A reduction method for local sensitivity analyses of network equilibrium arc flows

Hsun-Jung Cho [a], Tony E. Smith [b], Terry L. Friesz [c,*]

[a] *Department of Transportation Engineering and Management, National Chiao Tung University, Hsinchu, Taiwan*
[b] *Department of Systems Engineering, University of Pennsylvania, Philadelphia, PA 19104, USA*
[c] *Department of Systems Engineering and Operations Research, George Mason University, Fairfax, VA 22030, USA*

Abstract

A reduction method is proposed which allows standard sensitivity techniques for variational inequalities to applied to equilibrium network flow problems without additional assumptions on either the underlying network or the numbers of active paths. In particular it is shown that under mild regularity conditions, small perturbations of equilibria can be given an explicit arc-flow representation which is free of path-flow variables. It is also shown that this reduced form allows the differentiability of perturbations to be studied by standard methods. These results are illustrated by a small numerical example. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Methods of sensitivity analysis for nonlinear programming problems (Fiacco, 1983) and for variational inequality problems (Dafermos, 1988; Kyparisis, 1987; Kyparisis, 1988; Tobin, 1986; Qiu and Magnanti, 1987; Pang, 1990) have been applied to spatial price equilibrium problems (Chao and Friesz, 1984; Dafermos and Nagurney, 1984a; and Tobin, 1987). However, direct application of these methods to the variational inequality formulation of the equilibrium network flow problem is not feasible since its solutions do not typically satisfy the required local uniqueness conditions. This is primarily due to the presence of path variables in the problem formulation. As a consequence, computational procedures which have thus far been proposed to find the gradients of arc-flow variables with respect to parameter perturbations

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* Corresponding author. Present address: SEOR Department, George Mason University, Farifax, VA 22030, USA.
Tel.: 703 993 1658; e-mail: tfriesz@gmu.edu

require the determination of an unperturbed equilibrium path-flow vector with a restricted number of active paths (as for example in Dafermos and Nagurney (1984c) and Tobin and Friesz (1988)). However, since traditional algorithms (such as the Frank–Wolfe feasible directions algorithm) usually terminate with approximate solutions which do not satisfy these restrictions, one must employ auxiliary search procedures to find such path-flow vectors (as in the linear-programming approach of Tobin and Friesz (1988)). Hence, our main objective in this paper is to develop an alternative approach which permits standard gradient-based methods of sensitivity analysis to be applied directly to the approximate solutions obtained by existing algorithms. The method we introduce in this paper finds a perturbed path-flow vector which is the "least distance" from an unperturbed path-flow vector (path-flow information, see also Cho (1991)). Our central result is to show that if this unperturbed vector is positive in all minimum-cost path components (i.e., if all such paths are used in equilibrium), then the resulting minimum-distance perturbed path-flow vectors can be used to formulate feasible arc-flow sets entirely in terms of arc variables. In particular, it is shown that in a sufficiently small neighborhood of the unperturbed equilibrium, the only relevant flow information required for sensitivity analysis is the basic set of flow-conservation conditions which must hold at each node in the network.

2. Equilibrium network flow problems

Consider a transportation network–consisting of a finite set of *nodes*, $i \in N$, and *arcs* $a \in A$, together with a nonempty set of *origin–destination pairs*, $w \in W \subseteq N^2$. Each $w \in W$ is joined by a nonempty finite set of *paths*, $p \in P_w$. If $P = \cup_{w \in W} P_w$ denotes the set of all paths, then each $p \in P$ consists of a finite sequence of connected arcs, and is representable by a zero-one column vector, $\Delta_p = (\Delta_{ap} : a \in A)$, with $\triangle_{ap} = 1$ if *arc*, a, belongs to *path* p, and, $\Delta_{ap} = 0$ otherwise. If the cardinalities of A, W and P are denoted respectively by $\alpha = |A|$, $\omega = |W|$, and $\rho = |P|$, then the $(\alpha \times \rho)$-*matrix*, $\Delta = (\Delta_p : p \in P)$ is designated as the (A, P)-*matrix*. Similarly, if for each $p \in P$ and $w \in W$ we let $\Lambda_{wp} = 1$ if $p \in P_w$ and $\Lambda_{wp} = 0$ otherwise, then the $(\omega \times \rho)$-*matrix*, $\Lambda = (\Lambda_{wp})$, is designated as the (W, P)-*matrix*. If the real numbers, nonnegative reals, and positive reals are denoted respectively by R, R_+ and R_{++}, then each positive column vector, $T = (T_w : w \in W) \in R_{++}^\omega$, is designated as a possible *travel-demand vector*. Each nonnegative column vector, $h = (h_p : p \in P) \in R_+^\rho$, is designated as a *path-flow vector* consistent with T iff $\Lambda h = T$. Similarly, each nonnegative column vector, $f = (f_a : a \in A) \in R_+^\alpha$, is designated as an *arc-flow vector* consistent with T iff $f = \Delta h$ for some path-flow vector consistent with T. Equivalently, if the set of path-flows consistent with both f and T is denoted by

$$H(f, T) = \{h \in R_+^\rho : f = \Delta h \text{ and } \Lambda h = T\}, \tag{1}$$

then f is consistent with T iff $H(f, T) \neq 0$. Each function, $c : R_+^\alpha \to R_+^\alpha$ is designated as a possible *arc-cost function*, where $c(f) = (c_a(f) : a \in A)$ denotes the vector of per-unit flow costs on each arc, $a \in A$, for the arc flow pattern f. For each path, $p \in P$, the associated *path cost* is assumed to be the sum of its arc costs, as given by

$$c_p(f) = \Delta_p^T c(f). \tag{2}$$

If for each $f \in R_+^\alpha$ and $w \in W$, we let the *minimum path cost* in P_w be denoted by

$$c_w(f) = \min\{c_p(f): p \in P_w\}, \tag{3}$$

then a (fixed-travel-demand) user equilibrium can be defined for such network flows as follows. For any given arc-cost function, c, and travel demand vector, T, a path-flow vector h^* consistent with T is designated as a *user equilibrium* for (c, T) iff for all $p \in P_w$ and $w \in W$

$$h_p^* > 0 \Rightarrow c_p(\Lambda h^*) = c_w(\Lambda h^*). \tag{4}$$

The associated arc-flow vector, $f = \Lambda h^*$, is then designated as a *equilibrium arc-flow vector* for (c, T). As is well known (see for example Dafermos, 1980), the set of equilibrium arc-flow vectors for (c, T) can be identified with the solutions of a certain variational inequality problem (VI-problem). To define this problem in a convenient manner for our purposes, we employ the following notation. For any nonempty sets, $S \subseteq X \subseteq R^n$, and function $\rho: X \to R^n$, the set

$$\mathrm{VI}(\rho, S) = \{x \in S: \rho(x)^{\mathrm{T}}(y - x) \geqslant \text{ for all } y \in S\} \tag{5}$$

is designated as the *solution set* for the VI-problem defined by ρ and S. In particular, for any travel demand vector, T, we designate the set

$$\Omega(T) = \{f \in R^\alpha: H(f, T) \neq 0\} \tag{6}$$

of all arc-flow vectors consistent with T as the *feasible arc-flow set* for T, then for any arc-cost function, c, the solution set, $\mathrm{VI}[c, \Omega(T)]$, corresponds precisely to the set of *equilibrium arc-flow vectors* for (c, T).

Hence, all sensitivity questions relating to the influence of arc-cost and travel-demand parameters on equilibrium arc-flows can be studied in terms of perturbations of this VI-problem.

3. Perturbation systems

To formulate this perturbation problem, suppose that the flow-cost function and travel-demand vector are influenced by some finite-dimensional vector of parameters, $\theta \in R^k$. In particular, suppose that a given function, c_0, and demand vector, T_0, are determined by parameter values, θ_0, and that it is meaningful to consider changes in C_0 and T_0 corresponding to parameter values in some neighborhood, Θ, of θ_0 in the parameter space R^k. Then, if for each $\theta \in \Theta$, we define the corresponding *perturbation vector*, $\varepsilon = \theta - \theta_0$, we may reparameterize these functions in terms of the associated set of *perturbation vectors*, $D = \{\varepsilon \in R^k: \theta_0 + \varepsilon \in \Theta\}$. Of special interest is the *zero perturbation vector*, $0 \in D$, which corresponds to the initial (unperturbed) parameter vector, θ_0. In particular, we shall be primarily concerned with small perturbations in this initial vector θ_0, and hence assume for convenience that all sufficiently small perturbations are possible. To be more precise, if for each $x \in R^n$ and positive scalar, $\delta > 0$, we designate the set $B(x) = \{y \in R^n: \|x - y\| < \delta\}$, as an *x-neighborhood* in R^n, then we now assume that D contains some *0-neighborhood*, $B(0)$, in R^k. In addition, since we are only interested in small perturbations, we assume that D is bounded (i.e., is contained in some 0-neighborhood). Finally, to study the

continuity properties of perturbations, it is convenient to assume that D is a closed set (and hence that D is compact in R^k). In summary then, we now say that:

Definition 3.1. Each compact set, $D \subseteq R^k$, containing a 0-neighborhood is designated as an admissible *perturbation domain*.

