JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 114 (2000) 333–343

www.elsevier.nl/locate/cam

An a posteriori finite element error analysis for the Stokes equations ☆

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Received 22 December 1998; received in revised form 27 May 1999

Abstract

Conforming and nonconforming error estimators are presented and analyzed for the mixed finite element approximation of the Stokes problem. The estimators are obtained by solving local Poisson-type problems that do not involve boundary conditions, compatibility and balancing conditions, incompressibility constraint, or flux jumps across inter-element boundaries. The estimators are bounded from above and below by constant multiples of the actual error in an energy-like norm and can be used in adaptive h, p, and hp computations. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 65N15; 65N30

Keywords: Stokes equation; Finite element; A posteriori error estimate

1. Introduction

Adaptive computation with reliable error controls is now a practical reality in modern numerical simulations. A posteiori error estimation is a driving force of adaptive computation and is an essential part of error controls. Error estimators are well established for self-adjoint, elliptic partial differential equations. However, the issues particularly associated with the mixed finite element approximation of the Stokes equations such as incompressibility condition, inf–sup condition, equilibrium (compatibility) condition and boundary conditions of the local residual problems, etc. make the error estimation more difficult and complicated than that of self-adjoint PDEs.

Error estimators for the Stokes problem can be categorized into two classes. The first class (e.g. [11,23,24]) is based on the local evaluation of residuals whereas the second one is based on the

☆ This work was supported by NSC under grant NSC87-2115-M-009-005, Taiwan

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PII: S0377-0427(99)00271-X

solution of local problems. The second class can be further divided into two subclasses. One of which is dealing with local Stokes-type problems [10,23] whereas the other is dealing with local Poisson-type problems [2,21]. The first class is less expensive but provides only refinement indicators. The second class gives more accurate global error estimators as well as local error indicators. The present estimators belong to the last subclass.

Our formulation of the local problems reduces all the above issues to only one condition on the construction of the complementary finite element subspaces for the approximation of the local problems. Both resulting conforming and nonconforming estimators are shown to be bounded from above and below by constant multiples of the actual error in an energy-like norm.

We summarize the main features of the present work as follows:

- The estimators are independent of the type of element used to approximate the original Stokes problem and therefore can be used in all adaptive h, p, and hp computations.
- The formulation does not explicitly involve the flux jumps across inter-element boundaries and do not require any local boundary conditions.
- The size of the system of linear equations resulting from the local problems is the smallest when compared with all the previous systems of the second class.
- The error analysis presented here is in finite-dimensional setting whereas that of Ainsworth and Oden [2] is in infinite setting.

2. Preliminaries

We consider a mixed finite element approximation of the Stokes equations

$$\begin{aligned} &-\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega,\\ &\operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, \\ &\boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is an open bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$, $\nu > 0$ a viscosity parameter, $\boldsymbol{u} \in (H^1(\Omega))^2$ the velocity field, $p \in L^2(\Omega)$ the pressure, and $\boldsymbol{f} : \Omega \to \mathbb{R}^2$ the body force. Here $H^r(\Omega)$, $r \in \mathbb{R}$, denotes a usual Sobolev space equipped with the norm $\|\cdot\|_r$ and with the semi-norm $|\cdot|_r$.

To formulate (2.1) in a weak form, we use the following notation:

$$\begin{aligned} &\boldsymbol{H} = (H_0^1(\Omega))^2,\\ &H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\},\\ &|\boldsymbol{v}|_1^2 = \int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \sum_{i,j=1}^2 \left(\frac{\partial v_i}{\partial x_j}\right)^2 \mathrm{d}x,\\ &\mathscr{M} = \left\{q \in L^2(\Omega) : \int_\Omega q\,\mathrm{d}x = 0\right\}, \end{aligned} \tag{2.2}$$

$$
\begin{aligned}
&a : \boldsymbol{H} \times \boldsymbol{H} \to \mathbb{R}, \quad a(\boldsymbol{u},\boldsymbol{v}) = v \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, \mathrm{d}x, \\
&b : \mathscr{M} \times \boldsymbol{H} \to \mathbb{R}, \quad b(q,\boldsymbol{v}) = \int_\Omega q \operatorname{div} \boldsymbol{v} \, \mathrm{d}x, \\
&c : \mathscr{M} \times \mathscr{M} \to \mathbb{R}, \quad c(p,q) = \int_\Omega pq \, \mathrm{d}x, \\
&L : \boldsymbol{H} \to \mathbb{R}, \quad L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x,
\end{aligned} \tag{2.3}
$$

