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W. L. Pearn [a] & G. H. Lin [b]

[a] Department of Industrial Engineering & Management , National Chiao Tung University , Hsinchu, Taiwan, ROC

[b] Department of Telecommunication Engineering , National Penghu Institute of Technology , Penghu, Taiwan, ROC

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ESTIMATING CAPABILITY INDEX C_{pk} FOR PROCESSES WITH ASYMMETRIC TOLERANCES

W. L. Pearn
Department of Industrial Engineering & Management
National Chiao Tung University
Hsinchu, Taiwan ROC

G. H. Lin
Department of Telecommunication Engineering
National Penghu Institute of Technology
Penghu, Taiwan ROC

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ABSTRACT

Pearn and Chen (1996) considered the process capability index C_{pk}, and investigated the statistical properties of its natural estimator under various process conditions. Their investigation, however, was restricted to processes with symmetric tolerances. Recently, Pearn and Chen (1998) considered a generalization of C_{pk}, referred to as C^{*}_{pk}, to cover processes with asymmetric tolerances. They investigated the statistical properties of the natural estimator of C^{*}_{pk}, and obtained the exact formulae for the expected value and variance. In this paper, we consider a new estimator of C^{*}_{pk}, assuming the knowledge on $P(\mu \geq T) = p$ is available, where $0 \leq p \leq 1$, which can be obtained from historical information of a stable process. We obtain the exact distribution of the

new estimator assuming the process characteristic follows the normal distribution. We show that the new estimator is consistent, asymptotically unbiased, which converges to a mixture of two normal distributions. We also show that by adding suitable correction factors to the new estimator, we may obtain the UMVUE and the MLE of the generalization C_{pk}^{*}.

1. INTRODUCTION

Process capability indices C_p and C_{pk}, have been widely used in the manufacturing industry providing numerical measures on process potential and performance. Application examples include the manufacturing of semiconductor products (Hoskins *et al.* (1988)), head/gimbals assembly for memory storage systems (Rado (1989)), jet-turbine engine components (Hubele *et al.* (1991)), flip-chips and chip-on-boards (Noguera and Nielson (1992)), rubber surrounds (Pearn and Kotz (1994)), wood products (Lyth and Rabiej (1995)), audio-speaker drivers (Chen and Pearn (1997)) electrolytic capacitors (Pearn and Chen (1997)) and many others. The two indices C_p and C_{pk} have been defined as the following (Kane (1986)):

$$C_p = \frac{USL - LSL}{6\sigma},$$

$$C_{pk} = \min\left\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\},$$

where USL and LSL are the upper and the lower specification limits, μ and σ are the process mean and the process standard deviation. The natural estimators of the two indices can be expressed as:

$$\hat{C}_p = \frac{USL - LSL}{6S},$$

$$\hat{C}_{pk} = \min\left\{\frac{USL - \bar{X}}{3S}, \frac{\bar{X} - LSL}{3S}\right\},$$

where $\bar{X} = (\sum_{i=1}^{n} X_i)/n$ and $S = \{(n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\}^{1/2}$ are conventional estimators of μ and σ. We note that C_{pk} can be rewritten as

$C_{pk} = \{d - |\mu - m|\}/3\sigma$, where $d = (USL - LSL)/2$, and $m = (USL + LSL)/2$. Thus, $\hat{C}_p$ and $\hat{C}_{pk}$ may be expressed as:

$$\hat{C}_p = \frac{d}{3S}, \quad \text{and } \hat{C}_{pk} = \frac{d - |\overline{X} - m|}{3S} = \left\{1 - \left|\frac{\overline{X} - m}{d}\right|\right\}\hat{C}_p.$$

Under normality assumption, $\hat{C}_p$ is distributed as $(n-1)^{1/2} C_p(\chi_{n-1}^{-1})$, and $n^{1/2}|\overline{X} - m|/\sigma$ is distributed as the folded normal distribution with parameter $n^{1/2}|\mu - m|/\sigma$. Thus, $\hat{C}_{pk}$ is a convolution of χ_{n-1}^{-1} and the folded normal distributions (Pearn *et al.* (1992)). If the knowledge on $P(\mu \geq m) = p$ is available, where $0 \leq p \leq 1$, Pearn and Chen (1996) proposed a Bayesian-like estimator which is more reliable (with smaller variance) than the natural estimator.

