(2,T)$ is a piecewise convex function of T on $T \in (0,K)$, where $r_2(T)$ is the failure rate function for state 2.

Proof: Since the time of minimal repair is negligible, then $\delta_2(T) = \gamma\phi_2(T) \to 0$ ($\phi_2(T)$ is finite.) By (22) and the definition of function $\psi(T-\delta_2(T))$, we directly obtain the results of (a) and (b). Moreover, the first derivative of $LCC_K(2,T)$ at points other than $\{K,(K-\tau)/2,(K-2\tau)/3,\ldots\} \cup \{a_{23},a_{24},\ldots,a_{2n}\}$ is

$$\frac{\mathrm{d}}{\mathrm{d}T}LCC_K(2,T) = \theta\left(\frac{\lambda^w-1}{\lambda-1}r_2(T) - w\lambda^w r_2(K-w(T+\tau))\right) + \frac{\beta^w-1}{\beta-1}\psi'(T),$$

where

$$\psi'(T-\tau) = \begin{cases} \sum_{j=3}^{n-1} c_{j-2}\left(\frac{1}{a_{2j}}-\frac{1}{a_{2,j+1}}\right) + c_{n-2}\frac{1}{a_{2n}} + c_1\left(-\frac{1}{a_{23}}\right), & \text{if } T < a_{23}, \\ \left(c_1-\frac{c_1}{a_{24}}\right) + \sum_{j-4}^{n-1} c_{j-2}\left(\frac{1}{a_{2j}}-\frac{1}{a_{2,j+1}}\right) + c_{n-2}\frac{1}{a_{2n}}, & \text{if } a_{23} < T < a_{24}, \\ \vdots & \\ c_{n-3} + (c_{n-2}-c_{n-3})\frac{1}{a_{2n}}, & \text{if } a_{2,n-1} < T < a_{2n}, \\ 0, & \text{if } a_{2n} < T. \end{cases}$$

For a fixed w $(w \neq 0)$,

(i) $r_2(K-w(T+\tau)) \to 0$ if $T \to (K-w\tau)/w$, and $r_2(K-w(T+\tau)) \to r_2(T)$ if $T \to (K-w\tau)/w+1$,
(ii) $(\lambda^w-1)(\lambda-1) < w\lambda^w$ for each $\lambda > 1$ and $w \neq 0$,
(iii) $(\lambda^w-1)/(\lambda-1)r_2(T)$ is strictly increasing in T for $T \in ((K-w\tau)/(w+1),(K-w\tau)/w)$,
(iv) $w\lambda^w r_2(K-w(T+\tau))$ is strictly decreasing in T for $T \in ((K-w\tau)/(w+1),(K-w\tau)/w)$, and
(v) $(\beta^w-1)/(\beta-1)\psi'(T)$ is a positive constant for a fixed range of T.

Thus we obtain that $LCC_K(2,T)$ is a piecewise convex function of T on $T \in (0,K)$. □

Remark 2:

(i) A situation in which $\lambda > 1$ and $\beta > 1$, implies that states cannot precisely account for the system failure. A larger amount of planned maintenance actions regarding policy 2 implies a higher failure rate for the system under a specific underlying state. This phenomenon is referred to herein as the 'action-amount-dependent' case.
(ii) By Theorem 1, for each $w = 0,1,2,\ldots$, it is easy to obtain T_w^*, thereby minimizing

$$\theta\left(\frac{\lambda^w-1}{\lambda-1}\phi_2(T) + \lambda^w\phi_2(K-w(T+\tau))\right) + wc_0 + \frac{\beta^w-1}{\beta-1}\psi(T)$$

where $(K-w\tau)/(w+1) \leq T < (K-w\tau)/w$ We have $LCC_K(2,T) \to \infty$ if $w \to \infty$. Thus there exists a finite w^* such that $T_{w^*}^*$ is optimal to minimize $LCC_K(2,T)$.
(iii) This strong assumption that the repair time of minimal repair is negligible verifies that system failure is normally only the damage of a list of parts of a component. In addition, unplanned action 'minimal repair' only attempts to replace those failed parts and the time is usually negligible for any replacement action of failed parts. □