where $\mathrm{d}x = \mathrm{d}x_1 \, \mathrm{d}x_2$. The weak formulation of the Stokes problem is to find $(\boldsymbol{u}, p) \in \boldsymbol{H} \times \mathscr{M}$ such that

$$
\begin{aligned}
&a(\boldsymbol{u},\boldsymbol{v}) - b(p,\boldsymbol{v}) = L(\boldsymbol{v}) \ \forall \boldsymbol{v} \in \boldsymbol{H}, \\
&b(q,\boldsymbol{u}) = 0 \ \forall q \in \mathscr{M}.
\end{aligned} \tag{2.4}
$$

The existence and uniqueness of a solution $(\boldsymbol{u}, p)$ of the weak problem is guaranteed (see e.g. [15]) since $L(\cdot)$ is continuous and $a(\cdot,\cdot)$ is continuous and coercive while $b(\cdot,\cdot)$ is continuous and satisfies an inf–sup condition, namely, there exist positive constants α_1, α_2 and β such that

$$
\begin{aligned}
&a(\boldsymbol{v},\boldsymbol{v}) = v|\boldsymbol{v}|_1^2 \quad \forall \boldsymbol{v} \in \boldsymbol{H}, \\
&a(\boldsymbol{u},\boldsymbol{v}) \leqslant \alpha_1 \, |\boldsymbol{u}|_1 \, |\boldsymbol{v}|_1 \quad \forall \boldsymbol{u}, \ \boldsymbol{v} \in \boldsymbol{H}, \\
&\inf_{q \in \mathscr{M}, q \neq 0} \ \sup_{\boldsymbol{v} \in \boldsymbol{H}, \boldsymbol{v} \neq \boldsymbol{0}} \frac{b(q,\boldsymbol{v})}{\|q\|_0 |\boldsymbol{v}|_1} \geqslant \beta, \\
&b(q,\boldsymbol{v}) \leqslant \alpha_2 \, \|q\|_0 |\boldsymbol{v}|_1, \quad q \in \mathscr{M}, \quad \boldsymbol{v} \in \boldsymbol{H}.
\end{aligned} \tag{2.5}
$$

We note that the uniqueness of the pressure p is unique up to an additive constant.

For the approximation of (2.4), we introduce two families of finite-dimensional subspaces $\boldsymbol{H}_h \subset \boldsymbol{H}$ and $\mathscr{M}_h \subset \mathscr{M}$ which are associated with a partition $T_h = \{t_i \mid i = 1, 2, \ldots, m\}$ on $\bar{\Omega}$. The partition is characterized by a mesh size h. For any two distinct elements (triangles or rectangles or both) t_i and t_j in T_h, $t_i \cap t_j$ is either empty, a single vertex, or a common edge. Two elements are said to be adjacent if they have a common edge. For a given rectangular element let $h_{\max}$ and $h_{\min}$ denote the largest and smallest edge lengths, respectively. Then the element edge ratio is defined by $h_{\min}/h_{\max}$. We always assume that the mesh T_h belongs to a *regular* family of structured or unstructured meshes on $\bar{\Omega}$. Recall that, see e.g. [3,7], the family is regular if all angles of its triangular elements and all edge ratios of rectangular elements are bounded below by some constant $\sigma > 0$. The constant is called *shape regularity parameter* of the mesh T_h. Shape regularity does not require a mesh to be globally quasi-uniform, but it does imply local quasi-uniformity of the mesh.

The mixed finite element approximation of (2.4) is to find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times \mathscr{M}_h$ such that

$$
\begin{aligned}
&a(\boldsymbol{u}_h,\boldsymbol{v}) - b(p_h,\boldsymbol{v}) = L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_h, \\
&b(q,\boldsymbol{u}_h) = 0 \quad \forall q \in \mathscr{M}_h.
\end{aligned} \tag{2.6}
$$

We note that if the finite-dimensional spaces $\boldsymbol{H}_h$ and $\mathscr{M}_h$ satisfy the discrete inf–sup condition, i.e., there exists a constant $\beta_h > 0$ such that

$$\inf_{q\in\mathscr{M}_h, q\neq 0} \sup_{\boldsymbol{v}\in\boldsymbol{H}_h, \boldsymbol{v}\neq\boldsymbol{0}} \frac{b(q,\boldsymbol{v})}{\|q\|_0|\boldsymbol{v}|_1} \geqslant \beta_h, \tag{2.7}$$

the existence and uniqueness of the solution of (2.6) are then ensured. The solution is stable if β_h is independent of h. Otherwise, or, even worse, (2.7) does not hold, the solution is unstable. Nevertheless, exactly the same as that of Ainsworth and Oden [2], condition (2.7) is not required for our error estimation. Of course, it is impractical to use unstable approximation.

