Efficient Formulations for the Manipulator Inertia Matrix in Terms of Minimal Linear Combinations of Inertia Parameters

Shir-Kuan Lin,* Chang-Jia Fang
Department of Electrical and Control Engineering
National Chiao Tung University
Hsinchu 30050, Taiwan, Republic of China
e-mail: sklin@cc.nctu.edu.tw

Received February 20, 1999; accepted August 10, 1999

This article summarizes four formulations of the composite body method for the inertia matrix of a manipulator in the earlier works and presents a new formulation. These five formulations all use the first moments and the inertia tensors of composite bodies about the origin of the local frame. This paper also presents an algorithm for computing these first moments and inertia tensors. This algorithm utilizes a set of minimal linear combinations of inertia parameters instead of the natural inertia parameters, so that a number of redundant computations are saved. It is found that the new algorithm for the first moments and the inertia tensors of composite bodies is computationally superior to the others in the literature. On the other hand, two among the five formulations for the inertia matrix are more efficient than the other three as well as the others in the literature. The new formulation is one of these two most efficient formulations, and is specially adequate to a manipulator with some translational joints.

1. INTRODUCTION

The forward dynamics of a manipulator is to solve the joint accelerations for given actuator forces and then to calculate the joint velocities and joint displacements by integrating the joint accelerations. The computation of the inertia matrix plays the crucial role in the forward dynamics. An efficient formulation for forming the inertia matrix is then the central topic of the forward dynamics.

There are three types of composite body methods for forming the inertia matrix of a manipulator in the literature. They are different in the point about which the inertia properties (first moments and inertia tensors) are measured: the center of mass,[1] the origin of the local frame,[2–6] and the origin of the base frame.[7] The concept of the composite body method was first presented by Walker

* To whom all correspondence should be addressed.

CCC 0741-2223/99/120679-17

and Orin.[1] However, they used the center of mass to measure the first moments and the inertia tensors of composite bodies and did not exploit the constant terms in the first moments and the inertia tensors. Renaud[2,3] used the first moments and the inertia tensors of composite bodies about the origin of the local frame to derive the so-called *generalized link* method of the inertia matrix. Renaud dissected the terms in the recursive formulation of the first moments and the inertia tensors into constant terms and varying terms, so that some computations are saved. His method was further extended and modified by Vukobratovik et al.[4] and Lin.[7] The method of Walker and Orin[1] utilizing the same inertia properties as those of Renaud, instead of those about the center of mass, was later rederived by Lilly and Orin[6] and found more efficient, although Lilly and Orin applied the spatial notation and named the resulting formulation as *spatial composite body* method. On the other hand, Lin[7] proposed another type of composite body method by measuring the first moments and the inertia tensors of composite bodies about the origin of the inertia frame on the base. His method was found the most efficient formulation for the inertia matrix at that time.

In this paper, we are concerned with the second type of composite body method, i.e., the one using the inertia properties about the origin of the local frame. There have been several alternate formulations based on this type of composite body method. It will be shown in this article that the efficiency of these formulations was underestimated in the earlier works. A new formulation of this type of composite body method is also presented in this article, which is specially adequate to a manipulator with some translational joints. One of the advantages of this type of composite body method is that a set of minimal inertia parameters for the manipulator dynamics is closely related to the first moments and the inertia tensors of composite bodies and can be utilized in the formulations of this type of composite body method.

A set of minimal linear combinations of inertia parameters (MLC's) is a set of *minimal* parameters that are *linear combinations* of the natural inertia parameters and sufficiently determine the manipulator dynamics. There are two different sets of MLC's. One was discovered by Khalil et al.[8–10] and Mayeda et al.[11,12] independently, the other was found by Lin.[13–15] Both sets were already used to reformulate the so-called recursive Newton–Euler formulation of the manipulator inverse dynamics.[15–17] It was exploited that using MLC's can reduce a lot of computations in the inverse dynamics. Kawasaki et al.[18] developed an efficient formulation for the inertia matrix in terms of the set of MLC's of Khalil et al. and Mayeda et al. and found that their new formulation is computationally superior to all the others in the literature. This paper will indicate that the formulation of Kawasaki et al. actually also belong to the second type of composite body method. On the other hand, we also try to develop an algorithm for computing the first moments and the inertia tensors of composite bodies using Lin's set of MLC's. It will be shown that this new algorithm is more efficient than that of Kawasaki et al.

This paper is organized as follows. Section 2 summarizes all formulations of the second type of composite body method in the literature and presents a new one. The efficiency of these formulations is compared also in this section under the assumption that the first moments and the inertia tensors of composite bodies are on hand. Section 3 briefly reviews the minimal linear combinations of inertia parameters. The algorithm for the first moments and the inertia tensors of composite bodies in terms of MLC's is proposed in section 4. Section 5 draws the conclusion.

2. FORMULATIONS

We consider a manipulator with n low-pair joints, which are labeled joint 1 to n outward from the base. Assign a body-fixed frame on each joint (i.e., frame E_i is fixed on joint i) in accord with the normal driving-axis coordinate system.[19,20] The distance from the origin of E_i to that of E_j is designated as ${}_i^j\mathbf{s}$, and that to the center of mass of link i as $\mathbf{c}_i$.

In the normal driving-axis coordinate system, the z-axis of a body-fixed frame is the driving axis of the corresponding link, i.e., the unit vector along joint i is $\mathbf{u}_i^{\langle i\rangle} = [0,0,1]^T$, where superscript "$\langle i\rangle$" denotes the representation of a vector with respect to frame E_i. The distance from the origin of frame E_{i-1} to frame E_i is

$$ {}_{i-1}^{\;\,i}\mathbf{s}^{\langle i-1\rangle} = \begin{bmatrix} b_i \\ -d_i S\beta_i \\ d_i C\beta_i \end{bmatrix}, \quad \text{or} \quad {}_{i-1}^{\;\,i}\mathbf{s}^{\langle i\rangle} = \begin{bmatrix} b_i C\theta_i \\ -b_i S\theta_i \\ d_i \end{bmatrix} \qquad (1) $$

where $S\theta_i \equiv \sin\theta_i$, $C\theta_i \equiv \cos\theta_i$, and b_i, d_i, β_i and θ_i are the geometrical parameters of the coordinate system. The coordinate transformation matrix from E_i to E_{i-1} is well known as

$$ {}_{i-1}^{\;\,i}\mathbf{R} = \begin{bmatrix} C\theta_i & -S\theta_i & 0 \\ C\beta_i S\theta_i & C\beta_i C\theta_i & -S\beta_i \\ S\beta_i S\theta_i & S\beta_i C\theta_i & C\beta_i \end{bmatrix} \tag{2} $$

The *composite body* i is defined as the union of link i to link n. Let the mass of the composite body i and the first moment of the composite body about the origin of E_i be denoted by $\hat{m}_i$ and $\hat{\mathbf{c}}_i$, respectively, to obtain

$$ \hat{m}_i = \sum_{j=i}^{n} m_j \tag{3} $$

$$ \hat{\mathbf{c}}_i^{\langle i\rangle} = \sum_{j=i}^{n} m_j \left({}_i^j\mathbf{s}^{\langle i\rangle} + \mathbf{c}_j^{\langle i\rangle} \right) \tag{4} $$

where m_j is the mass of link j. The inertia tensor of the composite body about the origin of frame E_i (denoted by $\hat{\mathbf{J}}_i$) results by using the Huygeno–Steiner formula[21] to obtain

$$ \hat{\mathbf{J}}_i^{\langle i\rangle} = \sum_{j=i}^{n} {}_i^j\mathbf{R}\mathbf{I}_j^{\langle i\rangle}{}_i^j\mathbf{R}^T - m_j\left[\left({}_i^j\mathbf{s}^{\langle i\rangle} + \mathbf{c}_j^{\langle i\rangle}\right)\times\right]\left[\left({}_i^j\mathbf{s}^{\langle i\rangle} + \mathbf{c}_j^{\langle i\rangle}\right)\times\right], \tag{5} $$

where $\mathbf{I}_j^{\langle j\rangle}$ is the representation of the inertia tensor of link j with respect to frame E_j and $[\mathbf{a}\times]$ denotes a skew-symmetric matrix representing vector multiplication, i.e., $[\mathbf{a}\times]\mathbf{b} = \mathbf{a}\times\mathbf{b}$. In the context, the overhead symbol "^" is used to denote the inertia parameters (mass, first moment and inertia tensor) of a composite body.