Theorem 2: *If the repair time of minimal repair is negligible, $r_2(T)$ is strictly increasing in T and $\lambda = \beta = 1$; then T^*, thereby minimizing $LCC_K(2,T)$, is at one of the points $\{K,(K-\tau)/2,(K-2\tau)/3,\ldots\}$.*

Proof: If the repair time of minimal repair is negligible and $\lambda = \beta = 1$, then $LCC_K(2,T)$ is written as

$$LCC_K(2,T) = \theta[w\phi_2(T) + \phi_2(K-w(T+\tau))] + w\psi(T)$$

for all $T \in (0,K]$, where

$$w = 0,1,2,\ldots,\frac{K-w\tau}{w+1} \leq T < \frac{K-w\tau}{w}.$$

The first derivative of $LCC_K(2,T)$ at points other than $\{K,(K-\tau)/2,(K-2\tau)/3,\ldots\} \cup \{a_{23},a_{24},\ldots,a_{2n}\}$ is

$$\frac{d}{dT}LCC_K(2,T) = \Theta[wr_2(T) - wr_2(K-w(T+\tau))] + wc_0 + w\psi'(T)$$

For a fixed w $(w \neq 0)$, since $r_2(T) > r_2(K-w(T+\tau))$ and $\psi'(T)$ is a positive constant, $LCC_K(2,T)$ is strictly increasing in T over $w = 1,2,\ldots$. Moreover, $LCC_K(2,T) = LCC_K(2,K)$ if $w = 0$. Therefore, T^* minimizes the $LCC_K(2,T)$ at one of the points $\{K,(K-\tau)/2,(K-2\tau)/3,\ldots\}$.

Remark 3:

(i) A situation in which $\lambda = 1$ and $\beta = 1$ implies that states can concisely account for the system failure. Moreover, the system deterioration process does not depend on the amount of maintenance actions regarding policy 2. This occurrence is referred to herein as the 'action-amount-independent' case.

(ii) By theorem 2, minimizing $LCC_K(2, T)$ reduces to minimizing

$$Q_K(w) = \theta\left\{(w+1)\phi_2\left(\frac{K-w\tau}{w+1}\right)\right\} + w\left(c_0 + \psi\left(\frac{K-w\tau}{w+1}\right)\right)$$

over $w = 0, 1, 2, \ldots$, that is

$$w^* = \arg\min_{w\in\{0,1,2,\ldots\}}\{Q_K(w)\}.$$

$Q_K(w) \to \infty$ when $w \geq M$ (M is a large integer). □

Therefore, given finite computation, the optimal solution can be obtained.

5. Numerical illustration

Consider a satellite maintenance problem. The satellite can be categorized as either of two modes: 'safe hold mode' and 'science mode'.The satellite in 'safe hold mode' implies its inability to perform the given missions owing to malfunctioning of components. The malfunctioning of components may originate from variations in environmental stress such as pressure, temperature and impacts of natural scenarios such as water impact and ground impact. Whenever the satellite is in 'safe hold mode'', a recovery procedure (an unplanned maintenance action) must be performed to reverse the 'safe hold mode' to the 'science mode (normal operations)'. Moreover, the 'science mode' can be categorized as finite novel states according to its technical properties. Let $S = \{1, 2, \ldots, n\}$ be the normal operational space, where 1 denotes the best science mode of the used satellite, and state n refers to the worst science mode of the used satellite. Given $n = 5$, that is $S = \{1, 2, 3, 4, 5\}$, assume that another new type will replace the satellite when the usage reaches 15 years ($K = 15$). Also, allow the planned action space $A = \{a_1, a_2, a_3\}$, where action a_1 costs c_1 that deterministically moves the satellite to state 2 from state 2 or 3. Action a_2 costs c_2 which deterministically moves the satellite to state 2 from state j, $j \in \{2, 3, 4\}$; action a_3 costs c_3 which deterministically moves the satellite to state 2 from state $j, j \in \{2, 3, 4, 5\}$. For easy computation and illustration, set $\gamma \to 0, \tau \to 0$ and then $\delta_2(T-\tau) = \gamma\phi_2(T-\tau) \to 0$. Moreover, assume that the interarrival times from state i of 'science mode' to 'safe hold mode' follows the Weibull distribution with a scale parameter $]_i$ and a shape parameter $\alpha = 2$. Restated, it has a probability density function