Given any perturbation domain, D, it is postulated that for each $\varepsilon \in D$ we may associate a unique arc-cost function, $c(\cdot, \varepsilon)$, and positive travel-demand vector $T(\varepsilon)$. Hence if for each $\varepsilon \in D$ and $f \in R^\alpha$, we now let

$$H(f, \varepsilon) = \{h \in R_+^\rho : f = \Delta h \text{ and } \Lambda = T(\varepsilon)\} \tag{7}$$

and define the *feasible arc-flow set* corresponding to Eq. (1) by

$$\Omega(\varepsilon) = \{f \in R^\alpha : H(f, \varepsilon) \neq \phi\} \subseteq R_+^\alpha \tag{8}$$

(where $H(f, 0) = H[f, T, (0))$ in Eq. (1) and $\Omega[T(0)]$ in Eq. (6)) then the set of equilibrium arc-flow vectors for the equilibrium problem defined by $c(\cdot, \varepsilon)$ and $T(\varepsilon)$ is now given by the solution set for the associated VI-problem, i.e., by

$$\text{VI}[c(\cdot, \varepsilon), \Omega(\varepsilon)] = \{f \in \Omega(\varepsilon)] : c(f, \varepsilon)^\text{T}(g - f) \geqslant 0 \text{ for all } g \in \Omega(\varepsilon)\}. \tag{9}$$

Our primary concern in the present paper is with those perturbation problems for which these equilibrium arc-flow vectors are at least *locally unique*. Hence, we now define a general class of perturbation systems which have this property. If the *closure* of a set $X \subseteq R^n$, is denoted by $cl(X)$, then we now say that:

Definition 3.2. For any perturbation domain, D, continuous functions, $c : R_+^\alpha \times D \to R_+^\alpha$, $T : D \to R_{++}^\omega$, and open set, $F \subseteq R^\alpha$, the ordered collection (D, F, T, c) is designated as a *perturbation system* iff the following local uniqueness condition is satisfied:

C (*Local uniqueness*). For *all perturbation vectors*, $\varepsilon \in D$,

$$|\text{VI}[c(\cdot, \varepsilon), \Omega(\varepsilon) \cap F]| = 1 = |\text{VI}[c(\cdot, \varepsilon), \Omega(\varepsilon)] \cap cl(F)|.$$

Condition C asserts that for each perturbation vector, $\varepsilon \in D$, the VI-problem in Eq. (9) has exactly one solution, $f(\varepsilon) \in \Omega(\varepsilon) \cap F$, and that there exist no other solutions in $\Omega(\varepsilon) \cap cl(F)$. If $\text{VI}[c(\cdot, \varepsilon), \Omega(\varepsilon)] - F \neq \phi$ then $f(\varepsilon)$ is *locally unique* with respect to F, and if $\text{VI}[c(\cdot, \varepsilon), \Omega(\varepsilon)] - F \neq \phi$, then $f(\varepsilon)$ is *globally unique* in $\Omega(\varepsilon)$. In all cases, the solution vectors, $f(\varepsilon)$, define a unique *equilibrium arc-flow function*, $f : D \to F$, with respect to the region $F \subseteq R^\alpha$. Our main result is to show that small perturbations in such systems always yield small changes in the associated local equilibrium arc-flows.

Theorem 3.1 [Continuity of arc-flows] *For each perturbation system*, (D, F, T, c), *the associated equilibrium arc-flow function*, $f : D \to F$ *is continuous.*

Notice also that the continuity of perturbed equilibrium arc-flows depends only on the continuity of c and T. In particular, this continuity property is independent of any monotonicity

properties of c, as employed for example in the continuity theorems of Fang (1979), Dafermos and Nagurney (1984b) and Dafermos (1986). [1]

But while this result may be said to provide a satisfactory conceptual framework for the analysis of small perturbations in equilibrium arc-flows, it fails to yield any operational procedures for doing so. Hence, our main objective is to impose stronger structural conditions on perturbation systems, (D, F, T, c), which, will yield a procedure for approximating the equilibrium arc-flow function, $f: D \to F$, in some small neighborhood of the unperturbed state, $\varepsilon = 0$.

3.1. The reduction method

As mentioned in the introduction, the central difficulty in applying standard approximation techniques to the present problem is the presence of path-flow sets, $H(f, \varepsilon)$, in the definition of feasible arc-flow sets in Eq. (8). Hence, our first objective is to construct a reduced version of the feasible arc-flow set which has a simple explicit representation solely in terms of arc-flow variables. To do so, we employ a "minimum-distance" technique to select a unique equilibrium path-flow vector for each equilibrium arc-flow vector, $f(\varepsilon)$. These minimum-distance solutions will then be employed to construct a reduced version of the VI-problem in Eq. (9) which continues to characterize arc-flow equilibria, $f(\varepsilon)$, in some neighborhood of the unperturbed state, $\varepsilon = 0$.

To do so, it is necessary to impose further regularity conditions on the perturbation system, (D, F, T, c), which will allow such a local reduction procedure. To motivate our first regularity condition, suppose that we are able to reduce the feasible arc-flow set, $\Omega(\varepsilon)$ in Eq. (9) to a smaller set, $S \subseteq \Omega(\varepsilon)$, for which $f(\varepsilon)$, is also a solution, i.e. is an element of VI$[c(\cdot, \varepsilon), S]$. Then in general, $f(\varepsilon)$, may no longer be the unique solution to this reduced problem. To preserve uniqueness, we thus require a stronger condition on $c(\cdot, \varepsilon)$. In particular, a function, $\rho: R_+^\alpha \to R_+^\alpha$, will be said to be *strictly monotone* on $S \subseteq R_+^\alpha$ iff $[\rho(x) - \rho(y)]^T (x - y) > 0$ for the distinct $x, y \in S$, and we now require that (D, F, T, c) satisfy the following *local strict monotonicity* condition:

C1 (Local strict monotonicity). There exists some 0-neighborhood, $B_s \subseteq D$ such that $c(\cdot, \varepsilon)$ is *strictly monotone* on $\Omega(\varepsilon) \cap cl(F)$ for all $\varepsilon \in B_s$.

As an immediate consequence of this regularity condition, we have the following useful property:

Lemma 3.1. *If (D, F, T, c) is a perturbation system satisfying* C1, *then for each $\varepsilon \in B_s$ and set $S \subseteq R_+^\alpha$ with*

$$f(\varepsilon) \in S \subseteq \Omega(\varepsilon), \tag{10}$$

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[1] It should be noted however that such assumptions yield stronger results. In particular it is shown in Dafermos and Nagurney (1984b) (Theorems 3.1 and 4.1) and Dafermos (1980) (Theorem 2.4) that if c is strongly monotone on R_+^α, and if both c and T are Lipschitz continuous on D, then f must also be Lipschitz continuous on D.

it must be true that

$$\mathrm{VI}[c(\cdot, \varepsilon), S] \cap F = \{f(\varepsilon)\}. \tag{11}$$

Proof. First observe from Eq. (9) that $f(\varepsilon) \in \mathrm{VI}[c(\cdot, \varepsilon), \Omega(\varepsilon)] \Rightarrow c[f(\varepsilon), \varepsilon]^{\mathrm{T}} (f - f(\varepsilon)) \geqslant 0$ for all $f \in S \subseteq \Omega(\varepsilon) \Rightarrow f(\varepsilon) \in \mathrm{VI}[c(\cdot, \varepsilon), S]$. Moreover by Eq. (5), if there is some $f \in \mathrm{VI}[c(\cdot, \varepsilon), S] \cap F$ with $f \neq f(\varepsilon)$ then (as in Kinderlehrer and Stampacchia, 1980, p. 14) we must have, $0 \leqslant c[f(\varepsilon), \varepsilon]^{\mathrm{T}}(f - f(\varepsilon)) + c[f, \varepsilon]^{\mathrm{T}}(f(\varepsilon) - f) = -(c[f(\varepsilon), \varepsilon] - c[f, \varepsilon])^{\mathrm{T}}(f(\varepsilon) - f)$, which together with $f, f(\varepsilon) \in \Omega(\varepsilon) \cap F$, contradicts the strict monotonicity of $c(\cdot, \varepsilon)$ in $\Omega(\varepsilon) \cap F$. $\square$