3. A conforming error estimator

Usually, the norm on $\boldsymbol{H}\times\mathscr{M}$ can be taken as

$$\|(\boldsymbol{v},q)\|^2_{\boldsymbol{H}\times\mathscr{M}} = |\,\|\boldsymbol{v}\|\,|^2 + \|q\|_0^2 \quad \forall \boldsymbol{v}\in\boldsymbol{H}, \quad q\in\mathscr{M},$$

where

$$|\,\|\boldsymbol{v}\|\,|^2 = a(\boldsymbol{v},\boldsymbol{v}) = \nu|\boldsymbol{v}|_1^2.$$

Following [2], the discretization error $(\boldsymbol{e}_u, e_p)\in\boldsymbol{H}\times\mathscr{M}$, where $\boldsymbol{e}_u = \boldsymbol{u}-\boldsymbol{u}_h$ and $e_p = p - p_h$, defines a pair $(\boldsymbol{e},\eta)\in\boldsymbol{H}\times\mathscr{M}$ such that

$$a(\boldsymbol{e},\boldsymbol{v}) = a(\boldsymbol{e}_u,\boldsymbol{v}) - b(e_p,\boldsymbol{v}) = L(\boldsymbol{v}) - a(\boldsymbol{u}_h,\boldsymbol{v}) + b(p_h,\boldsymbol{v}), \tag{3.1}$$

$$c(\eta,q) = -b(q,\boldsymbol{e}_u) = b(q,\boldsymbol{u}_h) \tag{3.2}$$

for all $(\boldsymbol{v},q)\in\boldsymbol{H}\times\mathscr{M}$. Moreover, we have the equivalence relation

$$C_1\|(\boldsymbol{e},\eta)\|_{\boldsymbol{H}\times\mathscr{M}} \leqslant \|(\boldsymbol{e}_u,e_p)\|_{\boldsymbol{H}\times\mathscr{M}} \leqslant C_2\|(\boldsymbol{e},\eta)\|_{\boldsymbol{H}\times\mathscr{M}}. \tag{3.3}$$

The computation of the residual norm

$$\|\eta\|_0 = \|\operatorname{div}\boldsymbol{u}_h\|_0 = \left\{\sum_{t_i\in T_h}\int_{t_i}(\operatorname{div}\boldsymbol{u}_h)^2\,\mathrm{d}x\right\}^{1/2} \tag{3.4}$$

is straightforward. The work is left to estimate $|\,\|\boldsymbol{e}\|\,|$.

Since

$$a(\boldsymbol{e},\boldsymbol{v}) = a(\boldsymbol{e}_u,\boldsymbol{v}) - b(e_p,\boldsymbol{v}) = L(\boldsymbol{v}) - a(\boldsymbol{u}_h,\boldsymbol{v}) + b(p_h,\boldsymbol{v}) = 0 \tag{3.5}$$

for all $(\boldsymbol{v},q)\in\boldsymbol{H}_h\times\mathscr{M}_h$, we should consider the discrete problem of (3.1) in a richer space $\boldsymbol{H}_{\bar h}$, $\boldsymbol{H}_h\subset\boldsymbol{H}_{\bar h}\subset\boldsymbol{H}$, namely, determine $\bar{\boldsymbol{e}}\in\boldsymbol{H}_{\bar h}$ such that

$$a(\bar{\boldsymbol{e}},\boldsymbol{v}) = L(\boldsymbol{v}) - a(\boldsymbol{u}_h,\boldsymbol{v}) + b(p_h,\boldsymbol{v}) \tag{3.6}$$

for all $\boldsymbol{v}\in\boldsymbol{H}_{\bar h}$.