$$g_i(t) = 2]_i(]_i t)\exp([-(]_i t)^2] \quad \text{for } i \in S,]_i, t > 0,$$

and a cumulative failure rate function $\phi_i(y) = \int_0^y 2]_i(]_i t)\,dt = (]_i y)^2$.

By assuming that $c_1 = c_2$ (i.e. action a_1 is the same as action a_2) and $\lambda = \beta = 1$ (i.e. the failure rate of a system does not depend on the amount of maintenance regarding policy 2, and by theorem 2 (see Remark 3(ii)), minimizing $LCC_K(2, T)$ reduces to minimizing

$$\begin{aligned} Q_K(w) &= (w+1)\theta\phi_2\left(\frac{K}{w+1}\right) + wc_0 + w\psi\left(\frac{K}{w+1}\right) \\ &= (w+1)\theta\left(\frac{]_2 K}{(w+1)}\right)^2 + w(c_0 + c_1) \\ &\quad + w\frac{K}{a_{25}(w+1)}(c_3 - c_1) \end{aligned} \tag{23}$$

over $w = 0, 1, 2, \ldots$. Since

$$\frac{d}{dw}Q_K(w) = \frac{-\theta]_2^2K^2}{(w+1)^2} + c_0 + c_1 + \frac{K(c_3 - c_1)}{a_{25}(w+1)^2}.$$

where w is treated as a real number and if $(d/dw)Q_K(w) = 0$, then

$$w^* = \left(\frac{\theta]_2^2K^2 - K(c_3 - c_1)/a_{25}}{c_0 + c_1}\right)^{1/2} - 1.$$

Let

$$w_1 = \left(\frac{\theta]_2^2K^2 - K(c_3 - c_1)/a_{25}}{c_0 + c_1}\right)^{1/2} - 1 \tag{24}$$

and

$$w_2 = \left[\left(\frac{\theta]_2^2K^2 - K(c_3 - c_1)/a_{25}}{c_0 + c_1}\right)^{1/2} - 1\right] + 1 \tag{25}$$

Because w is an integer, the optimal maintenance number w^* can be rewritten as

$$w^* = \arg\min_{w\in\{w_1,w_2\}}\{Q_K(w)\}. \tag{26}$$

(Note that $Q_K(w)$ is a convex function of w.) Consequently, $T^* = K/(w^* + 1)$ minimizes the $LCC_K(2, T)$.

Given the other relevant numerical data $]_2 = 1.2$, $a_{25} = 1.3$, $\theta = $ US 120\$, $c_0 = $ US 5\$, $c_1 = $ US 85\$ $c_3 = $ US 400\$, on substitution into (24) and (25) yields, it is found that

$$w_1 = \left(\frac{120 \times 1.2^2 \times 15^2 - 15(400-85)/1.3}{5+85} \right)^{1/2} - 1$$

$$= [18.79] = 18$$

and

$$w_2 = \left[\left(\frac{120 \times 1.2^2 \times 15^2 - 15(400-85)/1.3}{5+85} \right)^{1/2} - 1 \right] + 1$$

$$= [18.39] + 1 = 19.$$

Also, substituting the above results for w_1 and w_2 into (23) leads to

$$Q_{15}(18) = (18+1) \times 120 \times \left(\frac{1.2 \times 15}{18+1} \right)^2 + 18 \times (5+85) + \frac{18 \times 15}{1.3(18+1)}(400-85)$$

$$= 7109.63$$

and

$$Q_{15}(19) = (19+1) \times 120 \times \left(\frac{1.2 \times 15}{19+1} \right)^2 + 19 \times (5+85) + \frac{19 \times 15}{1.3(19+1)}(400-85)$$

$$= 7106.88.$$

Therefore, according to (26), we obtain $w^* = \arg\min\{Q_{15}(18), Q_{15}(19)\} = 19$ and the optimal PM cycle time can be obtained as

$$T^* = \frac{K}{w^*+1} = \frac{15}{19+1} = 0.75(\text{years}) = 273.75(\text{days}).$$

This result suggests that the decision makers should implement a specific PM to recover the present state of the satellite to the extremely better state 2 for every 273.75 days until the satellite becomes technologically obsolete.