We introduce the notation of

$$ K_i^* \equiv (1 - K_i) \equiv \begin{cases} 1, & \text{for rotational joint } i, \\ 0, & \text{for translational joint } i. \end{cases} \tag{6} $$

The concept of the composite-body method[1] is that the (j,i)th, $j \le i$, entry of the inertia matrix is the actuator force of joint j to support the inertia force and torque of the composite body i due to a unit joint acceleration of joint i (i.e., when only joint i moves with a unit joint acceleration and the other joints are stationary). Applying Newton–Euler equations (see Appendix A), we obtain the following recursive form for computing the inertia matrix:

$$ \mathbf{f}_{i,i}^{\langle i\rangle} = K_i^*\mathbf{u}_i^{\langle i\rangle} \times \hat{\mathbf{c}}_i^{\langle i\rangle} + K_i\hat{m}_i\mathbf{u}_i^{\langle i\rangle}, \qquad i = 1,\ldots,n \tag{7} $$

$$ \mathbf{t}_{i,i}^{\langle i\rangle} = K_i^*\hat{\mathbf{J}}_i^{\langle i\rangle}\mathbf{u}_i^{\langle i\rangle} + K_i\hat{\mathbf{c}}_i^{\langle i\rangle} \times \mathbf{u}_i^{\langle i\rangle}, \qquad i = 1,\ldots,n \tag{8} $$

$$ \mathbf{f}_{j,i}^{\langle j\rangle} = {}_j^{j+1}\mathbf{R}\mathbf{f}_{j+1,i}^{\langle j+1\rangle}, \qquad 1 \le j \le i-1, \qquad i = 1,\ldots,n \tag{9} $$

$$ \mathbf{t}_{j,i}^{\langle j\rangle} = {}_j^{j+1}\mathbf{R}\mathbf{t}_{j+1,i}^{\langle j+1\rangle} + {}_j^{j+1}\mathbf{s}^{\langle j\rangle} \times \mathbf{f}_{j,i}^{\langle j\rangle}, \quad 1 \le j \le i-1, \quad i = 1,\ldots,n \tag{10} $$

$$ D_{ji} = \mathbf{u}_j^{\langle j\rangle} \cdot \left(K_j^*\mathbf{t}_{j,i}^{\langle j\rangle} + K_j\mathbf{f}_{j,i}^{\langle j\rangle} \right), \quad 1 \le j \le i, \quad i = 1,\ldots,n \tag{11} $$

where D_{ji} is the (j,i)th entry of the inertia matrix $\mathbf{D}$. Note that $\mathbf{D}$ is symmetrical and $\mathbf{u}_i^{\langle i\rangle} = [0,0,1]^T$.

If we define the notation of spatial dynamics as

$$ \mathbf{K}_i^{\langle i\rangle} \equiv \begin{bmatrix} \hat{\mathbf{J}}_i^{\langle i\rangle} & \left[\hat{\mathbf{c}}_i^{\langle i\rangle}\times\right] \\ \left[\hat{\mathbf{c}}_i^{\langle i\rangle}\times\right]^T & \mathrm{diag}\left(\hat{m}_i, \hat{m}_i, \hat{m}_i\right) \end{bmatrix} \tag{12} $$

$$ {}_i^{i+1}\mathbf{X} \equiv \begin{bmatrix} {}_i^{i+1}\mathbf{R} & \left[{}_i^{i+1}\mathbf{s}^{\langle i\rangle}\times\right]{}_i^{i+1}\mathbf{R} \\ \mathbf{0} & {}_i^{i+1}\mathbf{R} \end{bmatrix} \tag{13} $$

$$ \phi_i^{\langle i\rangle} \equiv \begin{bmatrix} K_i^*\mathbf{u}_i^{\langle i\rangle} \\ K_i\mathbf{u}_i^{\langle i\rangle} \end{bmatrix} \tag{14} $$

Then (7)–(11) turns out to be the spatial composite-body method in Lilly and Orin[6] as follows:

$$ \tilde{\mathbf{f}}_{i,i}^{\langle i\rangle} = \mathbf{K}_i^{\langle i\rangle}\phi_i^{\langle i\rangle}, \qquad i = 1,\ldots,n \tag{15} $$

$$ \tilde{\mathbf{f}}_{j,i}^{\langle j\rangle} = {}_j^{j+1}\mathbf{X}\tilde{\mathbf{f}}_{j+1,i}^{\langle j+1\rangle}, \qquad 1 \le j \le i-1, \quad i = 1,\ldots,n \tag{16} $$

$$ D_{ji} = \phi_j^{\langle j\rangle} \cdot \tilde{\mathbf{f}}_{j,i}^{\langle j\rangle}, \qquad 1 \le j \le i, \quad i = 1,\ldots,n \tag{17} $$

This shows that Method IV of Lilly and Orin[6] is the spatial form of (7)–(11). However, the computation redundancy is more easily explored from the form of (7)–(11). We denote this form as *Formulation 1*, which is listed in Table I with the consideration of computation redundancy. In Table I, (7) and (8) are expanded to reveal that no computations are required for them. The computations of $\mathbf{f}_{i-1,i}^{\langle i-1\rangle}$ and $\mathbf{t}_{i-1,i}^{\langle i-1\rangle}$ are separated from those of $\mathbf{f}_{j,i}^{\langle j\rangle}$ and $\mathbf{t}_{j,i}^{\langle j\rangle}$, $j < i$, since the vectors $\mathbf{f}_{i,i}^{\langle i\rangle}$ and $\mathbf{t}_{i,i}^{\langle i\rangle}$ have zero components and the computations with them can be saved. If joint 1 is rotational, $\mathbf{f}_{1,i}^{\langle 1\rangle}$ is redundant, while $\mathbf{t}_{1,i}^{\langle 1\rangle}$ is redundant for translational joint 1. This

Table I. Formulation 1.[a]

	Number of operations
$\mathbf{f}_{i,i}^{\langle i\rangle} = K_i^* \begin{bmatrix} -(\hat{\mathbf{c}}_i^{\langle i\rangle})_y \\ (\hat{\mathbf{c}}_i^{\langle i\rangle})_x \\ 0 \end{bmatrix} + K_i \begin{bmatrix} 0 \\ 0 \\ \hat{m}_i \end{bmatrix}, \quad i = 1,\ldots,n$	0
$\mathbf{t}_{i,i}^{\langle i\rangle} = K_i^* \begin{bmatrix} (\hat{\mathbf{J}}_i^{\langle i\rangle})_{13} \\ (\hat{\mathbf{J}}_i^{\langle i\rangle})_{23} \\ (\hat{\mathbf{J}}_i^{\langle i\rangle})_{33} \end{bmatrix} + K_i \begin{bmatrix} (\hat{\mathbf{c}}_i^{\langle i\rangle})_y \\ -(\hat{\mathbf{c}}_i^{\langle i\rangle})_x \\ 0 \end{bmatrix}, \quad i = 1,\ldots,n$	0
$D_{ii} = K_i^*(\mathbf{t}_{i,i}^{\langle i\rangle})_z + K_i(\mathbf{f}_{i,i}^{\langle i\rangle})_z, \quad i = 1,\ldots,n$	0
$\mathbf{f}_{i-1,i}^{\langle i-1\rangle} = K_i^* \, {}_{i-1}^{\;\;i}\mathbf{R}\mathbf{f}_{i,i}^{\langle i\rangle} + K_i[0, -\hat{m}_i S\beta_i, \hat{m}_i C\beta_i]^T, \quad i = 3,\ldots,n$	K_i^* 6M2A
$\mathbf{t}_{i-1,i}^{\langle i-1\rangle} = {}_{i-1}^{\;\;i}\mathbf{R}\left(\mathbf{t}_{i,i}^{\langle i\rangle} + \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix} \times \mathbf{f}_{i,i}^{\langle i\rangle}\right) + \begin{bmatrix} b_i \\ 0 \\ 0 \end{bmatrix} \times \mathbf{f}_{i-1,i}^{\langle i-1\rangle}, \quad i = 3,\ldots,n$	K_i^* 12M8A + K_i8M6A
$\mathbf{f}_{j,i}^{\langle j\rangle} = {}^{j+1}_{\;\;j}\mathbf{R}\mathbf{f}_{j+1,i}^{\langle j+1\rangle}, \quad i = 4,\ldots,n;\; j = 2,\ldots,i-2$	8M4A
$\mathbf{t}_{j,i}^{\langle j\rangle} = {}^{j+1}_{\;\;j}\mathbf{R}\left(\mathbf{t}_{j+1,i}^{\langle j+1\rangle} + \begin{bmatrix} 0 \\ 0 \\ d_{j+1} \end{bmatrix} \times \mathbf{f}_{j+1,i}^{\langle j+1\rangle}\right) + \begin{bmatrix} b_{j+1} \\ 0 \\ 0 \end{bmatrix} \times \mathbf{f}_{j,i}^{\langle j\rangle},$ $i = 4,\ldots,n;\; j = 2,\ldots,i-2$	$12M8A$
$\mathbf{f}_{1,i}^{\langle 1\rangle} = K_1 {}_1^2\mathbf{R}\mathbf{f}_{2,i}^{\langle 2\rangle}, \quad i = 2,\ldots,n$	$\begin{cases} K_1 8\text{M}4\text{A}, & i \neq 2 \\ K_1 6\text{M}2\text{A}, & i = 2 \end{cases}$
$\mathbf{t}_{1,i}^{\langle 1\rangle} = K_1^* {}_1^2\mathbf{R}(\mathbf{t}_{2,i}^{\langle 2\rangle} + {}_1^2\mathbf{s}^{\langle 2\rangle} \times \mathbf{f}_{2,i}^{\langle 2\rangle}), \quad i = 2,\ldots,n$	$\begin{cases} K_1^* 14\text{M}10\text{A}, & i \neq 2 \\ K_1^* 12\text{M}8\text{A}, & i = 2 \end{cases}$
$D_{ji} = K_j^*(\mathbf{t}_{j,i}^{\langle j\rangle})_z + K_j(\mathbf{f}_{j,i}^{\langle j\rangle})_z, \quad i = 2,\ldots,n;\; j = 1,\ldots,i-1$	0