Note that (3.6) is a standard finite element approximation of (3.1) and that the bilinear form $a(\cdot,\cdot)$ is symmetric. It follows Céa's Lemma [13] that

$$|\,\|\boldsymbol{e}-\bar{\boldsymbol{e}}\|\,| = \inf_{\boldsymbol{v}\in\boldsymbol{H}_{\bar h}} |\,\|\boldsymbol{e}-\boldsymbol{v}\|\,|. \tag{3.7}$$

Let $\boldsymbol{e}_0 \in \boldsymbol{H}_h$ denote the solution of (3.6) for all $v \in \boldsymbol{H}_h$. By (3.5), it is a trivial solution, i.e., $\boldsymbol{e}_0 = 0$. Hence, by (3.7) we have

$$| \, \|\boldsymbol{e} - \bar{\boldsymbol{e}}\| \, | \leqslant \rho | \, \|\boldsymbol{e}\| \, | \tag{3.8}$$

with $\rho \leqslant 1$. The inequality will assume the equality if and only if

$$a(\bar{\boldsymbol{e}}, \bar{\boldsymbol{e}}) = 2a(\boldsymbol{e}, \bar{\boldsymbol{e}}) = 2a(\bar{\boldsymbol{e}}, \bar{\boldsymbol{e}}) = 0$$

which implies that $\rho < 1$ provided $\boldsymbol{H}_{\bar{h}} \neq \boldsymbol{H}_h$. This suggests that the enlarged space $\boldsymbol{H}_{\bar{h}}$ can be defined on the current mesh T_h, i.e., more basis functions that do not belong to $\boldsymbol{H}_h$ are constructed on the present mesh without any re-meshing. These functions constitute a complementary subspace $\boldsymbol{H}_h^{\mathrm{c}}$ to $\boldsymbol{H}_h$ in $\boldsymbol{H}_{\bar{h}}$. Apparently, for any fixed mesh or equivalently any fixed mesh parameter h, the constant ρ is independent of the h and depends only on how many or how these complementary basis functions are constructed as long as $\boldsymbol{H}_h^{\mathrm{c}} \neq \emptyset$. We thus define the enlarged space by

$$\boldsymbol{H}_{\bar{h}} = \boldsymbol{H}_h \oplus \boldsymbol{H}_h^{\mathrm{c}}, \qquad \boldsymbol{H}_h \cap \boldsymbol{H}_h^{\mathrm{c}} = \{0\}, \qquad \boldsymbol{H}_h^{\mathrm{c}} \neq \emptyset \tag{3.9}$$

such that (3.8) holds for $\rho < 1$ independent of h. This is a saturation assumption frequently used in a posteriori error analysis [7,8,16].

Since (3.6) is a Poisson-type problem, the space $\boldsymbol{H}_{\bar{h}}$ can be defined via, for instance, the standard hierarchical basis functions on T_h [22] without any stability restriction.

The definition of the enlarged space also implies the strengthened Cauchy–Schwarz inequality (3.10) for which a proof can be found, for example, in [7,14].

Lemma 3.1. *Let $\boldsymbol{H}_{\bar{h}}$ be defined by* (3.9). *Then there exists a constant $\gamma \in [0,1)$ independent of the mesh size h such that*

$$|a(\boldsymbol{v}, \boldsymbol{w})| \leqslant \gamma | \, \|\boldsymbol{v}\| \, | \, | \, \|\boldsymbol{w}\| \, | \; \forall \boldsymbol{v} \in \boldsymbol{H}_h, \quad \forall \boldsymbol{w} \in \boldsymbol{H}_h^{\mathrm{c}}. \tag{3.10}$$

The approximation of $\boldsymbol{e}$ can be reduced to solving the *conforming* error problem: Determine $\boldsymbol{e}^{\mathrm{c}} \in \boldsymbol{H}_h^{\mathrm{c}}$ such that

$$a(\boldsymbol{e}^{\mathrm{c}}, \boldsymbol{v}) = L(\boldsymbol{v}) - a(\boldsymbol{u}_h, \boldsymbol{v}) + b(p_h, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_h^{\mathrm{c}}. \tag{3.11}$$

By (3.8) and (3.10) and the standard argument given, for instance, in [7,8,12,16,19], we have the following result.

Lemma 3.2. *Let $\boldsymbol{e}$ be defined by* (3.1). *If* (3.9) *holds for $\boldsymbol{H}_{\bar{h}}$, then* (3.11) *has a unique nontrivial solution $\boldsymbol{e}^{\mathrm{c}} \in \boldsymbol{H}_h^{\mathrm{c}}$ and*

$$(1-\rho)\sqrt{1-\gamma^2} | \, \|\boldsymbol{e}\| \, | \leqslant | \, \|\boldsymbol{e}^{\mathrm{c}}\| \, | \leqslant (1+\rho) | \, \|\boldsymbol{e}\| \, |, \tag{3.12}$$

where the constants $\rho, \gamma \in [0,1)$ are independent of the mesh size h.