As a first application of this result, we now show that for ε sufficiently close to zero, $\Omega(\varepsilon)$ can be reduced to a smaller set, $\Omega_0(\varepsilon)$, satisfying Eq. (11) which involves much lower dimensional path-flow vectors. To do so, recall from Eq. (4) that each user equilibrium path-flow vector for the unperturbed state, $[c(\cdot, 0), T(0)]$, must have zero flow on each path, $p \in P$, which is not a minimum-cost path. Moreover, as we now show, each such path must continue to have zero flow in all user equilibria for perturbed states, $\varepsilon \in D$, which are sufficiently close to zero. Hence if for each origin-destination pair, $w \in W$, we now let

$$c_w(\varepsilon) = \min\{c_p[f(\varepsilon), \varepsilon]: p \in P_w\} \tag{12}$$

denote the minimum path cost in P_w (as in expression (3) above), then the set of *minimum-cost paths* in P_w is given by:

$$P_w(\varepsilon) = \{p \in P_w: c_p[f(\varepsilon), \varepsilon] = c_w(\varepsilon)\} \tag{13}$$

and the corresponding *minimum-cost path set* for the entire network is given by

$$P(\varepsilon) = \cup_{w \in W} P_w(\varepsilon). \tag{14}$$

With this notation, we now show that:

Lemma 3.2. *For any perturbation system (D, F, T, c), there exists a 0-neighborhood, $B_0 \subseteq D$, such that for all $\varepsilon \in B_0$*

$$P(\varepsilon) \subseteq P(0). \tag{15}$$

Proof. It suffices to produce a 0-neighborhood, $B_0 \subseteq D$, such that for all $\varepsilon \in B_0$, $P - P(0) \subseteq P - P(\varepsilon)$. To do so, observe first that if $P(0) = P$ then the result is trivial. Hence, we may assume that $P(0) - P \neq \phi$. But for any $p \in P - P(0)$, we must have $c_p[f(0), 0] > c_w(0)$, which together with the continuity of c implies that $c_p[f(\varepsilon), \varepsilon] > c_w(\varepsilon)$ must hold for all ε in some 0-neigborhood, $B_p \subseteq D$. Hence it follows from the finite cardinality of $P - P(0)$ that the set, $B_0 = \cap_{p \in P - P(0)} B_p$, defines a 0-neighborhood in D with the desired property. $\square$

(In the above proof, and throughout the analysis to follow, we make constant use of the fact that each *finite* intersection of 0-neighborhoods is also a 0-neighborhood.) Next, to construct a reduced version of $\Omega(\varepsilon)$, we now let $\rho_0 = |P(0)| \leqslant \rho$, and let Δ^0 and Λ^0 denote the reductions of Δ and Λ obtained by eliminating all columns corresponding to paths, $p \in P - P(0)$. Then, by letting

$$H_0(f, \varepsilon) = \{h \in R_+^{\rho_0}: \Delta^0 h = f \text{ and } \Lambda^0 h = T(\varepsilon)\} \tag{16}$$

denote the corresponding reduced version of $H(f, \varepsilon)$ in Eq. (7) for each $\varepsilon \in D$ and $f \in \Omega(\varepsilon)$, we now show that $\Omega(\varepsilon)$ in Eq. (8) can be replaced by the corresponding smaller set of arc-flows

$$\Omega_0(\varepsilon) = \{f \in R^\alpha : H_0(f, \varepsilon) \neq \phi\} \tag{17}$$

for all ε sufficiently close to zero:

Lemma 3.3. *For any perturbation system,* (D, F, T, c), *satisfying* **C1**, *there exists a* 0-*neighborhood,* $B \subseteq D$, *such that for all* $\varepsilon \in B$

$$\text{VI}[c(\cdot, \varepsilon), \Omega_0(\varepsilon)] \cap F = \{f(\varepsilon)\}. \tag{18}$$

Proof. Observe first that given any $\varepsilon \in D$ and $f \in \Omega(\varepsilon)$, it follows by definition that each $h \in H_0(f, \varepsilon)$ can be uniquely extended to an element $\hat{h} \in H(f, \varepsilon)$, by setting $\hat{h}_p = h_p$ for all $p \in P(0)$ and $\hat{h}_p = 0$ otherwise. Hence, $f \in \Omega_0(\varepsilon) \Rightarrow H_0(f, \varepsilon) \neq \phi \Rightarrow H(f, \varepsilon) \neq \phi \Rightarrow f \in \Omega(\varepsilon)$, so that we must have $\Omega_0(\varepsilon) \subseteq \Omega(\varepsilon)$. Moreover, for $f(\varepsilon)$ in particular, it follows that each element, $h \in H[f(\varepsilon), \varepsilon]$, is by definition an equilibrium path-flow vector, and hence satisfies $h_p = 0$ for each $p \notin P(\varepsilon)$. But, by Lemma 3.2, $\varepsilon \in B_0 \Rightarrow P(\varepsilon) \subseteq P(0) \Rightarrow h_p = 0$ for each $p \notin P(0)$, so that each element $h \in H[f(\varepsilon), \varepsilon]$, yields a unique element of $H_0[f(\varepsilon), \varepsilon]$, obtained by removing all components $p \notin P(0)$. Hence for each $\varepsilon \in B_0$, we must also have $H[f(\varepsilon), \varepsilon] \neq \phi \Rightarrow H_0[f(\varepsilon), \varepsilon] \neq \phi \Rightarrow f(\varepsilon) \in \Omega_0(\varepsilon)$ and may conclude that $f(\varepsilon) \in \Omega_0(\varepsilon) \subseteq \Omega(\varepsilon)$ for all $\varepsilon \in B_0$. Finally, by letting $B = B_s \cap B_0$, we may conclude from Lemma 3.1 that Eq. (18) must hold for all $\varepsilon \in B$. $\square$

We are now ready to develop the minimum-distance construction. To do so, we begin by observing from the positivity of P_w that each set, $P_w(0)$ in Eq. (13) for the unperturbed state, $\varepsilon = 0$, is nonempty. But since each column, Λ_p^0 of Λ^0 with $p \in P_w$ is by definition an identity basis vector with 1 in the w position, it follows that this $(\omega \times \rho_0)$-matrix contains ω linearly independent columns, and hence is of full row rank, ω. Given this observation, we next select from Δ^0 a *maximal* set of rows, says Δ_1^0, for which the combined matrix $\begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix}$ is of *full row rank*. Here, there are two cases to consider. First suppose that Δ_1^0 is empty, i.e., that every row of Δ^0 lies in the row span of Λ^0. Then by definition there exists a matrix, M^0, such that $\Delta^0 = M^0 \Lambda^0$, so that by substituting this relation into Eq. (16), we see that $H_0(f, \varepsilon) \neq 0 \iff f = M^0 T(\varepsilon)$, and hence from Eq. (17) that $\Omega_0(\varepsilon)$ contains the unique element $M^0 T(\varepsilon)$. But since $f(\varepsilon) \in \Omega_0(\varepsilon)$ by Lemma 3.2, we may conclude that

$$f(\varepsilon) = M^0 T(\varepsilon). \tag{19}$$

Hence, for this case, sensitivity analysis of equilibrium arc-flows, $f(\varepsilon)$, is trivially determined by the travel demands, $T(\varepsilon)$, and we need proceed no further. It is thus assumed throughout the rest of the analysis that Δ^0 is *not* contained in the row span of Λ^0, so that Δ_1^0 is well defined. With this in mind, we now partition Δ^0 as

$$\Delta^0 = \begin{bmatrix} \Delta_1^0 \\ \Delta_2^0 \end{bmatrix}, \tag{20}$$

where Δ_1^0 is chosen to be a maximal set of say, α_1, rows of Δ^0 such that $\begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix}$ is of full row rank, and where Δ_2^0 is defined by the set of $\alpha_2 = \alpha - \alpha_1$ remaining rows of Δ^0. As is shown following expression (22) below, α_2 is always positive, so that Δ_2^0 is always defined. Moreover, Δ_2^0 by construction must be contained in the row span of $\begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix}$, so that there must exist matrices M_1 and M_2 such that

$$\Delta_2^0 = M_1 \Delta_1^0 + M_2 \Lambda^0. \tag{21}$$

In particular this implies that if we partition each arc-flow vector, $f \in \Omega_0(\varepsilon)$, in a manner compatible with Eq. (20) as $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ then it follows at once from Eqs. (16) and (21) that for any choice of $h \in H_0(f, \varepsilon)$ we must have

$$f_2 = \Delta_2^0 h = M_1 \Delta_1^0 h + M_2 \Lambda^0 h = M_1 f_1 + M_2 T(\varepsilon). \tag{22}$$