2. THE GENERALIZATION C_{pk}^*

The indices C_p and C_{pk} are appropriate for processes with symmetric tolerances, but have been shown to be inappropriate for processes with asymmetric tolerances. For processes with asymmetric tolerances, Pearn and Chen (1998) considered a generalization of C_{pk}, referred to as C_{pk}^*, which is defined as :

$$C_{pk}^* = \frac{d^* - A^*}{3\sigma},$$

where $d^* = \min\{d_u, d_l\}$, $A^* = \max\{d^*(\mu - T)/d_u, d^*(T - \mu)/d_l\}$, $d_u = USL - T$, and $d_l = T - LSL$. Clearly, if $T = m$ (symmetric case) then $d^* = d$, $A^* = |\mu - m|$ and the generalization C_{pk}^* reduces to the original index C_{pk}. The factors d^* and A^* ensure that the generalization C_{pk}^* obtain its maximal value at T (process is on-target in this case) regardless of whether the tolerances are symmetric or asymmetric.

The natural estimator of C_{pk}^* can be obtained by replacing μ and σ by $\overline{X}$ and S as defined earlier. Thus the natural estimator can be expressed as the following, where $\hat{A}^* = \max\{d^*(\overline{X} - T)/d_u, d^*(T - \overline{X})/d_l\}$. Pearn and Chen (1998) investigated the statistical properties of the natural estimator of C^*_{pk}, and obtained the exact formulae for the expected value and variance.

$$\hat{C}^*_{pk} = \frac{d^* - \hat{A}^*}{3S}.$$

3. A NEW ESTIMATOR OF C^*_{pk}

If the knowledge on $P(\mu \geq T) = p$ is available, where $0 \leq p \leq 1$, then we consider the following new estimator $\tilde{C}^*_{pk}$. The knowledge on the probability $P(\mu \geq T) = p$ may be obtained from historical information of a process that is demonstrably stable.

$$\tilde{C}^*_{pk} = \frac{d^* - \tilde{A}^*}{3S},$$

where $\tilde{A}^* = \max\ \{ (\bar{X}-T)I_{B^*}(\mu)d^*/d_u,\ (\bar{X}-T)I_{B^*}(\mu)d^*/d_l \}$, $I_{B^*}(.)$ is the indicator function defined as $I_{B^*}(\mu) = 1$ if $\mu \in B^*$, and $I_{B^*}(\mu) = -1$ if $\mu \notin B^*$, with $B^* = \{\mu \mid \mu \geq T\}$. In the following, we show that if the process characteristic follows the normal distribution $N(\mu, \sigma^2)$ then the new estimator $\tilde{C}^*_{pk}$ is distributed as $t_{n-1}(\delta^*)$, a non-central t distribution with n -1 degrees of freedom and non-centrality parameter $\delta^* = 3\sqrt{n}C^*_{pk}$.

Theorem 1. If the process characteristic follows the normal distribution, then $3n^{1/2}\tilde{C}^*_{pk}$ is distributed as $t_{n-1}(\delta^*)$, a non-central t distribution with n - 1 degrees of freedom and non-centrality parameter $\delta^* = 3\sqrt{n}C^*_{pk}$.

Proof: Case 1: If USL - T > T - LSL, then $\tilde{C}^*_{pk} = \{d_l - (\bar{X}-T)I_{B^*}(\mu)\}/(3S)$,

$$\tilde{C}^*_{pk}\frac{S}{\sigma} - C^*_{pk} = \frac{\{d_l - (\bar{X}-T)I_{B^*}(\mu)\}}{3S}\frac{S}{\sigma} - \frac{\{d_l - (\mu - T)I_{B^*}(\mu)\}}{3\sigma} = -\frac{(\bar{X}-\mu)I_{B^*}(\mu)}{3\sigma}.$$

Case 2: If USL - T < T - LSL, then $\tilde{C}^*_{pk} = \{d_u - (\bar{X}-T)I_{B^*}(\mu)\}/(3S)$,

$$\tilde{C}^*_{pk}\frac{S}{\sigma} - C^*_{pk} = \frac{\{d_u - (\bar{X}-T)I_{B^*}(\mu)\}}{3S}\frac{S}{\sigma} - \frac{\{d_u - (\mu - T)I_{B^*}(\mu)\}}{3\sigma} = -\frac{(\bar{X}-\mu)I_{B^*}(\mu)}{3\sigma}.$$

