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HOLOMORPHIC SECTIONS OF PRE-QUANTUM LINE BUNDLES ON $G/(P,P)$

MENG-KIAT CHUAH

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ABSTRACT. Let $G = KAN$ be the Iwasawa decomposition of a complex connected semi-simple Lie group G. Let $P \subset G$ be a parabolic subgroup containing AN, and let (P,P) be its commutator subgroup. In this paper, we characterize the K-invariant Kähler structures on $G/(P,P)$, and study the holomorphic sections of their corresponding pre-quantum line bundles.

1. Introduction

Let K be a compact connected semi-simple Lie group, let G be its complexification, and let $G = KAN$ be an Iwasawa decomposition. Let T be the centralizer of A in K, so that $H = TA$ is a Cartan subgroup, and $B = HN$ is a Borel subgroup of G. Let P be a parabolic subgroup of G containing B, and (P,P) its commutator subgroup. Each P determines a subgroup $A_P \subset A$ via Langlands decomposition $P = M_P A_P N_P$ ([7], p. 132). It also determines a subtorus $T_P \subset T$, which makes $H_P = T_P A_P$ complex. Since H_P normalizes (P,P), it has right action on $G/(P,P)$. In [3], we consider $K \times T_P$-invariant Kähler structures ω on $G/(P,P)$, and study the pre-quantum line bundle [8] corresponding to ω. We then observe that the holomorphic sections of the pre-quantum line bundle constitute a nice multiplicity-free K-representation. In this paper, we show that if the K-invariant ω is not preserved by the right T_P-action, then the pre-quantum line bundle has no holomorphic section other than the zero section.

The Lie algebra of a Lie group shall always be denoted by its lowercase German letter. For instance, $\mathfrak{h}$ and $\mathfrak{t}_P$ are the Lie algebras of H and T_P respectively.

Consider the root system in $\mathfrak{h}^*$. By declaring $\mathfrak{n}$ to be the negative root spaces, we obtain a system of positive roots in $\mathfrak{h}^*$. Let Δ be the simple roots. There is a natural bijective correspondence between the parabolic subgroups P containing B and the subsets of Δ. Namely, P corresponds to $\Delta_P \subset \Delta$ by

$$\Delta_P = \{\lambda \in \Delta \ ; \ (\lambda, v) \neq 0 \text{ for some } v \in \mathfrak{t}_P\}. \tag{1.1}$$

Note that as P grows bigger, Δ_P gets smaller. For example, $\Delta_B = \Delta$.

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Fix one parabolic subgroup P containing B, with corresponding simple roots $\Delta_P = \{\lambda_1, ..., \lambda_r\}$ via (1.1). Each λ_i is integral, in the sense that there is a multiplicative homomorphism $\chi_i : H \longrightarrow \mathbf{C}^\times$ such that

$$\chi_i(e^v) = \exp(\lambda_i, v) \tag{1.2}$$

for all $v \in \mathfrak{h}$. Let R_t denote the right action of $t \in T_P$.

Theorem 1. *Every K-invariant Kähler form on $G/(P,P)$ can be uniquely expressed as*

$$\omega = \sqrt{-1}\partial\bar{\partial}F + \sum_1^r d\beta_i,$$

where $R_t^*\beta_i = \chi_i(t)\beta_i$ for all $t \in T_P$. So ω has a potential function if and only if it is right T_P-invariant, if and only if $\sum_1^r d\beta_i$ vanishes.

Let ω be a K-invariant Kähler form on $G/(P,P)$. By Theorem 1, ω is exact. Therefore it is integral, and corresponds to a pre-quantum line bundle $\mathbf{L}$ in the sense of Kostant [8]. Namely the Chern class of $\mathbf{L}$ is $[\omega] = 0$, and $\mathbf{L}$ has a connection ∇ whose curvature is ω. A smooth section s of $\mathbf{L}$ is said to be holomorphic if $\nabla_v s = 0$ whenever v is an anti-holomorphic vector field [5]. Let $H(\mathbf{L})$ denote the holomorphic sections of $\mathbf{L}$. The K-action on $G/(P,P)$ lifts to a K-representation on $H(\mathbf{L})$. In [3], we show that if ω is right T_P-invariant, then every irreducible K-representation with highest weight in $\mathfrak{t}_P^*$ occurs exactly once in $H(\mathbf{L})$. The following theorem observes the opposite situation, when ω is not right T_P-invariant.

Theorem 2. *Suppose that ω does not satisfy the equivalent conditions given in Theorem 1. Then $H(\mathbf{L}) = 0$.*

We remark that partial results of Theorems 1 and 2 appear in [1] and [4], for the special case where P is the Borel subgroup HN. The present paper extends those results to general parabolic subgroups P.

2. Proofs of theorems

In this section, we prove the two theorems mentioned in the introduction. We start with the following topological property of $G/(P,P)$.

Proposition 3. $H^2(G/(P,P), \mathbf{R}) = 0$.