[a]This formulation is equivalent to the spatial composite-body method proposed by Lilly and Orin[6] that used the notation of spatial dynamics. In this formulation, vectors are represented with respect to the individual local frame E_j. In computing $\mathbf{t}_{i-1,i}^{\langle i-1\rangle}$, if joint i is a translational joint, then $[b_i, 0, 0]^T \times \mathbf{f}_{i-1,i}^{\langle i-1\rangle} = [0, -b_i\hat{m}_i C\beta_i, -b_i\hat{m}_i S\beta_i]^T$ is a constant vector.

fact is also taken in account in Table I. Examining (1) reveals

$$ {}^{j+1}_{\;\;j}\mathbf{s}^{\langle j\rangle} \times \mathbf{f}_{j,i}^{\langle j\rangle} = {}^{j+1}_{\;\;j}\mathbf{R}\begin{bmatrix} 0 \\ 0 \\ d_{j+1} \end{bmatrix} \times \mathbf{f}_{j+1,i}^{\langle j+1\rangle} + \begin{bmatrix} b_{j+1} \\ 0 \\ 0 \end{bmatrix} \times \mathbf{f}_{j,i}^{\langle j\rangle} \tag{18} $$

so that (10) requires 12M (multiplications) and 8A (additions) instead of 14M and 10A. This technique was already exploited in ref. 15 and is used in Table I.

Substituting (7)–(10) into (11), we obtain the closed-form formulation of

$$ \begin{aligned} D_{ji} = {} & K_j^* K_i^* \mathbf{u}_j^{\langle j\rangle} \cdot \left\{ \hat{\mathbf{J}}_i^{\langle i\rangle} \mathbf{u}_i^{\langle j\rangle} + {}_j^i\mathbf{s}^{\langle j\rangle} \times \left(\mathbf{u}_i^{\langle j\rangle} \times \hat{\mathbf{c}}_i^{\langle j\rangle} \right) \right\} \\ & + K_j K_i^* \mathbf{u}_j^{\langle j\rangle} \cdot \left(\mathbf{u}_i^{\langle j\rangle} \times \hat{\mathbf{c}}_i^{\langle j\rangle} \right) \\ & + K_j^* K_i \mathbf{u}_j^{\langle j\rangle} \cdot \left(\hat{m}_i {}_j^i\mathbf{s}^{\langle j\rangle} \times \mathbf{u}_i^{\langle j\rangle} + \hat{\mathbf{c}}_i^{\langle j\rangle} \times \mathbf{u}_i^{\langle j\rangle} \right) \\ & + K_j K_i \hat{m}_i \mathbf{u}_j^{\langle j\rangle} \cdot \mathbf{u}_i^{\langle j\rangle}, \qquad j \le i. \end{aligned} \tag{19} $$

We apply a coordinate transformation to (19) to reformulate it as

$$D_{ji} = K_j^* K_i^* \mathbf{u}_j^{\langle i\rangle} \cdot \left\{ \begin{bmatrix} \left(\hat{\mathbf{J}}_i^{\langle i\rangle}\right)_{13} \\ \left(\hat{\mathbf{J}}_i^{\langle i\rangle}\right)_{23} \\ \left(\hat{\mathbf{J}}_i^{\langle i\rangle}\right)_{33} \end{bmatrix} + \begin{bmatrix} -(\hat{\mathbf{c}}_i^{\langle i\rangle})_x \left({}_j^i\mathbf{s}^{\langle i\rangle}\right)_z \\ -(\hat{\mathbf{c}}_i^{\langle i\rangle})_y \left({}_j^i\mathbf{s}^{\langle i\rangle}\right)_z \\ (\hat{\mathbf{c}}_i^{\langle i\rangle})_x \left({}_j^i\mathbf{s}^{\langle i\rangle}\right)_x + (\hat{\mathbf{c}}_i^{\langle i\rangle})_y \left({}_j^i\mathbf{s}^{\langle i\rangle}\right)_y \end{bmatrix} \right\}$$
$$+ K_j K_i^* \mathbf{u}_j^{\langle i\rangle} \cdot \left(\mathbf{u}_i^{\langle i\rangle} \times \hat{\mathbf{c}}_i^{\langle i\rangle}\right)$$
$$+ K_j^* K_i \mathbf{u}_j^{\langle i\rangle} \cdot \begin{bmatrix} \hat{m}_i \left({}_j^i\mathbf{s}^{\langle i\rangle}\right)_y + (\hat{\mathbf{c}}_i^{\langle i\rangle})_y \\ -\hat{m}_i \left({}_j^i\mathbf{s}^{\langle i\rangle}\right)_x - (\hat{\mathbf{c}}_i^{\langle i\rangle})_x \\ 0 \end{bmatrix}$$
$$+ K_j K_i \hat{m}_i \left(\mathbf{u}_j^{\langle i\rangle}\right)_z, \qquad j \le i. \tag{20}$$

All vectors in (20) are represented with respect to frame E_i, instead of E_j, so that $\hat{\mathbf{J}}_i^{\langle i\rangle}$ and $\hat{\mathbf{c}}_i^{\langle i\rangle}$ can be directly used for all $j \le i$, while (19) needs to calculate $\hat{\mathbf{J}}_i^{\langle j\rangle}$ and $\hat{\mathbf{c}}_i^{\langle j\rangle}$, $j < i$.