The a posteriori (conforming) error estimate for the mixed finite element solution of (2.6) is then a direct consequence of (3.3) and (3.12).

Theorem 3.3. *Let* $(\boldsymbol{u}, p) \in \boldsymbol{H} \times \mathscr{M}$ *and* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times \mathscr{M}_h$ *be the solutions of* (2.4) *and* (2.6), *respectively. Let* $(\boldsymbol{e}_u, e_p) = (\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)$. *If* (3.9) *holds for* $\boldsymbol{H}_{\bar{h}}$, *then* (3.11) *has a unique nontrivial solution* $\boldsymbol{e}^c \in \boldsymbol{H}_h^c$ *and*

$$\frac{(1-\rho)\sqrt{1-\gamma^2}}{C_2}\|(\boldsymbol{e}_u, e_p)\|_{\boldsymbol{H}\times\mathscr{M}} \leqslant \|(\boldsymbol{e}^c, \eta)\|_{\boldsymbol{H}\times\mathscr{M}} \leqslant \frac{1+\rho}{C_1}\|(\boldsymbol{e}_u, e_p)\|_{\boldsymbol{H}\times\mathscr{M}}, \tag{3.13}$$

where the constants $\rho,\ \gamma \in [0,1)$ *and* $C_1,\ C_2 \in (0,\infty)$ *are all independent of the mesh size* h.

4. A nonconforming error estimator

Although the residual norm of (3.4) can be calculated on an element-by-element basis, the computation of $\boldsymbol{e}^c \in \boldsymbol{H}_h^c$ in (3.11) will result in a global solution if the basis functions of $\boldsymbol{H}_h^c$ have supports on more than one element. The error estimator will be impractical when the complementary space $\boldsymbol{H}_h^c$ is large as required for reliable error estimation. On the other hand, if the basis functions have supports only on their individual elements, the error estimator will not be effective to handle the errors across elements (flux jumps). In fact, most widely used error estimators, see e.g. [1,4–7,9,20,21,24,25], explicitly involve the jumps.

The nonconforming approach is devised to consider both efficient and effective aspects of practical error estimators. We now describe our nonconforming error estimator for the mixed FEM. It does not explicitly involve the jumps but retains the use of the weak residual term in (3.11). The errors occurring on the edges of elements are handled indirectly by a proper modification of the basis functions of the conforming complementary space $\boldsymbol{H}_h^c$ for which we need to be more specific.

For simplicity, we assume that the approximation space $\boldsymbol{H}_h$ consists of piecewise linears. The following results hold for more general approximation with some technical modifications. For any fixed mesh T_h, we construct a set $\mathscr{K}$ of shape functions such that each function $\phi \in \mathscr{K}$ has its support on a pair of two adjacent elements and its nodal point at the center of the common edge of the pair, namely, these functions are side modes [22]. Let

$$\boldsymbol{K} = \left\{\Phi\colon\ \Phi = \begin{pmatrix}\phi\\0\end{pmatrix} \text{ or } \begin{pmatrix}0\\\phi\end{pmatrix} \text{ for } \phi \in \mathscr{K}\right\},$$

$$\boldsymbol{K}(t_i) = \{\Phi\colon\ \Phi \in \boldsymbol{K},\ \operatorname{supp}(\Phi) \cap t_i \neq \emptyset,\ t_i \in T_h\}.$$

Define the conforming and nonconforming subspaces

$$\boldsymbol{H}_h^c = \operatorname{span}\{\Phi\colon\ \Phi \in \boldsymbol{K}\} \subset \boldsymbol{H}, \tag{4.1}$$

$$\boldsymbol{H}_h^c(t_i) = \operatorname{span}\{\Phi\colon\ \Phi \in \boldsymbol{K}(t_i)\} \subset \boldsymbol{H}_h^c \subset \boldsymbol{H}, \tag{4.2}$$

$$\boldsymbol{H}_h^n(t_i) = \operatorname{span}\left\{\tilde{\Phi}\colon\ \tilde{\Phi} = \begin{cases}\Phi \in \boldsymbol{K}(t_i) & \text{on } t_i,\\ 0 & \text{otherwise,}\end{cases}\right\} \not\subset \boldsymbol{H}, \tag{4.3}$$

$$\boldsymbol{H}_h^n = \boldsymbol{H}_h^n(t_1) \oplus \boldsymbol{H}_h^n(t_2) \oplus \cdots \oplus \boldsymbol{H}_h^n(t_m) \not\subset \boldsymbol{H}. \tag{4.4}$$

Exploiting the property of the shape regularity of T_h and the finite dimensionality $\boldsymbol{H}_h^c(t_i)$, the following lemma can be proved in a similar way as that of Bornemann et al. [12].