Proof. Let K^P be the centralizer of T_P in K, and $K_{ss}^P = (K^P, K^P)$ be its commutator subgroup. As a manifold, $G/(P,P) = (K/K_{ss}^P) \times A_P$ [3]. Since A_P has the topology of an Euclidean space, it suffices to show that $H^2(K/K_{ss}^P, \mathbf{R}) = 0$. But K is compact. So we only need to consider the DeRham subcomplex of K-invariant differential forms on K/K_{ss}^P, and show that the H^2 of this subcomplex vanishes. This is done via relative Lie algebra cohomology as follows.

We restrict the coadjoint representation of K to K_{ss}^P, and get

$$Ad^* : K_{ss}^P \longrightarrow Aut(\mathfrak{k}^*).$$

We extend this representation to exterior algebras, then differentiate to get the Lie algebra representation

$$ad^* : \mathfrak{k}_{ss}^P \longrightarrow End(\wedge^q \mathfrak{k}^*).$$

The relative Lie algebra cohomology is defined by the complex

$$\wedge^q\, (\mathfrak{k}, \mathfrak{k}_{ss}^P)^* = \{\omega \in \wedge^q \mathfrak{k}^* \ ; \ \iota(v)\omega = ad_v^*\omega = 0 \text{ for all } v \in \mathfrak{k}_{ss}^P\}. \tag{2.1}$$

Here $\iota(v)\omega$ denotes the interior product. We write $H^q(\mathfrak{k}, \mathfrak{k}_{ss}^P)$ for the cohomology resulting from (2.1). The elements in (2.1) can be naturally identified with K-invariant differential forms on K/K_{ss}^P. Hence to prove the proposition, it suffices to show that

$$H^2(\mathfrak{k}, \mathfrak{k}_{ss}^P) = 0. \tag{2.2}$$

Let $\omega \in \wedge^2(\mathfrak{k}, \mathfrak{k}_{ss}^P)^*$, and suppose that $d\omega = 0$. Since $\mathfrak{k}$ is semi-simple, $H^2(\mathfrak{k}) = 0$ by the Whitehead lemma [6]. So since $\omega \in \wedge^2\mathfrak{k}^*$, there exists $\beta \in \wedge^1\mathfrak{k}^*$ such that $d\beta = \omega$. To prove (2.2), we need to show that $\beta \in \wedge^1(\mathfrak{k}, \mathfrak{k}_{ss}^P)^*$; namely

$$\langle \beta, v \rangle = ad_v^* \beta = 0 \tag{2.3}$$

for all $v \in \mathfrak{k}_{ss}^P$.

Pick $v \in \mathfrak{k}_{ss}^P$. Up to linear combination, there exist $x, y \in \mathfrak{k}_{ss}^P$ such that $v = [x, y]$ because $\mathfrak{k}_{ss}^P$ is semi-simple. Then

$$\begin{aligned} \langle \beta, v \rangle &= \langle \beta, [x, y] \rangle \\ &= d\beta(x, y) \\ &= \omega(x, y) \\ &= (\iota(x)\omega)(y). \end{aligned} \tag{2.4}$$

Since $\omega \in \wedge^2(\mathfrak{k}, \mathfrak{k}_{ss}^P)^*$ and $x \in \mathfrak{k}_{ss}^P$, we get $\iota(x)\omega = 0$. Therefore, the last expression in (2.4) vanishes. This proves half of (2.3), and we next prove the other half of it.

Pick $x \in \mathfrak{k}_{ss}^P$ and $y \in \mathfrak{k}$. By following the arguments of (2.4), we get

$$\langle ad_x^* \beta, y \rangle = \langle \beta, [x, y] \rangle = (\iota(x)\omega)(y) = 0.$$

Hence $ad_x^* \beta = 0$. This completes the proof of (2.3), which implies (2.2). Proposition 3 follows. □

Let W be the Weyl group, acting on the roots in $\mathfrak{h}^*$. Given $\tau \in W$, we let $l(\tau)$ denote its length. Let ρ denote half the sum of all positive roots.

Proof of Theorem 1. Let ω be a K-invariant Kähler form on $G/(P, P)$. By Proposition 3, $\omega = d\beta$ for some real 1-form β. We write

$$\beta = \alpha + \bar{\alpha}, \tag{2.5}$$

where α is a $(0, 1)$-form. Since ω is a $(1, 1)$-form, it follows from $d\beta = \omega$ that $\bar{\partial}\alpha = \partial\bar{\alpha} = 0$. In other words, we get a Dolbeault cohomology class $[\alpha] \in H^{0,1}(G/(P, P))$. We suppress $G/(P, P)$ and write $H^{0,1}$ for simplicity.