These relations, which we henceforth designate as the *flow-conservation conditions*, will play a fundamental role in our subsequent analysis. For the moment it suffices to observe that this set of α equations must by definition summarize all flow-conservation relations at each node in the network, and hence must always be nonempty. In particular, if for any node, $i \in N$, we let $A(i, \cdot) \times [A(\cdot, i)]$ denote the sets of arcs in A which originate (terminate) at i, and set $W(i, \cdot) = \{j \in N : (i, j) \in W\}$ and $W(\cdot, i) = \{j \in N : (j, i) \in W\}$, then for each travel demand vector, $T(\varepsilon)$, and arc-flow vector, f, with $H[f, T(\varepsilon)] \neq \phi$, it must be true that

$$\sum_{w \in W(i, \cdot)} T_w(\varepsilon) + \sum_{a \in A(\cdot, i)} f_a = \sum_{w \in W(\cdot, i)} T_w(\varepsilon) + \sum_{a \in A(i, \cdot)} f_a.$$

Hence, for any network there must always be linear dependencies of the form (22), and we may conclude that $\alpha_2 > 0$.

Given these preliminary observations, we now consider the following *minimum-distance problem*, defined for any fixed $\varepsilon \in D$, $f_1 \in R^{\alpha_1}$ and $h_0 \in R^{\rho_0}$ by:

$$\text{min:} \quad \|h - h_0\|^2 = (h - h_0)^{\mathrm{T}} (h - h_0) \tag{23}$$

$$\text{s.t.:} \quad h \in R^{\rho_0} \quad \text{and} \quad \begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix} h = \begin{bmatrix} f_1 \\ T(\varepsilon) \end{bmatrix}. \tag{24}$$

Since the matrix $\begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix}$ is of full row rank, it follows at once that for each choice of (ε, f_1, h_0), a unique solution, $h(\varepsilon, f_1, h_0)$, to problem [(23), (24)] exists, and is given by

$$h(\varepsilon, f_1, h_0) = h_0 + \begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Delta_1^0 \Delta_1^{0\mathrm{T}} & \Delta_1^0 \Lambda^{0\mathrm{T}} \\ \Lambda^0 \Delta_1^{0\mathrm{T}} & \Lambda^0 \Lambda^{0\mathrm{T}} \end{bmatrix}^{-1} \left(\begin{bmatrix} f_1 \\ T(\varepsilon) \end{bmatrix} - \begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix} h_0 \right). \tag{25}$$

Moreover, if we let

$$\begin{bmatrix} \Delta_1^0 \Delta_1^{0\mathrm{T}} & \Delta_1^0 \Lambda^{0\mathrm{T}} \\ \Lambda^0 \Delta_1^{0\mathrm{T}} & \Lambda^0 \Lambda^{0\mathrm{T}} \end{bmatrix}^{-1} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{26}$$

where

$$M_{11} = \left[\Delta_1^0 \Delta_1^{0^T} - \Delta_1^0 \Lambda^{0^T} \left(\Lambda^0 \Lambda^{0^T} \right)^{-1} \Lambda^0 \Delta_1^{0^T} \right]^{-1}, \tag{27}$$

$$M_{12} = -\left(\Delta_1^0 \Delta_1^{0^T} \right)^{-1} \Delta_1^0 \Lambda^{0^T} \left[\Lambda^0 \Lambda^{0^T} - \Lambda^0 \Delta_1^{0^T} \left(\Delta_1^0 \Delta_1^{0^T} \right)^{-1} \Delta_1^0 \Lambda^{0^T} \right]^{-1}, \tag{28}$$

$$M_{21} = -\left[\Lambda^0 \Lambda^{0^T} - \Lambda^0 \Delta_1^{0^T} \left(\Delta_1^0 \Delta_1^{0^T} \right)^{-1} \Delta_1^0 \Lambda^{0^T} \right]^{-1} \Lambda^0 \Delta_1^{0^T} \left(\Delta_1^0 \Delta_1^{0^T} \right)^{-1}, \tag{29}$$

$$M_{22} = \left[\Lambda^0 \Lambda^{0^T} - \Lambda^0 \Delta_1^{0^T} \left(\Delta_1^0 \Delta_1^{0^T} \right)^{-1} \Delta_1^0 \Lambda^{0^T} \right]^{-1}, \tag{30}$$

then we may write Eq. (25) in compact form as

$$h(\varepsilon, f_1, h_0) = N_0 h_0 + N_1 f_1 + N_2 T(\varepsilon) \tag{31}$$

where

$$N_0 = \begin{bmatrix} I & \\ & I \end{bmatrix} - \begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix}^T \begin{bmatrix} \Delta_1^0 \Delta_1^{0^T} & \Delta_1^0 \Lambda^{0^T} \\ \Lambda^0 \Delta_1^{0^T} & \Lambda^0 \Lambda^{0^T} \end{bmatrix} \begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix}, \tag{32}$$

$$N_1 = \Delta_1^{0^T} M_{11} + \Lambda^{0^T} M_{21} \tag{33}$$

$$N_2 = \Delta_1^{0^T} M_{12} + \Lambda^{0^T} M_{22}. \tag{34}$$

Observe first from the full row rank condition on $\begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix}$ that one may employ Eqs. (21) and (26) to solve for M_1 and M_2 as follows.

$$\Delta_2^0 = M_1 \Delta_1^0 + M_2 \Lambda^0 = [M_1, M_2] \begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix}$$

$$\Rightarrow [M_1, M_2] = \Delta_2^0 \begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix}^T \left(\begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix} \begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix}^T \right)^{-1} = \Delta_2^0 \begin{bmatrix} \Delta_1^0 \\ \Lambda^0 \end{bmatrix}^T \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

Hence, M_1 and M_2 can be expressed explicitly in terms of Δ_2^0, and the matrices N_1 and N_2 in Eqs. (33) and (34) as $M_1 = \Delta_2^0 N_1$ and $M_2 = \Delta_2^0 N_2$, and we may conclude that the *flow-conservation conditions* in Eq. (22) can now be written explicitly as

$$\Delta_2^0 N_1 f_1 - f_2 + \Delta_2^0 N_2 T(\varepsilon) = 0. \tag{35}$$

Given the minimum-distance solution in Eq. (31), observe next that if we define $f_2 \in R^{\alpha - \alpha_1}$ by Eq. (22) and set $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, then it follows at once from the constraint Eq. (24) together with Eqs. (21) and (22) that

$$\Delta^0 h(\varepsilon, f_1, h_0) = f, \tag{36}$$

$$\Lambda^0 h(\varepsilon, f_1, h_0) = T(\varepsilon). \tag{37}$$

Thus, we see that for each f with $h(\varepsilon, f_1, h_0) \in R_+^{\rho_0}$, it must be true by definition that, $h(\varepsilon, f_1, h_0) \in H_0(f, \varepsilon)$, and hence from Eq. (17) that $f \in \Omega_0(\varepsilon)$. In other words, if we now write this nonnegativity condition in terms of Eq. (31) as

$$N_0 h_0 + N_1 f_1 + N_2 T(\varepsilon) \geqslant 0, \tag{38}$$

then these observations imply that the *reduced feasible arc-flow set* defined by

$$\Omega_\varepsilon(h_0) = \{f \in R_+^\alpha : (35) \text{ and } (38) \text{ hold}\} \tag{39}$$

must always satisfy $\Omega_\varepsilon(h_0) \subseteq \Omega_0(\varepsilon)$. Moreover, since the equilibrium arc-flow vector, $f(\varepsilon) = \begin{bmatrix} f_1(\varepsilon) \\ f_2(\varepsilon) \end{bmatrix}$, by definition satisfies Eq. (35), it follows that *if* it were true that $N_0 h_0 + N_1 f_1(\varepsilon) + N_2 T(\varepsilon) \geqslant 0$ then we would have $f(\varepsilon) \in \Omega_\varepsilon(h_0) \subseteq \Omega_0(\varepsilon)$ and could employ Lemma 3.1 to characterize $f(\varepsilon)$ in terms of the VI-problem for $\Omega_\varepsilon(h_0)$. Moreover, since Eqs. (35) and (38) are free of path-flow variables, this would in turn allow a direct application of the results of Section 2.