Lemma 4.1. *For any two adjacent elements t_i, $t_j \in T_h$, there exists a positive constant C_3 independent of h such that*

$$|\,\|\boldsymbol{v}_i^{\rm c}\|\,|_{t_j}^2 \leqslant C_3 |\,\|\boldsymbol{v}_i^{\rm c}\|\,|_{t_i}^2, \quad \forall \boldsymbol{v}_i^{\rm c} \in \boldsymbol{H}_h^{\rm c}(t_i), \tag{4.5}$$

where $|\,\|\boldsymbol{v}_i^{\rm c}\|\,|_{t_i}^2 = v\int_{t_i} \nabla \boldsymbol{v}_i^{\rm c} : \nabla \boldsymbol{v}_i^{\rm c}\,{\rm d}x$.

We now localize Eq. (3.11) on each element t_i as follows. The test and trial functions on the left-hand side are taken to be the halved-functions of $\boldsymbol{H}_h^n(t_i)$ while the test functions on the right-hand side are still conforming. More specifically, the nonconforming error estimator is obtained by determining $\tilde{\boldsymbol{e}}_i \in \boldsymbol{H}_h^n(t_i)$ such that, for each $t_i \in T_h$,

$$a(\tilde{\boldsymbol{e}}_i, \boldsymbol{v}_i^n) = \tfrac{1}{2}(L(\boldsymbol{v}_i^{\rm c}) - a(\boldsymbol{u}_h, \boldsymbol{v}_i^{\rm c}) + b(p_h, \boldsymbol{v}_i^{\rm c})) \quad \forall \boldsymbol{v}_i^n \in \boldsymbol{H}_h^n(t_i), \tag{4.6}$$

where $\boldsymbol{v}_i^{\rm c} = \boldsymbol{v}_i^n$ on t_i and $\boldsymbol{v}_i^{\rm c} \in \boldsymbol{H}_h^{\rm c}(t_i)$. Note that the conforming function $\boldsymbol{v}_i^{\rm c}$ is obtained by extending the basis functions of $\boldsymbol{v}_i^n$ from the element t_i to its neighbors. The factor $\frac{1}{2}$ on the right-hand side reflects the residual contribution to the element t_i since every basis function of $\boldsymbol{H}_h^{\rm c}(t_i)$ has its support on two adjacent elements and the size of the two elements does not differ too much, i.e., the mesh T_h is locally quasi-uniform. The uniqueness and existence of $\tilde{\boldsymbol{e}}_i$ is guaranteed since the bilinear form $a(\cdot,\cdot)$ induces a norm in the space $\boldsymbol{H}_h^n(t_i)$ and the space itself is finite dimensional. Thus

$$a(\boldsymbol{v}_i^n, \boldsymbol{w}_i^n) = v\int_{t_i} \nabla \boldsymbol{v}_i^{\rm c} : \nabla \boldsymbol{w}_i^{\rm c}\,{\rm d}x, \quad |\,\|\boldsymbol{v}_i^n\|\,|^2 = a(\boldsymbol{v}_i^n, \boldsymbol{v}_i^n)\ \forall \boldsymbol{v}_i^n, \boldsymbol{w}_i^n \in \boldsymbol{H}_h^n(t_i).$$

Define

$$|\,\|\boldsymbol{e}^n\|\,| := \left(\sum_{i=1}^m |\,\|\tilde{\boldsymbol{e}}_i\|\,|^2\right)^{1/2}, \qquad \boldsymbol{e}^n = \tilde{\boldsymbol{e}}_1 \oplus \tilde{\boldsymbol{e}}_2 \oplus \cdots \oplus \tilde{\boldsymbol{e}}_m. \tag{4.7}$$

Lemma 4.2. *Let* $\boldsymbol{e}$ *be defined by* (3.1). *If* (3.9) *holds for* $\boldsymbol{H}_{\bar{h}}$ *and* $\boldsymbol{H}_h^{\rm c}$ *and* $\boldsymbol{H}_h^n$ *are defined, respectively, by* (4.1) *and* (4.4), *then*

$$(1-\rho)\sqrt{1-\gamma^2}\,|\,\|\boldsymbol{e}\|\,| \leqslant |\,\|\boldsymbol{e}^n\|\,| \leqslant C_4(1+\rho)\,|\,\|\boldsymbol{e}\|\,|, \tag{4.8}$$

where C_4 is a positive constant, ρ and γ are given in Lemma 3.2, *and all the constants are independent of the mesh size h.*