By using the rule of scalar triple product, (20) yields

$$D_{ji} = K_j^* K_i^* \left\{ \mathbf{u}_j^{\langle i\rangle} \cdot \left(\hat{\mathbf{J}}_i^{\langle i\rangle} \mathbf{u}_i^{\langle i\rangle}\right) + \left(\mathbf{u}_j^{\langle i\rangle} \times {}_j^i\mathbf{s}^{\langle i\rangle}\right) \cdot \left(\mathbf{u}_i^{\langle i\rangle} \times \hat{\mathbf{c}}_i^{\langle i\rangle}\right) \right\} + K_j K_i^* \mathbf{u}_j^{\langle i\rangle} \cdot \left(\mathbf{u}_i^{\langle i\rangle} \times \hat{\mathbf{c}}_i^{\langle i\rangle}\right)$$
$$+ K_j^* K_i \left\{ \hat{m}_i \mathbf{u}_i^{\langle i\rangle} \cdot \left(\mathbf{u}_j^{\langle i\rangle} \times {}_j^i\mathbf{s}^{\langle i\rangle}\right) + \mathbf{u}_j^{\langle i\rangle} \cdot (\hat{\mathbf{c}}_i^{\langle i\rangle} \times \mathbf{u}_i^{\langle i\rangle}) \right\}$$
$$+ K_j K_i \hat{m}_i \mathbf{u}_j^{\langle i\rangle} \cdot \mathbf{u}_i^{\langle i\rangle}, \qquad j \le i \tag{21}$$

which is the formulation that Renaud[2] first derived for a manipulator with only rotational joints and was extended to a general manipulator by Lin.[7] These two forms (20) and (21) are designated as *Formulations 2* and *3*, respectively. Formulation 2 needs $\mathbf{u}_j^{\langle k\rangle}$ and ${}_j^k\mathbf{s}^{\langle k\rangle}$, $k > j$, while Formulation 3 requires $\mathbf{u}_j^{\langle k\rangle}$ and ${}_j^k\hat{\mathbf{s}}^{\langle k\rangle}$, $k > j$, where ${}_j^k\hat{\mathbf{s}}^{\langle k\rangle} \equiv \mathbf{u}_j^{\langle k\rangle} \times {}_j^k\mathbf{s}^{\langle k\rangle}$. Both formulations are listed in Tables II and III, respectively.

If we change the coordinate frame, with respect to which vectors are represented, to the end-effector frame, then (7)–(8) turns out to be the formulation of Fijany and Bejczy,[5] which is listed in Table IV and is called *Formulation 4*. The additionally required variables are ${}_n^i\mathbf{R}$, $\mathbf{u}_i^{\langle n\rangle}$, and ${}^{i+1}_{\ i}\mathbf{s}^{\langle n\rangle}$.

Finally, it is pointed in Appendix B that the formulation of Kawasaki et al.[18] is in fact a variant form of this type of composite-body method, too. We designate it as *Formulation 5* and list it in Table V. This formulation is different from Formulation 1 in using ${}^{i-1}\mathbf{h}_i^j$ [see (B3)] instead of the first moment of the composite body $\hat{\mathbf{c}}_i^{\langle i\rangle}$.

Although Formulations 1 and 3–5 were presented in earlier work, their efficiency was underestimated. Tables I–V are accompanied with the computation requirements along each equation, in order to give clear and precise computation estimates. We consider two types of manipulators: Type 1 is a manipulator with n rotational joints such as the Puma robot, and Type 2 is that with a translational joint as joint 3 and $n-1$ rotational joints such as the Stanford arm. Table VI summarizes the computation requirements of five formulations with the assumption that the values of $\hat{\mathbf{J}}_i^{\langle i\rangle}$ and $\hat{\mathbf{c}}_i^{\langle i\rangle}$ are on hand.

According to computational structures, Formulations 1, 4, and 5 can be categorized to one group, and Formulations 2 and 3 belong to another group. In the latter group, Formulation 2 is always more efficient than Formulation 3, since the computations of ${}_j^k\hat{\mathbf{s}}^{\langle k\rangle}$ in Formulation 3 entails $n(n-1)$ and $n(n-3)$ more multiplications for Type 1 and Type 2, respectively.

On the other hand, Formulation 1 is always computationally superior to Formulation 5. This is because Formulation 5 utilizes ${}^{i-1}\mathbf{h}_i^i$ and then requires $4n$ more additions. Formulation 4 trades less $O(n^2)$ computations with much more $O(n)$ due to the computations of ${}_n^i\mathbf{R}$, so it is the least efficient in the former group for $n \le 11$.

Formulation 1 is the best efficient for Type 1 with $5 \le n \le 11$ and for Type 2 with $9 \le n \le 11$. For $n \le 4$, Formulation 2 is superior to the others for both types, while it is the best efficient for Type 2 when $n \le 8$. In comparison with the computation requirements of Type 1, Formulation 2 reduces $(13n - 44)$M and $(11n - 36)$A for Type 2, while Formulation 1 just saves 10M and 4A, independently of n. It is then recommended to use Formulation 1 or 2, depending on efficiency, for a manipulator with less than 12 joints.

Although the above efficiency evaluation neglects the computations of $\hat{\mathbf{c}}_i^{\langle i\rangle}$ and $\hat{\mathbf{J}}_i^{\langle i\rangle}$, this does

Table II. Formulation 2.[a]

	Number of operations
$\mathbf{u}_j^{\langle j+1\rangle} = {}_{j+1}^{\;j}\mathbf{R}\mathbf{u}_j^{\langle j\rangle}, \quad j=1,\dots,n-1$	0
$\mathbf{u}_j^{\langle k\rangle} = {}^{k-1}_{\;k}\mathbf{R}\mathbf{u}_j^{\langle k-1\rangle}, \quad j=1,\dots,n-2;\ k=j+2,\dots,n$	8M4A
${}^k_j\mathbf{s}^{\langle k\rangle} = K_j^*\left\{{}^{k-1}_{\;k}\mathbf{R}\left({}^{k-1}_{\;j}\mathbf{s}^{\langle k-1\rangle} + \begin{bmatrix} b_k \\ 0 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} 0 \\ 0 \\ d_k \end{bmatrix}\right\}, \quad j=1,\dots,n-2;\ k=j+2,\dots,n$	K_j^* 8M6A
$D_{ii} = K_i^*(\hat{\mathbf{J}}_i^{\langle i\rangle})_{33} + K_i\hat{m}_i, \quad i=1,\dots,n$	0
$D_{ji} = K_j^*K_i^*\,\mathbf{u}_j^{\langle i\rangle}\cdot\begin{bmatrix} (\hat{\mathbf{J}}_i^{\langle i\rangle})_{13} - (\hat{\mathbf{c}}_i^{\langle i\rangle})_x\left({}^i_j\mathbf{s}^{\langle i\rangle}\right)_z \\ (\hat{\mathbf{J}}_i^{\langle i\rangle})_{23} - (\hat{\mathbf{c}}_i^{\langle i\rangle})_y\left({}^i_j\mathbf{s}^{\langle i\rangle}\right)_z \\ (\hat{\mathbf{J}}_i^{\langle i\rangle})_{33} + (\hat{\mathbf{c}}_i^{\langle i\rangle})_x\left({}^i_j\mathbf{s}^{\langle i\rangle}\right)_x + (\hat{\mathbf{c}}_i^{\langle i\rangle})_y\left({}^i_j\mathbf{s}^{\langle i\rangle}\right)_y \end{bmatrix}$	$K_j^*K_i^*$ 7M6A
$+K_jK_i^*\,\mathbf{u}_j^{\langle i\rangle}\cdot\begin{bmatrix} -(\hat{\mathbf{c}}_i^{\langle i\rangle})_y \\ (\hat{\mathbf{c}}_i^{\langle i\rangle})_x \\ 0 \end{bmatrix}$	$K_jK_i^*$ 2M1A
$+K_j^*K_i\mathbf{u}_j^{\langle i\rangle}\cdot\begin{bmatrix} \hat{m}_i\left({}^i_j\mathbf{s}^{\langle i\rangle}\right)_y + (\hat{\mathbf{c}}_i^{\langle i\rangle})_y \\ -\hat{m}_i\left({}^i_j\mathbf{s}^{\langle i\rangle}\right)_x - (\hat{\mathbf{c}}_i^{\langle i\rangle})_x \\ 0 \end{bmatrix}$	$K_j^*K_i$ 4M3A
$+K_jK_i\hat{m}_i(\mathbf{u}_j^{\langle i\rangle})_z, \quad i=2,\dots,n;\ j=1,\dots,i-1$	K_jK_i 1M

[a] In this formulation, vectors are represented with respect to the individual pivot frame E_i.

not affect the efficiency comparison, since all five formulations need the values of $\hat{\mathbf{c}}_i^{\langle i\rangle}$ and $\hat{\mathbf{J}}_i^{\langle i\rangle}$. It was pointed out[15] that an algorithm for computing $\hat{\mathbf{c}}_i^{\langle i\rangle}$ and $\hat{\mathbf{J}}_i^{\langle i\rangle}$ in terms of a set of minimal inertia parameters is more efficient than that in terms of the natural inertia parameters. Therefore, the following section briefly reviews the theory of minimal linear combinations of inertia parameters.[14,15]