3.2. Locally regular perturbation systems

Now, our present approach is to impose an additional regularity condition on the perturbation system, (D, F, T, c), which will ensure the existence of choices for h_0 in Eq. (23) which guarantee *positivity* of $h(\varepsilon, f_1, h_0)$ for all ε sufficiently close to zero, and all $f_1(\varepsilon)$ sufficiently close to the equilibrium arc-flow vector, $f(\varepsilon) \in \Omega_0(\varepsilon)$. To do so, it suffices to require that in the unperturbed network equilibrium problem, there exist at least one equilibrium path-flow vector in which all *minimum-cost paths are used*. Hence, if we now let

$$H_+(\varepsilon) = H_0[f(\varepsilon), \varepsilon] \cap R_{++}^{\rho_0} \tag{40}$$

denote the set of positive flow vectors in $H_0[f(\varepsilon), \varepsilon]$, then the desired *local positivity condition* can be stated concisely as follows.

C2 (Local positivity). $H_+(0) \neq \phi$.

Given these two regularity conditions, we are now ready to formalize the class of perturbation systems for which our reduction method is applicable:

Definition 3.3. For any perturbation domain, D, continuous functions, $c : R_+^\alpha \times D \to R_+^\alpha$, $T : D \to R_{++}^\omega$, and open set, $F \subseteq R^\alpha$, the ordered collection (D, F, T, c) is designated as a locally *regular perturbation system* iff (D, F, T, c) satisfies C1 and C2 together with the following local existence condition:

C3 (*Local existence*). *For all perturbation vectors*, $\varepsilon \in D$, $\mathrm{VI}[c(\cdot, \varepsilon), \Omega(\varepsilon)] \cap F \neq \phi$.

Note that conditions C1 and C3 together imply the local uniqueness condition (C), by the argument in Lemma 3.1 above. Hence each locally regular system (D, F, T, c), is indeed a perturbation system in the sense of Definition 3.2. Given this class of perturbation systems, our first

result is to show that for any choice of $h_0 \in H_+(0)$, the solution to Eqs. (23) and (24) obtained by setting f_1 equal to the equilibrium arc flow vector, $f_1(\varepsilon)$, is positive for all ε sufficiently close to zero.

Lemma 3.4. *For each locally regular perturbation system*, (D, F, T, c), *and vector*, $h_0 \in H_+(0)$, *there exists a 0-neighborhood*, $B \subseteq D$, *such that for all $\varepsilon \in B$,*

$$h[\varepsilon, f_1(\varepsilon), h_0] \in R_{++}^{\rho_0}. \tag{41}$$

Proof. First observe from Eq. (40) that $h_0 \in H_+(0) \Rightarrow h_0 \in R_{++}^{\rho_0} \Rightarrow B(h_0) \subseteq R_{++}^{\rho_0}$ for some h_0-neighborhood, $B(h_0)$, in R. Hence, it suffices to show that there is some 0-neighborhood, $B \subseteq D$, such that for all $\varepsilon \in D$,

$$h[\varepsilon, f(\varepsilon), h_0] \in B(h_0). \tag{42}$$

To do so, suppose to the contrary that Eq. (42) fails for each 0-neighborhood, $B_n = \{\varepsilon \in R^k : |\varepsilon\| < 1/n\}$. Then for each n there is some $\varepsilon_n \in B_n$ with $h[\varepsilon_n, f_1(\varepsilon_n), h_0] \neq B(h_0)$. But since $f(\varepsilon) = \begin{bmatrix} f_1(\varepsilon) \\ f_2(\varepsilon) \end{bmatrix}$, is continuous in ε by Theorem 3.1, and since $T(\varepsilon)$ is continuous in ε by hypothesis, it follows that

$$
\begin{aligned}
\varepsilon_n \to 0 &\Rightarrow (f_1(\varepsilon_n), T(\varepsilon_n)) \to (f_1(0), T(0)) \\
&\Rightarrow N_0 h_0 + N_1 f_1(\varepsilon_n) + N_2 T(\varepsilon_n) \to N_0 h_0 + N_1 f_1(0) + N_2 T(0) \\
&\Rightarrow h[\varepsilon_n, f_1(\varepsilon_n), h_0] \to h[0, f_1(0), h_0].
\end{aligned} \tag{43}
$$

Finally, since $h_0 \in H_+(0) \subseteq H_0[f(0), 0]$ implies that h_0 satisfies Eq. (24) with $f_1 = f_1(0)$ and $\varepsilon = 0$, it follows from Eq. (25) that $h_0 = h[0, f_1, h_0]$, and hence from Eq. (43) that $h[\varepsilon_n, f_1(\varepsilon_n), h_0] \in B(h_0)$ for all n sufficiently large. Thus we obtain a contradiction, and may conclude that Eq. (42) must hold. $\square$

As a direct consequence of this result, we now have the following local characterization of equilibrium arc-flows in terms of the *reduced feasible arc-flow sets* in Eq. (39) above

Theorem 3.2 [Reduction Theorem] *For each locally regular perturbation system*, (D, F, T, c) *and choice of $h_0 \in H_+(0)$, there exists a 0-neighborhood*, $B_0 \subseteq D$, *such that for all $\varepsilon \in B$*

$$\mathrm{VI}[c(\cdot, \varepsilon), \Omega_\varepsilon(h_0)] \cap F = \{f(\varepsilon)\}. \tag{44}$$

Proof. By Lemmas 3.1 and 3.3 it suffices to show that there exists some 0-neighborhood, $B_0 \subseteq B_s$ (as in condition C1) such that for all $\varepsilon \in B_0$,

$$f(\varepsilon) \in \Omega_\varepsilon(h_0) \subseteq \Omega_0(\varepsilon) \tag{45}$$

To do so, let $B \subseteq D$ be any 0-neighborhood as in Lemma 3.4 and set $B_0 \subseteq B \cap B_s$. Then $\varepsilon \in B \Rightarrow h[\varepsilon, f_1(\varepsilon), h_0] = N_0 h_0 + N_1 f_1(\varepsilon) + N_2 T(\varepsilon) > 0$, which together with (22), (35), (38) and (39) implies that $f(\varepsilon) \in \Omega_\varepsilon(h_0)$. Finally, since $f \in \Omega_\varepsilon(h_0) \Rightarrow h[\varepsilon, f_1, h_0] = N_0 h_0 + N_1 f_1 + N_2 T(\varepsilon) \geqslant 0 \Rightarrow h[\varepsilon, f_1, h_0] \in H_0(f, \varepsilon) \neq \phi \Rightarrow f \in \Omega_0(\varepsilon)$, it also follows that $\Omega_\varepsilon(h_0) \subseteq \Omega_0(\varepsilon)$, and we may conclude that Eq. (45) holds. $\square$

4. Local sensitivity analysis of equilibrium arc-flow functions

Given this reduced arc-flow characterization of the equilibrium arc-flow function, $f: D \to R_+^\alpha$, near zero, we are now ready to analyze the local properties of perturbed arc-flows in this region.