Proof. Let $\boldsymbol{e}^{\rm c} \in \boldsymbol{H}_h^{\rm c}$ be the solution of (3.11). Define $\boldsymbol{e}_i^{\rm c} \in \boldsymbol{H}_h^{\rm c}(t_i)$ and $\boldsymbol{e}_i^n \in \boldsymbol{H}_h^n(t_i)$ such that

$$\boldsymbol{e}_i^{\rm c} = \boldsymbol{e}_i^n = \boldsymbol{e}^{\rm c} \quad \text{on } t_i.$$

It then follows from (3.11) and (4.6) that

$$\begin{aligned}
|\,\|\boldsymbol{e}^{\rm c}\|\,|^2 &= a(\boldsymbol{e}^{\rm c}, \boldsymbol{e}^{\rm c}) \\
&= \frac{1}{2} a\left(\boldsymbol{e}^{\rm c}, \sum_i \boldsymbol{e}_i^{\rm c}\right) \\
&= \frac{1}{2}\sum_i a(\boldsymbol{e}^{\rm c}, \boldsymbol{e}_i^{\rm c})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_i a(\tilde{e}_i, e_i^n) \\
&\leqslant \sum_i |\,\|\tilde{e}_i\|\,|\,|\,\|e_i^n\|\,| \\
&= \sum_i |\,\|\tilde{e}_i\|\,|\,|\,\|e_i^c\|\,|_{t_i} \\
&\leqslant \frac{1}{2}\sum_i (|\,\|\tilde{e}_i\|\,|^2 + |\,\|e^c\|\,|_{t_i}^2) \\
&= \tfrac{1}{2}(|\,\|e^n\|\,|^2 + |\,\|e^c\|\,|^2),
\end{aligned}
$$

where the first factor $\frac{1}{2}$ accounts for a twice integration over each element. Hence

$$|\,\|e^c\|\,| \leqslant |\,\|e^n\|\,|,$$

which together with (3.12) proves the left inequality of (4.8). Similarly, for the right inequality, we have

$$
\begin{aligned}
|\,\|\tilde{e}_i\|\,|^2 &= a(\tilde{e}_i, \tilde{e}_i) \\
&= a(\tilde{e}_i, \tilde{e}_i^c) \\
&= \tfrac{1}{2}a(e^c, \tilde{e}_i^c) \\
&\leqslant \tfrac{1}{2}|\,\|e^c\|\,|_{S_i}|\,\|\tilde{e}_i^c\|\,|_{S_i},
\end{aligned}
$$

where $\tilde{e}_i^c \in H_h^c(t_i)$ and S_i are defined by

$$\tilde{e}_i^c = \tilde{e}_i \quad \text{on } t_i,$$

$$S_i = t_i \cup_{j\in J} t_j.$$

Here J is an index set of $j \neq i$ such that $t_j \in T_h$ is adjacent to t_i. Note that, by Lemma 4.1, we have

$$
\begin{aligned}
|\,\|\tilde{e}_i^c\|\,|_{S_i}^2 &= |\,\|\tilde{e}_i^c\|\,|_{t_i}^2 + \sum_{j\in J} |\,\|\tilde{e}_i^c\|\,|_{t_j}^2 \\
&= |\,\|\tilde{e}_i\|\,|^2 + \sum_{j\in J} |\,\|\tilde{e}_i^c\|\,|_{t_j}^2 \\
&\leqslant |\,\|\tilde{e}_i\|\,|^2 + C_3 \sum_{j\in J} |\,\|\tilde{e}_i^c\|\,|_{t_i}^2 \\
&= |\,\|\tilde{e}_i\|\,|^2 + 4C_3|\,\|\tilde{e}_i^c\|\,|_{t_i}^2 \\
&= (1+4C_3)|\,\|\tilde{e}_i\|\,|^2.
\end{aligned}
$$

Here we have used the fact that each element has at most four adjacent elements. Hence,

$$|\,\|\tilde{e}_i\|\,| \leqslant \frac{\sqrt{1+4C_3}}{2}|\,\|e^c\|\,|_{S_i}.$$

Therefore,

$$
\begin{aligned}
|\,\|e^n\|\,|^2 &= \sum_{i=1}^{m} |\,\|\tilde{e}_i\|\,|^2 \\
&\leqslant \frac{1+4C_3}{4} \sum_{i=1}^{m} |\,\|e^{c}\|\,|_{S_i}^2 \\
&= \frac{5(1+4C_3)}{4} |\,\|e^{c}\|\,|^2
\end{aligned}
$$

which together with (3.12) implies the right inequality of (4.8) with $C_4 = \sqrt{5(1+4C_3)/4}$. □

Following (3.3) and (4.8), we now state the main result of the nonconforming error estimator.