3. MINIMAL LINEAR COMBINATIONS OF INERTIA PARAMETERS

By the principle of mathematical induction, it has been shown[13] that the first moment and the inertia tensor of the composite body i can be expressed as the sum of a constant vector ($\mathbf{k}_i$ or $\mathbf{U}_i$) and a varying vector ($\boldsymbol{l}_i$ or $\mathbf{V}_i$) as follows:

$$\hat{\mathbf{c}}_i^{\langle i\rangle} = \mathbf{k}_i + \boldsymbol{l}_i \tag{22}$$

$$\hat{\mathbf{J}}_i^{\langle i\rangle} = \mathbf{U}_i + \mathbf{V}_i \tag{23}$$

where $\mathbf{k}_n = \hat{\mathbf{c}}_n^{\langle n\rangle}$, $\boldsymbol{l}_n = \mathbf{0}$, $\mathbf{U}_n = \hat{\mathbf{J}}_n^{\langle n\rangle}$, $\mathbf{V}_n = \mathbf{0}$, and

$$\boldsymbol{l}_i = K_{i+1}^*\,{}^{i+1}_{\;\;i}\mathbf{R}\begin{bmatrix} (\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_x \\ (\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_y \\ (\boldsymbol{l}_{i+1})_z \end{bmatrix} + K_{i+1}\,{}^{i+1}_{\;\;i}\mathbf{R}\left(\boldsymbol{l}_{i+1} + \begin{bmatrix} 0 \\ 0 \\ \hat{m}_{i+1}d_{i+1} \end{bmatrix}\right) \tag{24}$$

If joint $i+1$ is a rotational joint, then

$$\mathbf{V}_i = {}^{i+1}_{i}\mathbf{R}\left(\mathbf{V}_{i+1} + \begin{bmatrix} (\mathbf{U}_{i+1})_{11} - (\mathbf{U}_{i+1})_{22} & (\mathbf{U}_{i+1})_{12} & (\mathbf{U}_{i+1})_{13} \\ (\mathbf{U}_{i+1})_{12} & 0 & (\mathbf{U}_{i+1})_{23} \\ (\mathbf{U}_{i+1})_{13} & (\mathbf{U}_{i+1})_{23} & (\mathbf{U}_{i+1})_{33} \end{bmatrix}\right) {}^{i+1}_{i}\mathbf{R}^T - \left[{}^{i+1}_{i}\mathbf{s}^{\langle i\rangle}\times\right][\,\boldsymbol{l}_i\times] - [\,\boldsymbol{l}_i\times]\left[{}^{i+1}_{i}\mathbf{s}^{\langle i\rangle}\times\right] \qquad (25)$$

while for translational joint $i+1$,

$$\begin{aligned}\mathbf{V}_i = {}^{i+1}_{i}\mathbf{R}\Big(\mathbf{V}_{i+1} &- \hat{m}_{i+1}\left[\mathbf{d}_{i+1}^{\langle i+1\rangle}\times\right]\left[\mathbf{d}_{i+1}^{\langle i+1\rangle}\times\right] \\ &- \left[\mathbf{d}_{i+1}^{\langle i+1\rangle}\times\right]\left[\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle}\times\right] \\ &- \left[\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle}\times\right]\left[\mathbf{d}_{i+1}^{\langle i+1\rangle}\times\right]\Big){}^{i+1}_{i}\mathbf{R}^T \\ &- \left[\mathbf{b}_{i+1}^{\langle i\rangle}\times\right][\,\boldsymbol{l}_i\times] - [\,\boldsymbol{l}_i\times]\left[\mathbf{b}_{i+1}^{\langle i\rangle}\times\right] \qquad (26)\end{aligned}$$

where $\mathbf{b}_{i+1}^{\langle i\rangle} = [\,b_{i+1}, 0, 0]^T$ and $\mathbf{d}_{i+1}^{\langle i+1\rangle} = [0, 0, d_{i+1}]^T$ (i.e., ${}^{i+1}_{i}\mathbf{s}^{\langle i\rangle} = \mathbf{b}_{i+1}^{\langle i\rangle} + \mathbf{d}_{i+1}^{\langle i\rangle}$). The terms of $\mathbf{k}_i$ and $\mathbf{U}_i$ can be found in Appendix C, which are not used in the following.

$\hat{m}_i$, the vectors $\mathbf{k}_i$ and the matrices $\mathbf{U}_i$ are invariant to manipulator motion and named as *inertia constants of composite bodies*. The varying terms in $\hat{\mathbf{c}}_i^{\langle i\rangle}$ and $\hat{\mathbf{J}}_i^{\langle i\rangle}$ can be calculated with only some (not

Table III. Formulation 3.[a]

	Number of operations
$\mathbf{u}_j^{\langle j+1\rangle} = {}_{j+1}^{j}\mathbf{R}\mathbf{u}_j^{\langle j\rangle}, \quad j = 1,\dots,n-1$	0
$\mathbf{u}_j^{\langle k\rangle} = {}^{k-1}_{k}\mathbf{R}\mathbf{u}_j^{\langle k-1\rangle}, \quad j = 1,\dots,n-2;\ k = j+2,\dots,n$	8M4A
${}^{j+1}_{j}\hat{\mathbf{s}}^{\langle j+1\rangle} = K_j^*\, {}_{j+1}^{j}\mathbf{R}[\,d_{i+1}S\beta_{j+1}, b_{j+1}, 0]^T, \quad j = 1,\dots,n-1$	K_j^*4M2A
${}^{k}_{j}\hat{\mathbf{s}}^{\langle k\rangle} = K_j^*\left\{{}^{k-1}_{k}\mathbf{R}\left({}^{k-1}_{j}\mathbf{s}^{\langle k-1\rangle} + \mathbf{u}_j^{\langle k-1\rangle}\times\begin{bmatrix} b_k \\ 0 \\ 0\end{bmatrix}\right) + \mathbf{u}_j^{\langle k\rangle}\times\begin{bmatrix}0\\0\\d_k\end{bmatrix}\right\}, \quad j = 1,\dots,n-2;\ k = j+2,\dots,n$	K_i^*12M8A
$D_{ii} = K_i^*(\hat{\mathbf{J}}_i^{\langle i\rangle})_{33} + K_i\hat{m}_i, \quad i = 1,\dots,n$	0
$D_{ji} = K_j^*K_i^*\,\mathbf{u}_j^{\langle i\rangle}\cdot\begin{bmatrix}(\hat{\mathbf{J}}_i^{\langle i\rangle})_{13}\\(\hat{\mathbf{J}}_i^{\langle i\rangle})_{23}\\(\hat{\mathbf{J}}_i^{\langle i\rangle})_{33}\end{bmatrix} + \begin{bmatrix}-(\hat{\mathbf{c}}_i^{\langle i\rangle})_y\left({}^{i}_{j}\hat{\mathbf{s}}^{i}\right)_x\\(\hat{\mathbf{c}}_i^{\langle i\rangle})_x\left({}^{i}_{j}\hat{\mathbf{s}}^{\langle i\rangle}\right)_y\\0\end{bmatrix}$	$K_j^*K_i^*$5M4A
$+K_jK_i^*\,\mathbf{u}_j^{\langle i\rangle}\cdot\begin{bmatrix}-(\hat{\mathbf{c}}_i^{\langle i\rangle})_y\\(\hat{\mathbf{c}}_i^{\langle i\rangle})_x\\0\end{bmatrix}$	$K_jK_i^*$2M1A
$+K_j^*K_i\left\{\hat{m}_i\left({}^{i}_{j}\hat{\mathbf{s}}^{\langle i\rangle}\right)_z + \mathbf{u}_j^{\langle i\rangle}\cdot\begin{bmatrix}-(\hat{\mathbf{c}}_i^{\langle i\rangle})_y\\(\hat{\mathbf{c}}_i^{\langle i\rangle})_x\\0\end{bmatrix}\right\}$	$K_j^*K_i$3M2A
$+K_jK_i\hat{m}_i(\mathbf{u}_j^{\langle i\rangle})_z, \quad i = 2,\dots,n;\ j = 1,\dots,i-1$	K_jK_i1M

[a]This formulation is equivalent to the generalized link method proposed by Renaud[2] for a manipulator with only rotational joints, which was extended to a general manipulator and named as Renaud's formulation by Lin.[7] In this formulation, vectors are represented with respect to the individual pivot frame E_i as those in formulation 2, but this formulation calculates ${}^{i}_{j}\hat{\mathbf{s}}^{\langle i\rangle} \equiv \mathbf{u}_j^{\langle i\rangle}\times{}^{i}_{j}\mathbf{s}^{\langle i\rangle}$ instead of ${}^{i}_{j}\mathbf{s}^{\langle i\rangle}$.

all) of the inertia constants of composite bodies. This property allows us to set forth the following set of minimal linear combinations of inertia parameters (MLC's)[14] that is a set of linear combinations of the natural inertia parameters and can determine the dynamics of a manipulator.