Lemma 4.1. *For each locally regular perturbation system*, (D, F, T, c), *there exists a* 0-*neighborhood*, $B \subseteq D$, *such that for all* $\varepsilon \in B$ *and* $a \in A$,

$$f_a(\varepsilon) = 0 \Longleftrightarrow f_a(0) = 0 \tag{46}$$

Proof. If we consider the 0-neighborhood, B_0, in Lemma 3.4 above for any fixed choice of h_0, and for each $\varepsilon \in B$ let $h(\varepsilon) \in H[f(\varepsilon), \varepsilon]$ be defined by $h_p(\varepsilon) = h_p[\varepsilon, f_1(\varepsilon), h_0]$ all $p \in P(0)$, and $h_p(\varepsilon) = 0$ for all $p \in P - P(0)$, then by Lemma 3.4 and condition C2 it follows that $h_p(\varepsilon) > 0 \Longleftrightarrow p \in P(0) \Longleftrightarrow h_p(0) > 0$. Hence for each $a \in A$ it must be true that

$$
\begin{aligned}
f_a(\varepsilon) > 0 &\Longleftrightarrow \Delta_{ap} = 1 \text{ for some } p \text{ with } h_p(\varepsilon) > 0 \\
&\Longleftrightarrow \Delta_{ap} = 1 \text{ for some } p \text{ with } h_p(0) > 0 \\
&\Longleftrightarrow f_a(0)
\end{aligned}
\tag{47}
$$

and Eq. (46) must hold. $\square$

Hence for purposes of local sensitivity analysis, we may eliminate all arcs, $a \in A$, with $f_a(0) = 0$ and focus only on the subset

$$A_0 = \left\{ a \in A: \sum_{p \in P(0)} \Delta_{ap} > 0 \right\}. \tag{48}$$

For notational simplicity we may simply eliminate all rows in Eq. (38) corresponding to arcs in $A - A_0$, and redefine A to be A_0. With this in mind, we henceforth assume that

$$f(0) \in R_{++}^\alpha. \tag{49}$$

4.1. Locally smooth perturbation systems

Up to this point, none of our results have required differentiability assumptions. But in order to exploit the Reduction Theorem for local analysis of equilibrium arc-flow functions near zero, it is convenient to focus on perturbation systems which satisfy local differentiability assumptions. For any open set, $S \subseteq R^n$, and differentiable function, $G: S \to R^m$, let the $(m \times n)$-matrix of partial derivatives (i.e., the *Jacobian matrix* of G), evaluated at the point $x_0 = (x_{01}, \dots, x_{0n}) \in S$ be denoted by

$$\nabla G(x_0) = \left[\frac{\partial}{\partial x_j} G_i(x_0): i = 1, \dots, m, j = 1, \dots, n \right]. \tag{50}$$

Then G is by definition *continuously differentiable* iff the function, $\nabla G: S \to R^m$, is continuous. We denote the partial derivative of G with respect to its y-components evaluated at $x_0 = (y_0, z_0)$ by

$$\nabla_y G(y_0, z_0) = \left[\frac{\partial}{\partial x_j} G_i(x_0) : i = 1, \dots, m, j = 1, \dots, k \right]. \tag{51}$$

Finally, if we designate a square $(m \times n)$-matrix, M, as *positive definite* iff for all $u \in R^m$,

$$u \neq 0 \Rightarrow u^T M u > 0, \tag{52}$$

then we may now define the relevant class of differentiable systems for our purposes as follows:

Definition 4.1. A locally regular perturbation system, (D, F, T, c) is said to be *locally smooth* iff there exists a 0-neighborhood, $B(0) \subseteq D$, and an $f(0)$-neighborhood, $F(0) \subseteq F \cap R^\varepsilon_{++}$, such that the restricted functions, $c : F(0) \times B(0) \rightarrow R^\alpha_+$ and $T : B(0) \rightarrow R^\omega_{++}$, are continuously differentiable, and the following additional condition is satisfied:

C4. (Local positive definiteness). $\nabla_f c[f(0), 0]$ is positive definite.

We may assume without loss of generality that both of the following conditions hold for all $\varepsilon \in B(0)$:

$$f(\varepsilon) \in F(0). \tag{53}$$

$$c(\cdot, \varepsilon) \text{ is strictly monotone on } \Omega(\varepsilon) \cap F(0). \tag{54}$$

Moreover, if we fix any choice of $h_0 \in H_+(0)$, then by Lemma 3.4 and Theorem 3.2, it may also be assumed that the following two additional conditions hold for all $\varepsilon \in B(0)$:

$$h[\varepsilon, f_1(\varepsilon), h_0] \in R^{\rho_0}_{++}, \tag{55}$$

$$\text{VI}[c(\cdot, \varepsilon), \Omega_\varepsilon(h_0)] \cap F(0) = \{f(\varepsilon)\}. \tag{56}$$

Hence, it follows in particular from Eq. (56) that for each $\varepsilon \in B(0)$, $f(\varepsilon)$ is the unique element of $\Omega_\varepsilon(h_0) \cap F(0)$ satisfying

$$c[f(\varepsilon), \varepsilon]^T (f - f(\varepsilon)) \geq 0 \text{ for all } f \in \Omega_\varepsilon(h_0). \tag{57}$$

Moreover, by Eqs. (35), (38) and (39), $\Omega_\varepsilon(h_0)$ is defined to be the set of all $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in R^\alpha$, satisfying the inequalities

$$N_0 h_0 + N_1 f_1 + N_2 T(\varepsilon) \geq 0, \tag{58}$$

$$f \geq 0, \tag{59}$$

together with the flow-conservation equalities

$$Mf + \Delta_2^0 N_2 T(\varepsilon) = 0, \tag{60}$$

where

$$M = [\Delta_2^0 N_1, -I] \tag{61}$$

4.2. Differentiation of equilibrium arc-flow functions

First we establish the existence of unique Lagrange multipliers, $\mu(\varepsilon)$, for a linear programming characterization of $\text{VI}[c(\cdot, \varepsilon), \Omega_\varepsilon(h_0)]$.

Lemma 4.2 [Lagrange multipliers] *For each locally regular perturbation system*, (D, F, T, c) *and* 0-*neighborhood*, $B(0)$, *satisfying* Eqs. (53), (55) *and* (56), *there exists a unique function*, $\mu : B(0) \rightarrow R^{\alpha + \alpha_2}$, *such that for all* $\varepsilon \in B(0)$,

$$c[f(\varepsilon), \varepsilon] - M^{\mathrm{T}} \mu(\varepsilon) = 0. \tag{62}$$

Proof. Observe from Eq. (56) that $f(\varepsilon)$ satisfies Eq. (57), and hence is by definition a solution to the linear programming problem:

$$\min \ c[f(\varepsilon), \varepsilon]^{\mathrm{T}} f \tag{63}$$

subject to $f \in \Omega_\varepsilon(h_0)$, i.e., subject to Eqs. (58)–(60). Moreover, since $f(\varepsilon) \in F(0) \subseteq R^{\alpha}_{++}$ by Eq. (53) and since $N_0 h_0 + N_1 f_1(\varepsilon) + N_2 T(\varepsilon) \in R^{\rho_0}_{++}$ by Eqs. (31) and (55), it follows that no component of the inequality constraint system (Eqs. (58) and (59)) is binding at the solution point, $f(\varepsilon)$. Hence, by the Lagrange multiplier theorem (see for example Theorem 28.3 in Rockafellar (1970)), there exists a vector, $\mu(\varepsilon) \in R^{\alpha_2}$, satisfying Eq. (62). Moreover, since the $(\alpha_2 \times \alpha)$-matrix, M, in Eq. (61), has α_2 linearly independent columns (given by the columns of $-I$), it follows that M is of full row rank. Hence, $\mu(\varepsilon)$ is unique (and, in particular, is given for all $\varepsilon \in B(0)$ by $\mu(\varepsilon) = (MM^{\mathrm{T}})^{-1} Mc[f(\varepsilon), \varepsilon]$). $\quad \square$

We now designate the function, $\mu : B(0) \rightarrow R^{\alpha_2}$, as the *multiplier function* for the system. We begin by considering the open set, $S = P(0) \times R^{\alpha_2} \times B(0) \subseteq R^{\alpha + \alpha_2 + k}$, and let the function, $G : S \rightarrow R^{\alpha + \alpha_2}$, be defined for all $(f, \mu, \varepsilon) \in S$ by

$$G(f, \mu, \varepsilon) = \begin{bmatrix} c(f, \varepsilon) - M^{\mathrm{T}} \mu \\ Mf + \Delta_2^0 N_2 T(\varepsilon) \end{bmatrix}. \tag{64}$$

Then it follows at once from the properties of smooth systems that G is continuously differentiable on all of S. Hence for each fixed element, $(f_0, \mu_0, \varepsilon_0) \in S$, the $(\alpha + \alpha_2)$-square Jacobian matrix

$$\nabla_{(f, \mu)} G(f_0, \mu_0, \varepsilon_0) = \begin{bmatrix} \nabla_f c(f_0, \varepsilon_0) & -M^{\mathrm{T}} \\ M & 0 \end{bmatrix} \tag{65}$$

is well defined. Similarly, the $(\alpha + \alpha_2) \times \alpha$ Jacobian matrix,

$$\nabla_\varepsilon G(f_0, \mu_0, \varepsilon_0) = \begin{bmatrix} \nabla_\varepsilon c(f_0, \varepsilon_0) \\ \Delta_2^0 N_2 \nabla_\varepsilon T(\varepsilon_0) \end{bmatrix} \tag{66}$$

is also well defined. By employing this function, G, we now show that:

Theorem 4.1 [Differentiation Theorem] *For each locally smooth system*, (D, F, T, c) *there exists a* 0-*neighborhood*, $B_0 \subseteq B(0) \subseteq D$, *such that the restrictions of the multiplier function*, $\mu : B_0 \rightarrow R^{\alpha_2}$, *and the equilibrium arc-flow function*, $f : B_0 \rightarrow R^{\alpha}_+$, *are both continuously differentiable, and in particular*

$$\begin{bmatrix} \nabla f(0) \\ \nabla \mu(0) \end{bmatrix} = \begin{bmatrix} \nabla_f c[f(0), 0] & -M^{\mathrm{T}} \\ M & 0 \end{bmatrix}^{-1} \begin{bmatrix} -\nabla_\varepsilon c[f(0), 0] \\ -\Delta_2^0 N_2 \nabla T(0) \end{bmatrix}. \tag{67}$$

Proof. Since $0 \in B(0)$, it follows at once from the definition of G, together with Eqs. (60) and (62) that $G[f(0), \mu(0), 0] = 0$. Moreover, the Jacobian matrix in Eq. (65) is nonsingular at the point $(f_0, \mu_0, \varepsilon_0) = [f(0), \mu(0), 0]$. To see this, observe first that the matrix,

$$C = \nabla_f c[f(0), 0]. \tag{68}$$

is positive definite by C4, and hence must be nonsingular (since $x \neq 0 \Rightarrow x^T C x > 0 \Rightarrow C x \neq 0$). Next observe that each (possibly nonsymmetric) positive definite matrix, C, has a positive definite inverse (since $x \neq 0 \Rightarrow C^{-1} x \neq 0 \Rightarrow 0 < (C^{-1} x)^T C (C^{-1} x) = (C^{-1} x)^T x = x^T C^{-1} x$). But the full row rankness of M then implies that $M C^{-1} M^T$ must also be positive definite (since $x \neq 0 \Rightarrow M x \neq 0 \Rightarrow x^T (M C^{-1} M^T) x = (M^T x)^T C^{-1} (M^T x) > 0$), and hence nonsingular. Thus the determinants, $\det(C)$ and $\det(M C^{-1} M^T)$ are each nonzero, and we may conclude from Eq. (65) (together with standard determinantal identities for partitioned matrices) that

$$\det(\nabla_{(f,\mu)} G[f(0), \mu(0), 0]) = \det \begin{pmatrix} C & -M^T \\ M & 0 \end{pmatrix} = \det(C) \cdot \det(M C^{-1} M^T) \neq 0, \tag{69}$$

so that the Jacobian matrix $\nabla_{(f,\mu)} G[f(0), \mu(0), 0]$ is seen to be nonsingular. This in turn implies from the Implicit Function Theorem (as for example in Bartle (1964), Theorem 21.11)) that there exists a 0-neighborhood, $B_0 \subseteq B(0)$, and continuously differentiable functions, $f_0 : B(0) \to R_+^\alpha$ and $\mu_0 : B(0) \to R^{\alpha_2}$, such that for each $\varepsilon \in B(0)$, the point $[f_0(\varepsilon), \mu_0(\varepsilon), \varepsilon]$ is the unique solution to the equation, $G[f, \mu, \varepsilon] = 0$. But since Eqs. (60), (62) and (64) also imply that $G[f(\varepsilon), \mu(\varepsilon), \varepsilon] = 0$ for all $\varepsilon \in B_0 \subseteq B(0)$, we may thus conclude that $f = f_0$ and $\mu = \mu_0$ on B_0, so that in particular, f and μ must be continuously differentiable on B_0. Finally, it then follows from the standard corollary to the Implicit Function Theorem (as for example in Bartle (1964) Corollary 21.12)) that for the point, $0 \in B_0$,

$$\begin{bmatrix} \nabla f(0) \\ \nabla \mu(0) \end{bmatrix} = [\nabla_{(f,\mu)} G[f(0), \mu(0), 0]]^{-1} [-\nabla_\varepsilon G[f(0), \mu(0), 0]], \tag{70}$$

so that Eq. (67) follows by combining Eq. (70) with Eqs. (65) and (66) evaluated at $[f(0), \mu(0), 0]$.

In particular, it can be shown by direct calculation that if we employ Eq. (68) to write

$$M(0) = C^{-1} - C^{-1} M^T (M C^{-1} M^T)^{-1} M C^{-1}, \tag{71}$$

then $\nabla f(0)$ can be expressed in terms of Eqs. (67) and (71) as

$$\nabla f(0) = -M(0) \nabla_\varepsilon c[f(0), 0] - M^T (M M^T)^{-1} \Delta_2^0 N_2 \nabla T(0). \tag{72}$$

Hence, expression (72) yields a direct evaluation of the rates of change of the equilibrium arc-flows, f, with respect to the perturbation parameters, ε, at the unperturbed equilibrium point, $\varepsilon = 0$. Finally, this implies that for all ε sufficiently close to zero, the *local linearization*,

$$f(\varepsilon) = f(0) + \nabla f(0) \varepsilon \tag{73}$$

of f yields a good approximation to $f(\varepsilon)$ in the sense that for each $\delta > 0$ there is a 0-neighborhood, $B \subseteq B_0$ such that for all $\varepsilon \in B$

$$\|f(\varepsilon) - f(\varepsilon)\| \leqslant \delta \|\varepsilon\|. \tag{74}$$

Finally, it should be emphasized that the simplicity of the above result depends critically on the positivity condition (C2), which eliminates the complications of binding inequalities. As

mentioned in the introduction, this condition is almost always met in practice (and indeed is a *generic* property of arc-flow equilibria for any choice of reasonably behaved functions). Moreover, any exceptional cases violating C2 can in practice be handled by the auxiliary linear programming procedure of Tobin and Friesz (1988) mentioned in the introduction. However, from a theoretical viewpoint, the general differentiability results of Kyparisis (1987, 1988) and Pang (1990) (based on the "generalized equation" approach of Robinson (1979, 1980)) should allow a more unified approach to this problem, and will be explored in a subsequent paper. □
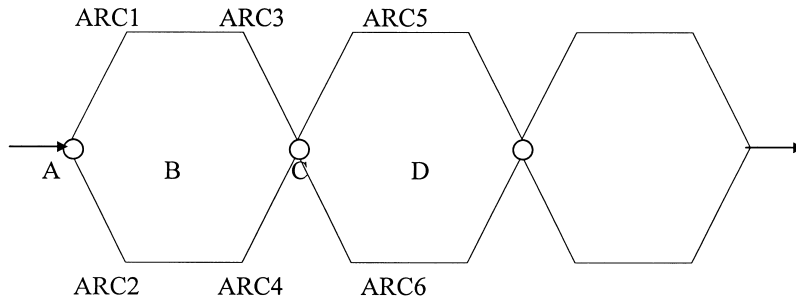
Fig. 1. Example network.

Table 1
Arc costs, travel demand, and equilibrium flows

ARC cost function	Unperturbed Equilibrium arc flow
$c_1 = 4 + \varepsilon_1^\alpha + (1 + \varepsilon_1^\beta)f_1^4$	$f_1(0) = 6$
$c_2 = 20 + \varepsilon_2^\alpha + (5 + \varepsilon_2^\beta)f_2^4$	$f_2(0) = 4$
$c_3 = 1 + \varepsilon_3^\alpha + (30 + \varepsilon_3^\beta)f_3^4$	$f_3(0) = 3$
$c_4 = 30 + \varepsilon_4^\alpha + (1 + \varepsilon_4^\beta)f_4^4$	$f_4(0) = 7$
$c_5 = 10 + \varepsilon_5^\alpha + (3 + \varepsilon_5^\beta)f_1^4$	$f_5(0) = 5$
$c_6 = 10 + \varepsilon_6^\alpha + (3 + \varepsilon_6^\beta)f_1^4$	$f_6(0) = 5$

O/D Travel demand
$T_{AD} = 10 + \varepsilon^D$

Path definitions	Positive equilibrium Path-flow
PATH 1 = $\{(1), (3), (5)\}$	$h_1(0) = 1.0$
PATH 2 = $\{(1), (3), (6)\}$	$h_2(0) = 1.0$
PATH 3 = $\{(1), (4), (5)\}$	$h_3(0) = 2.0$
PATH 4 = $\{(1), (4), (6)\}$	$h_4(0) = 2.0$
PATH 5 = $\{(2), (3), (5)\}$	$h_5(0) = 0.5$
PATH 6 = $\{(2), (3), (6)\}$	$h_6(0) = 0.5$
PATH 7 = $\{(2), (4), (5)\}$	$h_7(0) = 1.5$
PATH 8 = $\{(2), (4), (6)\}$	$h_8(0) = 1.5$