Case 3: If USL - T = T - LSL, then C^*_{pk} reduces to C_{pk},

$$\tilde{C}^*_{pk} = \{d - (\bar{X} - m)I_{B^*}(\mu)\}/(3S)\text{, and}$$

$$\tilde{C}^*_{pk}\frac{S}{\sigma} - C^*_{pk} = \frac{\{d - (\bar{X} - m)I_{B^*}(\mu)\}}{3S}\frac{S}{\sigma} - \frac{\{d - (\mu - m)I_{B^*}(\mu)\}}{3\sigma} = -\frac{(\bar{X} - \mu)I_{B^*}(\mu)}{3\sigma}.$$

Thus, for all three cases, and for all $x \in R$,

$$P\left\{3\sqrt{n}\left(\tilde{C}^*_{pk}\frac{S}{\sigma} - C^*_{pk}\right) \le x\right\}$$

$$= P\left\{3\sqrt{n}\left(\tilde{C}^*_{pk}\frac{S}{\sigma} - C^*_{pk}\right) \le x \middle| I_{B^*}(\mu) = 1\right\} \times P\{I_{B^*}(\mu) = 1\}$$

$$+ P\left\{3\sqrt{n}\left(\tilde{C}^*_{pk}\frac{S}{\sigma} - C^*_{pk}\right) \le x \middle| I_{B^*}(\mu) = -1\right\} \times P\{I_{B^*}(\mu) = -1\}$$

$$= P\left\{\sqrt{n}\left(\frac{\bar{X} - \mu}{\sigma}\right) \ge -x\right\} \times P\{I_{B^*}(\mu) = 1\}$$

$$+ P\left\{\sqrt{n}\left(\frac{\bar{X} - \mu}{\sigma}\right) \le x\right\} \times P\{I_{B^*}(\mu) = -1\}$$

$$= P\left\{\sqrt{n}\left(\frac{\bar{X} - \mu}{\sigma}\right) \le x\right\} = \Phi(x).$$

Under normality assumption, $\bar{X}$ and S^2 are mutually independent. Therefore, $Z = 3n^{1/2}\tilde{C}^*_{pk}S/\sigma$ and $W = (n-1)S^2/\sigma^2$ are also mutually independent, with Z distributed as $N(\delta^*, 1)$, a normal distribution with mean $\delta^* = 3\sqrt{n}C^*_{pk}$, and W distributed as χ^2_{n-1}, a chi-squared distribution with n - 1 degrees of freedom. Thus $3n^{1/2}\tilde{C}^*_{pk} = Z/\sqrt{W/(n-1)}$ is distributed as $t_{n-1}(\delta^*)$, a non-central *t* distribution with n - 1 degrees of freedom and non-centrality parameter $\delta^* = 3\sqrt{n}C^*_{pk}$.

The r-th moment (about zero), therefore, is:

$$E[\tilde{C}^*_{pk}]^r = \left(\frac{1}{3\sqrt{n}}\right)^r E\{t_{n-1}(\delta^*)\}^r.$$

By setting $r = 1$, and $r = 2$, we may obtain the expected value $E(\tilde{C}^*_{pk})$, and the variance $Var(\tilde{C}^*_{pk})$,

$$E[\tilde{C}^*_{pk}] = \left(\frac{1}{3\sqrt{n}}\right)E\{t_{n-1}(\delta^*)\} = \frac{1}{3\sqrt{n}}\frac{\delta^*}{b_{n-1}} = \frac{C^*_{pk}}{b_{n-1}},$$

Table 1. Values of $MSE(\tilde{C}^*_{pk})$ for n = 5(5)100 with $C^*_{pk} = 1$.