Theorem 4.3. *Let* $(\boldsymbol{u}, p) \in \boldsymbol{H} \times \mathcal{M}$ *and* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times \mathcal{M}_h$ *be the solutions of* (2.4) *and* (2.6), *respectively. Let* $(\boldsymbol{e}_u, e_p) = (\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)$. *If* (3.9) *holds for* $\boldsymbol{H}_{\bar{h}}$ *and* $\boldsymbol{H}_h^{c}$ *and* $\boldsymbol{H}_{\bar{h}}^{n}$ *are defined, respectively, by* (4.1) *and* (4.4), *then the computable error* $\boldsymbol{e}^n$ *is uniquely determined by* (4.6) *and it satisfies the estimate*

$$
\frac{(1-\rho)\sqrt{1-\gamma^2}}{C_2} \|(\boldsymbol{e}_u, e_p)\|_{\boldsymbol{H}\times\mathcal{M}} \leqslant \|(\boldsymbol{e}^n, \eta)\|_{\boldsymbol{H}\times\mathcal{M}} \leqslant \frac{C_4(1+\rho)}{C_1} \|(\boldsymbol{e}_u, e_p)\|_{\boldsymbol{H}\times\mathcal{M}}, \tag{4.9}
$$

where the constants ρ, $\gamma \in [0,1)$ *and* $C_1, C_2, C_4 \in (0,\infty)$ *are all independent of the mesh size* h.

5. Concluding remarks

If both conforming and nonconforming formulas, i.e., (3.11) and (4.6) are used, we obtain a generic method for all h, p, and hp adaptive computations. For example, assuming that the approximation order is p, if the next hierarchical shape functions of degree $p+1$ are internal modes [22], we use (3.11) to compute the error estimator. Otherwise, we use (4.6) for side modes.

Comparison among the first and second classes of error estimators has been addressed in [2]. We now stress the differences between the present estimators and that of [2,21] denoted by AO-estimator for convenience.

First of all, we note that our error problems do not explicitly contain the flux jumps which inevitably obliges one to solve an auxiliary system that balances interior and boundary residuals between two neighboring elements since both residuals appear on the right-hand side of the local equations for the AO-estimator. The balancing system is also a consequence of the compatibility condition which is a necessary and sufficient condition for the existence of the local problems. The extended shape functions on the residual term of (4.6) to the neighboring elements take both residuals into account. The residual term is in weak form which obviously involves the gradients of the computed solution on the subdomain associated with any particular element. In other words, the flux jumps are implicitly handled by the extended shape functions. The existence and uniqueness of the local problems (4.6) is solely determined by the complementary space in (3.9). In summary, the weak residual formulation of (3.11) and (4.6) transforms all the previous balancing and compatibility conditions to the construction of the complementary basis functions and their extensions.

We next remark on the a posteriori estimates (3.13) and (4.9). The most significant difference between our estimates and that of Ainsworth and Oden [2] is that our analysis is based on finite-dimensional local problems whereas the latter is in an infinite dimensional setting. However, our estimates require the saturation assumption which, together with the strengthened Cauchy–Schwarz inequality, results in a loss of the so-called upper bound property of the AO-estimator, i.e., the constants on the lower bound of (3.13) and (4.9) are not available explicitly whereas the corresponding constant for the AO-estimator is exactly equal to one.

Finally, we briefly discuss some numerical aspects of the estimators. The principal concept of using weak residual without flux jumps to estimate errors has been proposed in a general and abstract setting and been numerically verified by various boundary value problems such as linear elliptic problems, PDEs of mixed-type, and variational inequalities in [17]. Moreover, in [18], we also present some implementation details as well as numerical examples which include a driven cavity model and a semiconductor device model. All numerical evidence shows that both conforming and nonconforming estimators are effective and reliable for adaptive computation.

Acknowledgements

The authors would like to thank the referees for their helpful comments and suggestions on the manuscript.

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