Theorem 1 (refs. 14 and 15): *For a manipulator with n low-pair joints, in which joint r is the first rotational joint counting from the base and joint s is the nearest rotational joint not parallel to joint r, a set of MLC's for determining the actuator forces* τ *is the set* $\mathscr{S}$ *consisting of all nonzero elements of*

1. $K_j^*(\mathbf{U}_j)_{33}, \delta_j K_j^*(\mathbf{k}_j)_x, \delta_j K_j^*(\mathbf{k}_j)_y$ for $r \le j < s$,
2. $K_j^*((\mathbf{U}_j)_{11} - (\mathbf{U}_j)_{22}),\ K_j^*(\mathbf{U}_j)_{33},\ K_j^*(\mathbf{U}_j)_{12},\ K_j^*(\mathbf{U}_j)_{13},\ K_j^*(\mathbf{U}_j)_{23},\ K_j^*(\mathbf{k}_j)_x,\ K_j^*(\mathbf{k}_j)_y$ for $s \le j \le n$,
3. $K_i \hat{m}_i$ for $i = 1, \ldots, n$,
4. $K_i(\mathbf{k}_i)_x, K_i(\mathbf{k}_i)_y, K_i(\mathbf{k}_i)_z$ for $s < i \le n$,
5. $\sigma_j K_j \kappa_{1i}, \sigma_j K_j \kappa_{2i}$ for $r < i < s$,

where

$$\kappa_{1i} \equiv -(\mathbf{u}_r^{\langle i \rangle})_y(\mathbf{k}_i)_x + (\mathbf{u}_r^{\langle i \rangle})_x(\mathbf{k}_i)_y \tag{27}$$

$$\kappa_{2i} \equiv -(\mathbf{u}_r^{\langle i \rangle})_z\left[(\mathbf{u}_r^{\langle i \rangle})_x(\mathbf{k}_i)_x + (\mathbf{u}_r^{\langle i \rangle})_y(\mathbf{k}_i)_y\right] + \left(1 - (\mathbf{u}_r^{\langle i \rangle})_z^2\right)(\mathbf{k}_i)_z \tag{28}$$

and δ_i *and* σ_i *are either one or zero to denote the redundancy of the parameters, which are defined as follows:* $\delta_i = 0$ *for the case where* $\mathbf{u}_r // \mathbf{u}_k // (\mathbf{g} - \mathbf{a}_0)$, $\forall k < i < s$, *and* ${}^{i}_{m}\mathbf{s}$ *is zero or parallel to* $\mathbf{u}_r$ *for every rotational joint* $m, r \le m < i$, *otherwise* $\delta_i = 1$. *On the other hand,*

Table IV. Formulation 4.[a]

	Number of operations
${}^{i}_{n}\mathbf{R} = {}^{i+1}_{n}\mathbf{R}\,{}^{i}_{i+1}\mathbf{R}, \quad i = 1, \ldots, n-2$	24M12A
$\mathbf{u}_{n-1}^{\langle n \rangle} = {}^{n-1}_{n}\mathbf{R}\mathbf{u}_{n-1}^{\langle n-1 \rangle}$	0
$\mathbf{u}_i^{\langle n \rangle} = {}^{i}_{n}\mathbf{R}\mathbf{u}_i^{\langle i \rangle}, \quad i = 1, \ldots, n-2$	8M4A
${}^{n}_{n-1}\mathbf{s}^{\langle n \rangle} = [b_n C\theta_n, -b_n S\theta_n, d_n]^T$	2M
${}^{i+1}_{i}\mathbf{s}^{\langle n \rangle} = {}^{i}_{n}\mathbf{R}\begin{bmatrix} b_{i+1} \\ 0 \\ 0 \end{bmatrix} + {}^{i+1}_{n}\mathbf{R}\begin{bmatrix} 0 \\ 0 \\ d_{i+1} \end{bmatrix}, \quad i = 1, \ldots, n-2$	6M3A
$\mathbf{f}_{i,i}^{\langle i \rangle} = K_i^*\begin{bmatrix} -(\hat{\mathbf{c}}_i^{\langle i \rangle})_y \\ (\hat{\mathbf{c}}_i^{\langle i \rangle})_x \\ 0 \end{bmatrix} + K_i\begin{bmatrix} 0 \\ 0 \\ \hat{m}_i \end{bmatrix}, \quad i = 1, \ldots, n$	0
$\mathbf{t}_{i,i}^{\langle i \rangle} = K_i^*\begin{bmatrix} (\hat{\mathbf{J}}_i^{\langle i \rangle})_{13} \\ (\hat{\mathbf{J}}_i^{\langle i \rangle})_{23} \\ (\hat{\mathbf{J}}_i^{\langle i \rangle})_{33} \end{bmatrix} + K_i\begin{bmatrix} (\hat{\mathbf{c}}_i^{\langle i \rangle})_y \\ -(\hat{\mathbf{c}}_i^{\langle i \rangle})_x \\ 0 \end{bmatrix}, \quad i = 1, \ldots, n$	0
$D_{ii} = K_i^*(\mathbf{t}_{i,i}^{\langle i \rangle})_z + K_i(\mathbf{f}_{i,i}^{\langle i \rangle})_z, \quad i = 1, \ldots, n$	0
$\mathbf{f}_{i,i}^{\langle n \rangle} = {}^{i}_{n}\mathbf{R}\mathbf{f}_{i,i}^{\langle i \rangle}, \quad i = 2, \ldots, n-1$	K_i^* 6M3A + K_i 3M
$\mathbf{t}_{i,i}^{\langle n \rangle} = {}^{i}_{n}\mathbf{R}\mathbf{t}_{i,i}^{\langle i \rangle}, \quad i = 2, \ldots, n-1$	K_i^* 9M6A + K_i 6M3A
$\mathbf{f}_{j,i}^{\langle n \rangle} = \mathbf{f}_{j+1,i}^{\langle n \rangle}, \quad i = 2, \ldots, n;\ j = 2, \ldots, i-1$	0
$\mathbf{t}_{j,i}^{\langle n \rangle} = \mathbf{t}_{j+1,i}^{\langle n \rangle} + {}^{j+1}_{j}\mathbf{s}^{\langle n \rangle} \times \mathbf{f}_{j,i}^{\langle n \rangle}, \quad i = 2, \ldots, n;\ j = 1, \ldots, i-1$	6M6A
$D_{ji} = K_j^*\,\mathbf{u}_j^{\langle n \rangle} \cdot \mathbf{t}_{j,i}^{\langle n \rangle} + K_j \mathbf{u}_j^{\langle n \rangle} \cdot \mathbf{f}_{j,i}^{\langle n \rangle}, \quad i = 2, \ldots, n;\ j = 1, \ldots, i-1$	3M2A

[a] This formulation was proposed by Fijany and Bejczy[5] and is different from formulation 1 in that vectors are represented with respect to the end-effector frame E_n.