Sample size	MSE	Sample size	MSE
5	0.538	55	0.012
10	0.112	60	0.011
15	0.060	65	0.010
20	0.040	70	0.009
25	0.030	75	0.009
30	0.024	80	0.008
35	0.020	85	0.008
40	0.017	90	0.007
45	0.015	95	0.007
50	0.014	100	0.006

$$E[\tilde{C}^*_{pk}]^2 = \left(\frac{1}{3\sqrt{n}}\right)^2 E\{t_{n-1}(\delta^*)\}^2 = \frac{n-1}{n-3}\left\{[C^*_{pk}]^2 + \frac{1}{9n}\right\},$$

$$b_{n-1} = \frac{\Gamma[(n-1)/2]}{\Gamma[(n-2)/2]}\sqrt{\frac{2}{n-1}}.$$

Therefore, the mean-squared error $MSE(\tilde{C}^*_{pk})$ can be obtained as:

$$MSE(\tilde{C}^*_{pk}) = E[\tilde{C}^*_{pk} - C^*_{pk}]^2 = E[\tilde{C}^*_{pk}]^2 - 2C^*_{pk}E[\tilde{C}^*_{pk}] + [C^*_{pk}]^2$$

$$= 2\left(\frac{n-2}{n-3} - \frac{1}{b_{n-1}}\right)[C^*_{pk}]^2 + \frac{n-1}{n-3} \times \frac{1}{9n}.$$

It can be shown that the coefficient of $E(\tilde{C}^*_{pk})$, $b_{n-1} = [2/(n-1)]^{1/2} \times \Gamma[(n-1)/2]/\Gamma[(n-2)/2] < 1$ for all n, which in fact converges to 1 as n approaches to infinity. Thus, the estimator $\tilde{C}^*_{pk}$ is biased, which over-estimates the actual value of C^*_{pk}. Table 1 displays various values of $MSE(\tilde{C}^*_{pk})$ for sample sizes n = 5(5)100 under the condition of $C^*_{pk} = 1$. For sample size n $>$ 65, the mean-squared error is negligibly small (less than 0.01).

Theorem 2. If the process characteristic follows the normal distribution, then (a) $\tilde{C}^*_{pk}$ is asymptotically unbiased, (b) $\tilde{C}^*_{pk}$ is consistent.

Proof: (a) By Stirling's formula, it is easy to show that b_{n-1} converges to 1. Therefore, $E(\tilde{C}^*_{pk}) = C^*_{pk}/b_{n-1}$ converges to C^*_{pk}. Hence, the estimator $\tilde{C}^*_{pk}$ is asymptotically unbiased. (b) Since $\overline{X}$ converges to μ in probability, and S converges to σ in probability, then $\tilde{C}^*_{pk}$ must converges to C^*_{pk} in probability. Therefore, the estimator $\tilde{C}^*_{pk}$ is consistent.

Theorem 3. If the process characteristic follows the normal distribution, then $n^{1/2}(\tilde{C}^*_{pk} - C^*_{pk})$ converges to the following in distribution:

(a) $p \cdot N(0,\ [\sigma^*_{l1}]^2) + (1-p) \cdot N(0,\ [\sigma^*_{l2}]^2)$, if $USL - T > T - LSL$,

(b) $p \cdot N(0,\ [\sigma^*_{u1}]^2) + (1-p) \cdot N(0,\ [\sigma^*_{u2}]^2)$, if $USL - T < T - LSL$,

(c) $p \cdot N(0,\ [\sigma^*_{m1}]^2) + (1-p) \cdot N(0,\ [\sigma^*_{m2}]^2)$, if $USL - T = T - LSL$, where

$$[\sigma^*_{l1}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d_l - (\mu - T)}{3\sigma}\right\}^2,\quad [\sigma^*_{l2}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d_l + (\mu - T)}{3\sigma}\right\}^2,$$

$$[\sigma^*_{u1}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d_u - (\mu - T)}{3\sigma}\right\}^2,\quad [\sigma^*_{u2}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d_u + (\mu - T)}{3\sigma}\right\}^2,$$

$$[\sigma^*_{m1}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d - (\mu - m)}{3\sigma}\right\}^2,\quad [\sigma^*_{m2}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d + (\mu - m)}{3\sigma}\right\}^2.$$

Proof: See Appendix.

Theorem 4. If the process characteristic follows the normal distribution, then (a) $b_{n-1}\tilde{C}^*_{pk}$ is the UMVUE of C^*_{pk}, (b) $c_{n-1}\tilde{C}^*_{pk}$ is the MLE of C^*_{pk}, where $b_{n-1} = \Gamma[(n-1)/2]\{\Gamma[(n-2)/2]\}^{-1}[2/(n-1)]^{1/2}$, and $c_{n-1} = [n/(n-1)]^{1/2}$.