6. Discussion

Our model differs from the conventional periodic maintenance in the following ways:

(i) Most conventional periodic maintenance models have received extensive interest with respect to the infinite horizon (minimizing the cost rate function). However, because the system (e.g. the satellite) is technologically obsolete, a finite horizon should be considered (minimizing the life cycle cost).

(ii) For most conventional periodic maintenance models, a PM action to the condition of ‘as good as new’ or ‘a younger condition (imperfect maintenance)’ is performed for every T unit times. The failure rate and each PM cost depend on the system’s usage and/or the number of previous PMs. The imperfect maintenance seems to differentiate the various operational conditions depending on the various usages (e.g. using ‘the failure rate like usage t_0’ to represent an operational condition). However, if possible, it directly uses the characteristics of a system to categorize the operational state which may be better than usage. Restated, ‘multistate’ is more general than ‘usage’. The failure rate at any time interval can be more accurately estimated than the conventional models if the system’s operational state before this time interval is known. Also, the PM costs are independent of the system’s usage if the system’s current state is known. The variations in PM costs come from the differences in current operational states and the number of previous PMs. Consequently, using the multistate approach proposed herein provides further insight into some real world situations.

(iii) Although many imperfect-opportunistic maintenance models assume that a specific probability has perfect maintenance after PM, this paper considers that the system will reverse to an extremely better state after selecting a correct PM; the PM cost depends on the present state as well. In particular, if the number of PMs is not a significant factor of the system’s failure rate (action-amount-independent case), the PM is viewed as perfect maintenance. However, if the number of PMs is a significant factor (action-amount-dependent case), the PM is still imperfect.

Most maintenance problems with the system’s own mechanical properties are considered as the action-amount-dependent maintenance cases, for example airplanes and boats. The cases of action-amount-dependent maintenance are usually understood by most people. To illustrate a real example of action-amount-independent maintenance, we consider a manufacturing process in which a perfect product requires conditions which are as sterile as possible. We define the manufacturing process as a failure when the manufacturing process cannot be performed owing to equipment failure. If the manufacturing process is a failure, then the failed equipment is replaced (minimal repair of the manufacturing process). Moreover, the state 1 of normal operations represents the fact that there are no defective items for a period of time, state 2 denotes that the defective items in the manufacturing process converge to zero, and state n refers to when the number of expected defective items exceed a fixed number M for a period of time. Since the manufacturing process to produce a perfect item requires conditions that are as sterile as possible, it is reasonable to assume that the action space consists of different level of decontamination. In addition, the deterioration of the manufacturing process does not depend on the amount of decontamination because mechanical properties are not considered. Therefore, the 'action-amount-independent' model described in this study can be applied.

7. Conclusions

This study presents a periodic PM model for a multistate system over a finite horizon to fill the gap between multistate-based Markov maintenance models and conventional periodic maintenance models. The results presented herein demonstrate that, under the assumption of action-amount-independent maintenance, the model herein reduces to a simple form and has more easily computable structural characteristics to obtain the optimal PM cycle time. Without this assumption, the structural characteristics are much weaker. However, for the action-amount-dependent maintenance case, we can recast the problem in a form that permitted computation of the optimal PM cycle time by solving finite number of convex functions. The proposed model can be more easily understood and implemented than Markov-chain-based models. Also, owing to the consideration of multistate, this model is more general than conventional periodic replacement–maintenance models. To provide useful assistance to owners, the results presented herein must be extended in several ways. More specifically, only the repairable is considered herein. If the proportion of non-repairable failure is non-negligible, future efforts should concentrate on developing a useful model and solution procedure to resolve this problem.

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