Table V. Formulation 5.[a]

	Number of operations
${}^{i}\mathbf{k}_{i}=K_i^*\hat{\mathbf{J}}_i^{\langle i\rangle}\mathbf{u}_i^{\langle i\rangle},\quad i=1,\ldots,n$	0
$D_{ii}=K_i^*({}^{i}\mathbf{k}_{i})_z+K_i\hat{m}_i,\quad i=1,\ldots,n$	0
$\mathbf{f}_{i-1,i}^{\langle i-1\rangle}=-K_i^*({}^{i-1}\mathbf{h}_i^i-\hat{m}_{i}\,{}_{i-1}^{\;i}\mathbf{s}^{\langle i-1\rangle})\times\mathbf{u}_i^{\langle i-1\rangle}$	K_i^*4M4A
$+K_i\hat{m}_i\mathbf{u}_i^{\langle i-1\rangle},\quad i=2,\ldots,n$	0
$\mathbf{t}_{i-1,i}^{\langle i-1\rangle}=K_i^*\,{}_{i-1}^{\;i}\mathbf{R}\,{}^{i}\mathbf{k}_i+K_i^{i-1}\mathbf{h}_i^i\times\mathbf{u}_i^{\langle i-1\rangle}$	K_i^*14M10A
$-\mathbf{f}_{i-1,i}^{\langle i-1\rangle}\times{}_{i-1}^{\;i}\mathbf{s}^{\langle i-1\rangle},\quad i=3,\ldots,n$; and when $K_1^*=1$ for $i=2$	K_i8M5A
$\mathbf{f}_{j,i}^{\langle j\rangle}={}_{j}^{j+1}\mathbf{R}\mathbf{f}_{j+1,i}^{\langle j+1\rangle},\quad i=4,\ldots,n;\ j=2,\ldots,i-2$	8M4A
$\mathbf{t}_{j,i}^{\langle j\rangle}={}_{j}^{j+1}\mathbf{R}\mathbf{t}_{j+1,i}^{\langle j+1\rangle}+{}_{j}^{j+1}\mathbf{s}^{\langle j\rangle}\times\mathbf{f}_{j,i}^{\langle j\rangle},\quad i=4,\ldots,n;\ j=2,\ldots,i-2$	12M8A
$\mathbf{f}_{1,i}^{\langle 1\rangle}=K_1\,{}_{1}^{2}\mathbf{R}\mathbf{f}_{2,i}^{\langle 2\rangle},\quad i=3,\ldots,n$	$K_1$8M4A
$\mathbf{t}_{1,i}^{\langle 1\rangle}=K_1^*\,{}_{1}^{2}\mathbf{R}(\mathbf{t}_{2,i}^{\langle 2\rangle}+{}_{1}^{2}\mathbf{s}^{\langle 2\rangle}\times\mathbf{f}_{2,i}^{\langle 2\rangle}),\quad i=3,\ldots,n$	K_1^*14M10A
$D_{ij}=K_i^*(\mathbf{t}_{j,i}^{\langle j\rangle})_z+K_i(\mathbf{f}_{j,i}^{\langle j\rangle})_z,\quad i=2,\ldots,n;\ j=1,\ldots,i-1$	0

[a]This formulation was proposed by Kawasaki et al.[18] However, Steps 4.1 and 4.2 in ref. 18 are combined together and the erratum in Step 4.2 is corrected.

$\sigma_j=0$ for the case of $\mathbf{u}_j//\mathbf{u}_r$, $r<j<s$, otherwise $\sigma_j=1$. Note that $\mathbf{u}^{\langle i\rangle}$ for $i<s$ is a constant factor.

Remark: The term $\mathbf{V}_i$ in (25) for rotational joint $i+1$ is slightly different from that in the earlier works.[13–15] It additionally requires the (3,3)th entry of $\mathbf{U}_{i+1}$ in the first term on the right-hand side of (25). Meanwhile, $\mathbf{U}_i$ does not need $(\mathbf{U}_{i+1})_{33}$ [see (C2) in Appendix C]. Such a modification makes the inverse dynamics in ref. 15 directly in terms of the inertia constants of composite bodies, i.e., the term $\mathbf{D}_i^{\langle i\rangle}$ in that work is now directly in terms of $\mathbf{U}_i$ in (C2), instead of $\mathbf{U}_i^*$ in that work [cf. (38), (42), and (43) in ref. 15]. However, this modification does not affect the MLC's (according to Corollary 6 in ref. 14) and the efficiency of the inverse dynamics.

According to Theorem 1, not all inertia constants of composite bodies that are required by the formulations of the inertia matrix can be calculated out with the set of MLC's. For $i<r$, both $\hat{\mathbf{c}}_i^{\langle i\rangle}$ and $\hat{\mathbf{J}}_i^{\langle i\rangle}$ are not required in the above five formulations, so the fact that they cannot be calculated out with the MLC's has no influence. However, $\hat{\mathbf{c}}_i^{\langle i\rangle}$ is required for translational joint i, $r<i<s$, in the above five formulations, but it cannot be calculated out with the MLC's, since κ_{1i} and κ_{2i}, instead of $\mathbf{k}_i$, for translational joint i, $r<i<s$, are in the MLC's. It is then necessary to modify the formulations.

We consider only Formulations 1 and 2. In Formulation 1, $\mathbf{t}_{i,i}^{\langle i\rangle}$ and $\mathbf{t}_{j,i}^{\langle j\rangle}$ for $i<r$ are redundant, so that (8) and (10) are discarded for $i<r$. However, for $r\le i<s$ the modification of (8), (10), and (11) is necessary. Since all rotational joints in front of joint s are parallel to one another, only the z-components of the representations of the vectors $\mathbf{t}_{i,i}$ and $\mathbf{t}_{j,i}$ with respect to frame E_r are required for computing D_{ji}, $j\le i<s$. Thus, (8), (10), and (11) for $r\le i<s$ are replaced by

$$(\mathbf{t}_{i,i}^{\langle r\rangle})_z=K_i^*(\mathbf{u}_r^{\langle i\rangle})_z\big(\hat{\mathbf{J}}_i^{\langle i\rangle}\big)_{33}+K_i\big[\kappa_{1i}-(\mathbf{u}_r^{\langle i\rangle})_y(\boldsymbol{l}_i)_x+(\mathbf{u}_r^{\langle i\rangle})_x(\boldsymbol{l}_i)_y\big],\quad i=r,\ldots,s-1 \tag{29}$$

$$\big(\mathbf{t}_{j,i}^{\langle r\rangle}\big)_z=\big(\mathbf{t}_{j+1,i}^{\langle r\rangle}\big)_z+\mathbf{u}_r^{\langle j\rangle}\cdot\big({}_{j}^{j+1}\mathbf{s}^{\langle j\rangle}\times\mathbf{f}_{j,i}^{\langle j\rangle}\big),\quad 1\le j\le i,\ i=r,\ldots,s-1 \tag{30}$$

$$D_{ji}=K_j^*(\mathbf{u}_r^{\langle i\rangle})_z\big(\mathbf{t}_{j,i}^{\langle r\rangle}\big)_z+K_j\mathbf{u}_j^{\langle j\rangle}\cdot\mathbf{f}_{j,i}^{\langle j\rangle},\quad r\le j\le i-1,\ i=r,\ldots,s-1 \tag{31}$$

Equation (29) follows from (8), (22), and (27). Note that $\mathbf{u}_r^{\langle j\rangle}$ is constant for $j<s$, and $(\mathbf{u}_r^{\langle j\rangle})_z=\pm1$ for rotational joint j, $j<s$. In summary, Table I is still

retained for $i \geq s$, whereas $\mathbf{t}_{i,i}^{\langle i \rangle}$ and $\mathbf{t}_{j,i}^{\langle j \rangle}$ in Table I are, respectively, replaced by $(\mathbf{t}_{i,i}^{\langle r \rangle})_z$ and $(\mathbf{t}_{j,i}^{\langle r \rangle})_z$ in (29) and (30) for $r \leq i < s$ and are discarded for $i < r$.