Proof: (a) Since $E[\tilde{C}^*_{pk}] = C^*_{pk}/b_{n-1}$, then $b_{n-1}\tilde{C}^*_{pk}$ is an unbiased estimator of C^*_{pk}. Since the unbiased estimator $b_{n-1}\tilde{C}^*_{pk}$ is based on the complete and sufficient statistics $(\overline{X},\ S^2)$ only, then by Lehmann-Scheffe's theorem, $b_{n-1}\tilde{C}^*_{pk}$ is the UMVUE of C^*_{pk}.

(b) We first note that the statistic $(\overline{X},\ [(n-1)/n]S^2)$ is the MLE of $(\mu,\ \sigma^2)$. By the invariance property of the MLE, $c_{n-1}\tilde{C}^*_{pk}$ is the MLE of C^*_{pk}, where $c_{n-1} = [n/(n-1)]^{1/2}$.

Applying the Slutsky's theorem (Arnold (1990)), it is straightforward to show that the UMVUE $b_{n-1}\tilde{C}^*_{pk}$, and the MLE $c_{n-1}\tilde{C}^*_{pk}$ both converge to the mixture of two normal distributions as stated in Theorem 3.

4. CONCLUSIONS

Pearn and Chen (1998) proposed a generalization of C_{pk}, referred to as C^*_{pk}, for processes with asymmetric tolerances. They investigated the statistical properties of the natural estimator of C^*_{pk}, and obtained the exact formulae for the expected value and variance. In this paper, we considered a new estimator of C^*_{pk}, assuming the knowledge on $P(\mu \geq T) = p$ is available, where $0 \leq p \leq 1$, which can be obtained from historical information of a stable process. We showed that under normality assumption the new estimator is distributed as $t_{n-1}(\delta^*)$, a non-central t distribution with n - 1 degrees of freedom and non-centrality parameter $\delta^* = 3\sqrt{n}C^*_{pk}$. We also showed that the new estimator is consistent, asymptotically unbiased, which converges to a mixture of two normal distributions. In addition, we showed that by adding suitable correction factors to the new estimator, we may obtain the UMVUE and the MLE of the generalization C^*_{pk}.

APPENDIX

Theorem 3. If the process characteristic follows the normal distribution, then $\sqrt{n}(\tilde{C}^*_{pk} - C^*_{pk})$ converges to the following in distribution:

(a) $p \cdot N(0,\ [\sigma^*_{l1}]^2) + (1-p) \cdot N(0,\ [\sigma^*_{l2}]^2)$, if $USL - T > T - LSL$,

(b) $p \cdot N(0,\ [\sigma^*_{u1}]^2) + (1-p) \cdot N(0,\ [\sigma^*_{u2}]^2)$, if $USL - T < T - LSL$,

(c) $p \cdot N(0,\ [\sigma^*_{m1}]^2) + (1-p) \cdot N(0,\ [\sigma^*_{m2}]^2)$, if $USL - T = T - LSL$, with

$$[\sigma^*_{l1}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d_l - (\mu - T)}{3\sigma}\right\}^2,\quad [\sigma^*_{l2}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d_l + (\mu - T)}{3\sigma}\right\}^2,$$

$$[\sigma^*_{u1}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d_u - (\mu - T)}{3\sigma}\right\}^2,\quad [\sigma^*_{u2}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d_u + (\mu - T)}{3\sigma}\right\}^2,$$

$$[\sigma^*_{m1}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d - (\mu - m)}{3\sigma}\right\}^2,\quad [\sigma^*_{m2}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d + (\mu - m)}{3\sigma}\right\}^2.$$

Proof: (a) If $USL - T > T - LSL$, then

$$\tilde{C}^{*}_{pk} = \frac{d_l - (\overline{X} - T)}{3S}, \text{ for } \mu \geq T \text{ and } \tilde{C}^{*}_{pk} = \frac{d_l + (\overline{X} - T)}{3S}, \text{ for } \mu < T.$$

(1) For $\mu > T$, we define $g_{l1}(x,y) = \dfrac{d_l - (x - T)}{3\sqrt{y}}$, $x > T$, $y > 0$. Then,