Consider $H^{0,1}$ as a $K \times T_P$-representation space. Let $H_K^{0,1} \subset H^{0,1}$ denote the K-invariant cohomology classes. Since ω is K-invariant, averaging by K if necessary, we may assume that β and α of (2.5) are also K-invariant. In other words, $[\alpha] \in H_K^{0,1}$. For an integral weight $\lambda \in \mathfrak{t}_P^*$, let $H_\lambda^{0,1} \subset H^{0,1}$ be the cohomology classes which transform by λ under the right T_P-action. By Theorem 2 of [2], $H_K^{0,1}$ splits into 1-dimensional subrepresentations $H_\lambda^{0,1}$ for all $\lambda \in \mathfrak{t}_P^*$ in which we can find $\tau \in W$ satisfying

$$\tau(\lambda + \rho) - \rho = 0 \ , \ l(\tau) = 1. \tag{2.6}$$

But condition (2.6) simply means that $-\lambda$ is a simple root which lies in $\mathfrak{t}_P^*$. Equivalently $-\lambda \in \Delta_P$, where $\Delta_P = \{\lambda_1, ..., \lambda_r\}$ consists of the simple roots in (1.1).

Therefore, there exist $\bar{\partial}$-closed (0,1)-forms $\alpha_1, ..., \alpha_r$ such that

$$[\alpha] = [\sum_1^r \alpha_i] \in H_K^{0,1} \tag{2.7}$$

and

$$[\alpha_i] \in H_{-\lambda_i}^{0,1} \subset H_K^{0,1}. \tag{2.8}$$

Here (2.7) says that

$$\alpha = \bar{\partial} f + \sum_1^r \alpha_i \tag{2.9}$$

for some smooth function f. For the negative root $-\lambda_i$, the character corresponding to it via (1.2) is χ_i^{-1}. Therefore, (2.8) says that for all right action of R_t of $t \in T_P$,

$$R_t^* \alpha_i = \chi_i^{-1}(t^{-1})\alpha_i = \chi_i(t)\alpha_i. \tag{2.10}$$

Let $\beta_i = \alpha_i + \bar{\alpha}_i$ for all $i = 1, ..., r$. Then by (2.5) and (2.9),

$$\begin{aligned}\beta &= \alpha + \bar{\alpha} \\ &= \bar{\partial} f + \partial \bar{f} + \sum_1^r (\alpha_i + \bar{\alpha}_i) \\ &= \bar{\partial} f + \partial \bar{f} + \sum_1^r \beta_i.\end{aligned}$$

Therefore, by setting $F = \sqrt{-1}(\bar{f} - f)$,

$$\begin{aligned}\omega = d\beta &= \partial\bar{\partial} f + \bar{\partial}\partial \bar{f} + \sum_1^r d\beta_i \\ &= \sqrt{-1}\partial\bar{\partial} F + \sum_1^r d\beta_i.\end{aligned}$$

Since ω, β and α are K-invariant, we can take β_i, α_i and f to be K-invariant too. Since f is a K-invariant function on $G/(P,P) = (K/K_{ss}^P) \times A_P$, it is automatically $K \times T_P$-invariant. Therefore, F and $\sqrt{-1}\partial\bar{\partial}F$ are also $K \times T_P$-invariant.

Each β_i behaves by χ_i in (2.10) under the right T_P-action. If $\sum_1^r d\beta_i$ does not vanish and has a potential function, then it is right T_P-invariant, which is impossible. Therefore, ω has a potential function if and only if $\sum_1^r d\beta_i$ vanishes. This can also be seen from the nontrivial Dolbeault cohomology classes $[\alpha_i] \neq 0$. Equivalently, the vanishing of $\sum_1^r d\beta_i$ leaves $\omega = \sqrt{-1}\partial\bar{\partial}F$ to be right T_P-invariant. This proves Theorem 1. □

Let ω be a K-invariant Kähler form on $G/(P,P)$. By Theorem 1 ω is exact, so it is in particular integral. Let $\mathbf{L}$ be the pre-quantum line bundle [8] corresponding to ω. We now prove Theorem 2, concerning the holomorphic sections on $\mathbf{L}$.

Proof of Theorem 2. Since $B \subset P$ and $N = (B,B) \subset (P,P)$, we have the natural fibration

$$\pi : G/N \longrightarrow G/(P,P).$$

Suppose that ω is not invariant under the right T_P-action. Since π intertwines with the $K \times T_P$-action, $\pi^*\omega$ is K-invariant but not right T_P-invariant. Although

$\pi^*\omega$ is not Kähler, it is a closed (1,1)-form on G/N. Therefore, $\pi^*\omega$ accepts most arguments in [4], including Theorem 1 there. Namely, the only holomorphic section on the pre-quantum line bundle of $\pi^*\omega$ is the zero section.

Let $\mathbf{L}$ be the pre-quantum line bundle corresponding to ω. Then $\pi^*\mathbf{L}$ is the pre-quantum line bundle corresponding to $\pi^*\omega$. If s is a holomorphic section on $\mathbf{L}$ and $s \neq 0$, then π^*s is a holomorphic section on $\pi^*\mathbf{L}$ and $\pi^*s \neq 0$. This is a contradiction, so the only holomorphic section on $\mathbf{L}$ is the zero section. Hence Theorem 2 is proved. □

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Department of Applied Mathematics, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, Taiwan

E-mail address: chuah@math.nctu.edu.tw