5. Numerical example

In this section we provide a simple numerical example involving an extension of the network employed by Tobin and Friesz (1988). This network, which is illustrated in Fig. 1 involves a set of four nodes, $N = \{n_1, n_2, n_3, n_4\}$ together with a set of six arcs, $A = \{a_1, \ldots, a_6\}$, yielding a set of eight possible paths, $P = \{p_1, \ldots, p_8\}$, between the single origin–destination pair (n_1, n_4) shown in Table 1. The arc-cost function, $c: R_+^6 \times D \to R_+^6$ and travel-demand function, $T: D \to R_+$, are also depicted in Table 1, where each component arc-cost function, $c_i: R_+^6 \times D \to R_+$ involves an in-tercept perturbation parameter, ε_i^α, and slope perturbation parameter, ε_i^β, where the travel demand between n_1 and n_4 involves a single perturbation parameter, ε_d. Hence the relevant perturbation domain, D, is taken to consist of a (closed) zero neighborhood in R^{13} which is sufficiently small to

Table 2
Arc-path incidence matrix and its partition

ARC – PATH INCIDENCE MATRIX

	PATH	1	2	3	4	5	6	7	8
ARC									
	1	1	1	1	1	0	0	0	0
	2	0	0	0	0	1	1	1	1
$\Delta^o =$ 3		1	1	0	0	1	1	0	0
	4	0	0	1	1	0	0	1	1
	5	1	0	1	0	1	0	1	0
	6	0	1	0	1	0	1	0	1

PARTITION of ARC – PATH INCIDENCE MATRIX

	PATH	1	2	3	4	5	6	7	8
ARC									
$\Delta_1^o =$ 1		1	1	1	1	0	0	0	0
	3	1	1	0	0	1	1	0	0
	5	1	0	1	0	1	0	1	0

		1	2	3	4	5	6	7	8
$\Lambda^o =$	T_{AD}	1	1	1	1	1	1	1	1

	PATH	1	2	3	4	5	6	7	8
ARC									
$\Delta_2^o =$ 2		0	0	0	0	1	1	1	1
	4	0	0	1	1	0	0	1	1
	6	0	1	0	1	0	1	0	1

ensure that each perturbation vector $\varepsilon = (\varepsilon_1^\alpha, \ldots, \varepsilon_6^\alpha, \varepsilon_1^\beta, \ldots, \varepsilon_6^\beta, \varepsilon_d) \in D$ yields positive travel demands, $T(\varepsilon)$ and positive increasing arc-cost functions, $c(\cdot, \varepsilon)$. By definition, c and T are continuously differentiable functions. Moreover, the strictly increasing separable forms of the component arc-cost functions ensures global strict monotonicity of c. Thus $|\mathrm{VI}[c(\cdot, \varepsilon), \Omega(\varepsilon)]| = 1$ for all $\varepsilon \in D$, and it follows by setting $F = R^6$, that we obtain a well-defined *perturbation system*, (D, F, T, c) which satisfies C1 and C3. Moreover, it may readily be verified that the unique unperturbed equilibrium arc-flow vector, $f(0)$, shown in Table 1 is consistent with the positive equilibrium path-flow vector, $h(0)$, shown in Table 1, so that (D, F, T, c) also satisfies the local positivity condition, C2, and must be locally regular. Finally, since the Jacobian matrix, $\nabla_f c[f(0), 0]$, in Table 3 is a positive diagonal matrix, and hence is positive definite, we may conclude that (D, F, T, c) satisfies all the conditions for a *locally smooth perturbation system*.

This implies from Theorem 4.1 that the local linearization, $f(\varepsilon)$, of the equilibrium arc-flow vector, $f(\varepsilon)$, for each ε sufficiently close to zero is given by Eq. (73) together with Eq. (72). To compute this local linearization, we have chosen the maximal row-rank partition of $\Delta^0(= \Delta)$ shown in Table 2. By employing this partition, one may compute the relevant Jacobian matrices for expression (67) as in Table 3, and hence evaluate the local linearization, $f(\varepsilon)$, in Eq. (73).

Within this general computational framework, a number of specific sensitivity estimates are computed in Tables 4–6. For example, Table 6 illustrates a case involving simultaneous perturbations of the travel demand, and slopes of the arc-cost functions on arcs a_1, a_2, and a_3. These calculations (which involve only *columns* 1, 2, 3, and 13 of the matrices $\nabla_\varepsilon c[f(0), 0]$ and

Table 3
Jacobian matrices for Eq. (67)

$$\nabla_f c[f(0), 0] = \begin{bmatrix} 864 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1280 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3240 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1372 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1500 \end{bmatrix}$$

$$M = [\Delta_2^0 N_1 - I] = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\nabla_\varepsilon c[f(0), f] = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & -1296 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -256 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -81 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -2401 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -625 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -625 & 0 \end{bmatrix}$$

$$\Delta_2^0 \nabla_2 T(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Table 4
Comparison of estimated solutions with actual solutions for $\varepsilon_1^\alpha = 0.2$ and $\varepsilon_1^\alpha = 0.4$

Solutions with $\varepsilon = 0$		Solutions with $\varepsilon_1^\alpha = 0.2$		Solutions with $\varepsilon_1^\alpha = 0.4$	
		Actual	Estimated	Actual	Estimated
f_1	6	5.99991	5.99991	5.99981	5.99981
f_2	4	4.00009	4.00009	4.00019	4.00019
f_3	3	3	3	3	3
f_4	7	7	7	7	7
f_5	5	5	5	5	5
f_6	5	5	5	5	5

Table 5
Comparison of estimated solutions with actual solutions for $\varepsilon_1^\beta = 0.2$ and $\varepsilon_1^\beta = 0.4$

Solutions with $\varepsilon = 0$		Solutions with $\varepsilon_1^\beta = 0.2$		Solutions with $\varepsilon_1^\alpha = 0.4$	
		Actual	Estimated	Actual	Estimated
f_1	6	5.88935	5.8791045	5.79516	5.758209
f_2	4	4.11065	4.1209855	4.20484	4.241791
f_3	3	3	3	3	3
f_4	7	7	7	7	7
f_5	5	5	5	5	5
f_6	5	5	5	5	5

Table 6
Comparison of estimated solutions with actual solutions for $\varepsilon_1^\beta = 0.2$, $\varepsilon_3^\beta = 0.2$, $\varepsilon_5^\beta = 0.2$, $\varepsilon^D = 2$ and $\varepsilon^D = 4$

Solutions with $\varepsilon = 0$		Solutions with $\varepsilon_1^\beta = 0.2$ ε_3^β, $\varepsilon_5^\beta = 0.2 = 0.2$, $\varepsilon^D = 2$		Solutions with $\varepsilon_1^\beta = 0.2$ $\varepsilon_3^\beta = 0.2$, $\varepsilon_5^\beta = 0.2$, $\varepsilon^D = 4$	
		Actual	Estimated	Actual	Estimated
f_1	6	7.06303	7.07313	8.23813	8.26716
f_2	4	4.93697	4.92687	5.76187	5.73284
f_3	3	3.5919	3.59146	4.1886	4.18643
f_4	7	8.4081	8.40854	9.8114	9.81357
f_5	5	5.9516	5.9583	6.94353	6.9583
f_6	5	6.0484	6.0417	7.05647	7.0417

$\Delta_2^0 N_2 \nabla T(0)$ in Table 3) show that even for reasonably large perturbations, the local linear approximations are quite good. Note finally that while the present network is of course very simple, it nonetheless illustrates a case in which the number of paths in the unperturbed equilibrium can exceed the number of arcs plus origin–destination pairs, and hence in which the path restrictions of Dafermos and Nagurney (1984c) are violated. Moreover, while there do exist equilibrium path-flow vectors which meet these conditions such as $h(0) = (0, 2, 4, 0, 1, 0, 0, 3)$, the determination of such vectors can require lengthy search procedures in more complex networks. Hence, this simple

example serves to illustrate the potential usefulness of the present reduction methods for sensitivity analyses.

6. Further Reading

Berge (1963), Dugundji (1966), Friesz (1985), Harker and Pang (1987, 1988) and Hildenbrand (1974).

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