$\sqrt{n}(\tilde{C}^{*}_{pk} - C^{*}_{pk}) = \sqrt{n}\{g_{l1}(\overline{X}, S^2) - g_{l1}(\mu, \sigma^2)\}$ converges to $N(0, [\sigma^{*}_{l1}]^2)$

in distribution, with

$$[\sigma^{*}_{l1}]^2 = \left[\left.\frac{\partial g_{l1}}{\partial x}\right|_{(\mu,\sigma^2)}, \left.\frac{\partial g_{l1}}{\partial y}\right|_{(\mu,\sigma^2)} \right] \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix} \begin{bmatrix} \left.\frac{\partial g_{l1}}{\partial x}\right|_{(\mu,\sigma^2)} \\ \left.\frac{\partial g_{l1}}{\partial y}\right|_{(\mu,\sigma^2)} \end{bmatrix}$$

$$= \left[-\frac{1}{3\sigma}, -\frac{C^{*}_{pk}}{2\sigma^2} \right] \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix} \begin{bmatrix} -\frac{1}{3\sigma} \\ -\frac{C^{*}_{pk}}{2\sigma^2} \end{bmatrix} = \frac{1}{9} + \frac{C^{*}_{pk}}{2} = \frac{1}{9} + \frac{1}{2}\left\{ \frac{d_l - (\mu - T)}{3\sigma} \right\}^2.$$

(2) For $\mu < T$, we define $g_{l2}(x,y) = \dfrac{d_l + (x - T)}{3\sqrt{y}}$, $x < T$, $y > 0$. Then,

$\sqrt{n}(\tilde{C}^{*}_{pk} - C^{*}_{pk}) = \sqrt{n}\{g_{l2}(\overline{X}, S^2) - g_{l2}(\mu, \sigma^2)\}$ converges to $N(0, [\sigma^{*}_{l2}]^2)$

in distribution, with

$$[\sigma^{*}_{l2}]^2 = \left[\left.\frac{\partial g_{l2}}{\partial x}\right|_{(\mu,\sigma^2)}, \left.\frac{\partial g_{l2}}{\partial y}\right|_{(\mu,\sigma^2)} \right] \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix} \begin{bmatrix} \left.\frac{\partial g_{l2}}{\partial x}\right|_{(\mu,\sigma^2)} \\ \left.\frac{\partial g_{l2}}{\partial y}\right|_{(\mu,\sigma^2)} \end{bmatrix}$$

$$= \left[\frac{1}{3\sigma}, -\frac{C^{*}_{pk}}{2\sigma^2} \right] \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix} \begin{bmatrix} \frac{1}{3\sigma} \\ -\frac{C^{*}_{pk}}{2\sigma^2} \end{bmatrix} = \frac{1}{9} + \frac{C^{*}_{pk}}{2} = \frac{1}{9} + \frac{1}{2}\left\{ \frac{d_l + (\mu - T)}{3\sigma} \right\}^2.$$

(3) For $\mu = T$, we have $\sqrt{n}(\tilde{C}^{*}_{pk} - C^{*}_{pk}) = -\dfrac{\sqrt{n}(\overline{X} - \mu)}{3S} - \dfrac{d_l\sqrt{n}(S^2 - \sigma^2)}{3\sigma(\sigma + S)S}$.

Since $\sqrt{n}(\overline{X} - \mu, S^2 - \sigma^2)$ converges in distribution to (V, W), a bivariate normal distribution $N((0, 0), \Sigma)$, with variance-covariance matrix

$\Sigma = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}$, and $\left(-\dfrac{1}{3S}, -\dfrac{d_l}{3\sigma(\sigma + S)S} \right)$ converges to $\left(-\dfrac{1}{3\sigma}, -\dfrac{d_l}{6\sigma^3} \right)$

in probability, then $\sqrt{n}(\tilde{C}^*_{pk} - C^*_{pk})$ converges to Y_l in distribution, where $Y_l = -\frac{1}{3\sigma}V - \frac{d_l}{6\sigma^3}W$ is distributed as a normal distribution with mean $EY_l = -\frac{1}{3\sigma}EV - \frac{d_l}{6\sigma^3}EW = 0$, and variance

$$\mathrm{Var}(Y_l) = \left(-\frac{1}{3\sigma}\right)^2 \mathrm{Var}(V) + \left(-\frac{d_l}{6\sigma^3}\right)^2 \mathrm{Var}(W) = \frac{1}{9} + \frac{1}{2}\left(\frac{d_l}{3\sigma}\right)^2.$$

We note that for all $z \in R$,

$$P\left\{\sqrt{n}(\tilde{C}^*_{pk} - C^*_{pk}) \le z\right\} = P\{\mu \ge T\} \cdot P\left\{\sqrt{n}(\tilde{C}^*_{pk} - C^*_{pk}) \le z \mid \mu \ge T\right\}$$
$$+ P\{\mu < T\} \cdot P\left\{\sqrt{n}(\tilde{C}^*_{pk} - C^*_{pk}) \le z \mid \mu < T\right\}.$$

Hence, if $USL - T > T - LSL$, then $\sqrt{n}(\tilde{C}^*_{pk} - C^*_{pk})$ converges to $P\{\mu \ge T\} \cdot N(0,\ [\sigma^*_{l1}]^2) + P\{\mu < T\} \cdot N(0,\ [\sigma^*_{l2}]^2)$ in distribution, where

$$[\sigma^*_{l1}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d_l - (\mu - T)}{3\sigma}\right\}^2 \text{ and } [\sigma^*_{l2}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d_l + (\mu - T)}{3\sigma}\right\}^2.$$

(b) If $USL - T < T - LSL$, then

$$\tilde{C}^*_{pk} = \frac{d_u - (\bar{X} - T)}{3S}, \text{ for } \mu \ge T \text{ and } \tilde{C}^*_{pk} = \frac{d_u + (\bar{X} - T)}{3S}, \text{ for } \mu < T.$$

Applying the same technique used in (a) with $g_{u1}(x, y) = \frac{d_u - (x - T)}{3\sqrt{y}}$ for $\mu > T$, $g_{u2}(x, y) = \frac{d_u + (x - T)}{3\sqrt{y}}$ for $\mu < T$, and $Y_u = -\frac{1}{3\sigma}V - \frac{d_u}{6\sigma^3}W$ for $\mu = T$. Then, $\sqrt{n}(\tilde{C}^*_{pk} - C^*_{pk})$ converges to $N(0,\ [\sigma^*_{u1}]^2)$ and $N(0,\ [\sigma^*_{u2}]^2)$ in distribution for $\mu \ge T$ and $\mu < T$ respectively.

Therefore, if $USL - T < T - LSL$, then $\sqrt{n}(\tilde{C}^*_{pk} - C^*_{pk})$ converges to $P\{\mu \ge T\} \cdot N(0,\ [\sigma^*_{u1}]^2) + P\{\mu < T\} \cdot N(0,\ [\sigma^*_{u2}]^2)$ in distribution, where

$$[\sigma^*_{u1}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d_u - (\mu - T)}{3\sigma}\right\}^2,\ [\sigma^*_{u2}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d_u + (\mu - T)}{3\sigma}\right\}^2.$$

(c) If $USL - T = T - LSL$ ($T = m$ in this case), then

$\widetilde{C}^{*}_{pk} = \dfrac{d-(\overline{X}-m)}{3S}$, for $\mu \geq m$ and $\widetilde{C}^{*}_{pk} = \dfrac{d+(\overline{X}-m)}{3S}$, for $\mu < m$.

Applying the same technique used in (a) with $g_{m1}(x,y) = \dfrac{d-(x-m)}{3\sqrt{y}}$ for $\mu > m$, $g_{m2}(x,y) = \dfrac{d+(x-m)}{3\sqrt{y}}$ for $\mu < m$, and $Y_m = -\dfrac{1}{3\sigma}V - \dfrac{d}{6\sigma^3}W$ for $\mu = m$. Then, $\sqrt{n}(\widetilde{C}^{*}_{pk} - C^{*}_{pk})$ converges to $N(0, [\sigma^{*}_{m1}]^2)$ and $N(0, [\sigma^{*}_{m2}]^2)$ in distribution for $\mu \geq m$ and $\mu < m$ respectively.

Therefore, if $USL - T = T - LSL$, we have $\sqrt{n}(\widetilde{C}^{*}_{pk} - C^{*}_{pk})$ converges to $P\{\mu \geq m\} \cdot N(0, [\sigma^{*}_{m1}]^2) + P\{\mu < m\} \cdot N(0, [\sigma^{*}_{m2}]^2)$ in distribution, where

$$[\sigma^{*}_{m1}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d-(\mu-m)}{3\sigma}\right\}^2, \quad [\sigma^{*}_{m2}]^2 = \frac{1}{9} + \frac{1}{2}\left\{\frac{d+(\mu-m)}{3\sigma}\right\}^2.$$

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