In Formulation 2, only the first and third terms on the right-hand side of (20) should be modified. For rotational joints i and j, $j < i \leq s-1$, $\mathbf{u}_j^{\langle i \rangle} = [0, 0, \pm 1]^T$, so $\mathbf{u}_j^{\langle i \rangle} \cdot [(\hat{\mathbf{c}}_i^{\langle i \rangle})_y, -(\hat{\mathbf{c}}_i^{\langle i \rangle})_x, 0]^T$ can be in terms of κ_{1i} and $\boldsymbol{l}_i$ by using (22) and (27). Thus, (20) is modified for $i \leq s-1$ as

$$D_{ji} = K_j^* K_i^* \left(\mathbf{u}_j^{\langle i \rangle}\right)_z \Big\{ \left(\hat{\mathbf{J}}_i^{\langle i \rangle}\right)_{33} + \delta_i \Big[(\hat{\mathbf{c}}_i^{\langle i \rangle})_x \left({}_j^i\mathbf{s}^{\langle i \rangle}\right)_x + (\hat{\mathbf{c}}_i^{\langle i \rangle})_y \left({}_j^i\mathbf{s}^{\langle i \rangle}\right)_y \Big] \Big\} + K_j K_i^* \mathbf{u}_j^{\langle i \rangle} \cdot \delta_i \left(\mathbf{u}_i^{\langle i \rangle} \times \hat{\mathbf{c}}_i^{\langle i \rangle}\right) + K_j^* K_i \left\{ \mathbf{u}_j^{\langle i \rangle} \cdot \begin{bmatrix} \hat{m}_i \left({}_j^i\mathbf{s}^{\langle i \rangle}\right)_y \\ -\hat{m}_i \left({}_j^i\mathbf{s}^{\langle i \rangle}\right)_x \\ 0 \end{bmatrix} + (\mathbf{u}_r^{\langle j \rangle})_z (\mathbf{t}_{i,i}^{\langle r \rangle})_z \right\} + K_j K_i \hat{m}_i \left(\mathbf{u}_j^{\langle i \rangle}\right)_z, \qquad j \leq i \leq s-1 \tag{32}$$

where $(\mathbf{t}_{i,i}^{\langle r \rangle})_z$ is that in (29). Note that δ_i is added in the above equation, because the terms associated with δ_i in (32) have no contribution to D_{ji} when $\delta_i = 0$ according to the definition of δ_i in Theorem 1. All equations in Table II are retained with the exception of D_{ji}, which is replaced by (32) for $i < s$.

Consequently, both Formulations 1 and 2 require

- $(\hat{\mathbf{c}}_i^{\langle i \rangle})_x$ and $(\hat{\mathbf{c}}_i^{\langle i \rangle})_y$ for $i \geq s$;
- $(\hat{\mathbf{J}}_i^{\langle i \rangle})_{13}$, $(\hat{\mathbf{J}}_i^{\langle i \rangle})_{23}$, and $(\mathbf{J}_i^{\langle i \rangle})_{33}$ for rotational joint i, $i \geq s$;
- $(\hat{\mathbf{J}}_i^{\langle i \rangle})_{33}$, $(\hat{\mathbf{c}}_i^{\langle i \rangle})_x$, and $(\hat{\mathbf{c}}_i^{\langle i \rangle})_y$ for rotational joint i, $r \leq i < s$;
- $(\mathbf{t}_{i,i}^{\langle r \rangle})_z$ [see (29)] for $r \leq i < s$.

4. ALGORITHM FOR FIRST MOMENTS AND INERTIA TENSORS

Equations (22) and (24) already construct an efficient recursive form for computing $\boldsymbol{l}_i$ and $\hat{\mathbf{c}}_i^{\langle i \rangle}$. In the following, we are only concerned with $\mathbf{V}_i$ for computing $\hat{\mathbf{J}}_i^{\langle i \rangle}$.

Analog to (18), we can decompose $[{}^{i+1}_{\ \ i}\mathbf{s}^{\langle i \rangle} \times] \times [\boldsymbol{l}_i \times]$ in (25) as

$$\left[{}^{i+1}_{\ \ i}\mathbf{s}^{\langle i \rangle} \times\right][\boldsymbol{l}_i \times] = \left[\mathbf{b}_{i+1}^{\langle i \rangle} \times\right][\boldsymbol{l}_i \times] + {}^{i+1}_{\ \ i}\mathbf{R}\left[\mathbf{d}_{i+1}^{\langle i+1 \rangle} \times\right]\left[\left({}^{i+1}_{\ \ i}\mathbf{R}^T \boldsymbol{l}_i\right) \times\right] \tag{33}$$

Substituting (24) and (33) into (25), we obtain the expanded form of

$$\mathbf{V}_i = {}^{i+1}_{\ \ i}\mathbf{R} \left(\mathbf{V}_{i+1} + \begin{bmatrix} (\mathbf{U}_{i+1})_{11} - (\mathbf{U}_{i+1})_{22} & (\mathbf{U}_{i+1})_{12} & (\mathbf{U}_{i+1})_{13} \\ & 0 & (\mathbf{U}_{i+1})_{23} \\ \text{Symmetry} & & (\mathbf{U}_{i+1})_{33} \end{bmatrix} + \begin{bmatrix} 2d_{i+1}(\boldsymbol{l}_{i+1})_z & 0 & -d_{i+1}(\hat{\mathbf{c}}_{i+1}^{\langle i+1 \rangle})_x \\ & 2d_{i+1}(\boldsymbol{l}_{i+1})_z & -d_{i+1}(\hat{\mathbf{c}}_{i+1}^{\langle i+1 \rangle})_y \\ \text{Symmetry} & & 0 \end{bmatrix} \right) {}^{i+1}_{\ \ i}\mathbf{R}^T + \begin{bmatrix} 0 & -b_{i+1}(\boldsymbol{l}_i)_y & -b_{i+1}(\boldsymbol{l}_i)_z \\ & 2b_{i+1}(\boldsymbol{l}_i)_x & 0 \\ \text{Symmetry} & & 2b_{i+1}(\boldsymbol{l}_i)_x \end{bmatrix} \tag{34}$$

for rotational joint $i+1$. The expanded form of (26) is

$$\mathbf{V}_i = {}^{i+1}_{\ \ i}\mathbf{R} \left(\mathbf{V}_{i+1} + d_{i+1} \begin{bmatrix} \hat{m}_{i+1} d_{i+1} + 2(\hat{\mathbf{c}}_{i+1}^{\langle i+1 \rangle})_z & 0 & -(\hat{\mathbf{c}}_{i+1}^{\langle i+1 \rangle})_x \\ & \hat{m}_{i+1} d_{i+1} + 2(\hat{\mathbf{c}}_{i+1}^{\langle i+1 \rangle})_z & -(\hat{\mathbf{c}}_{i+1}^{\langle i+1 \rangle})_y \\ \text{Symmetry} & & 0 \end{bmatrix} \right) {}^{i+1}_{\ \ i}\mathbf{R}^T + \begin{bmatrix} 0 & -b_{i+1}(\boldsymbol{l}_i)_y & -b_{i+1}(\boldsymbol{l}_i)_z \\ & 2b_{i+1}(\boldsymbol{l}_i)_x & 0 \\ \text{Symmetry} & & 2b_{i+1}(\boldsymbol{l}_i)_x \end{bmatrix} \tag{35}$$

Table VI. Efficiency comparison of five formulations when $\hat{\mathbf{J}}_i^{\langle i\rangle}$ and $\hat{\mathbf{c}}_i^{\langle i\rangle}$ are on hand.

	Type 1				Type 2			
	n	4	6	11	n	6	8	9
Form. 1 M	$10n^2 - 18n + 10$	98	262	1022	$10n^2 - 18n$	252	496	648
A	$6n^2 - 10n + 4$	60	160	620	$6n^2 - 10n$	156	304	396
Form. 2 M	$11.5n^2 - 27.5n + 16$	90	265	1105	$11.5n^2 - 40.5n + 60$	231	472	627
A	$8n^2 - 18n + 10$	66	190	780	$8n^2 - 29n + 46$	160	326	433
Form. 3 M	$12.5n^2 - 28.5n + 16$	102	295	1215	$12.5n^2 - 43.5n + 65$	254	517	686
A	$8n^2 - 18n + 10$	66	190	780	$8n^2 - 29n + 45$	159	325	432
Form. 4 M	$4.5n^2 + 48.5n - 108$	158	345	970	$4.5n^2 + 48.5n - 114$	339	562	687
A	$4n^2 + 24n - 56$	104	232	692	$4n^2 + 24n - 62$	226	386	478
Form. 5 M	$10n^2 - 18n + 14$	102	266	1016	$10n^2 - 18n + 4$	256	500	652
A	$6n^2 - 6n + 2$	74	182	662	$6n^2 - 6n - 7$	173	329	425

for translational joint $i+1$. In the algorithm, $\mathbf{V}_i$ for $i \geq s$ is computed by using (34) or (35).

For joint i, $i < s$, only the $(3,3)$th entry of $K_i^* \hat{\mathbf{J}}_i^{\langle i\rangle}$ is required for computing the inertia matrix. It can be shown that

$$\left(\hat{\mathbf{J}}_i^{\langle i\rangle}\right)_{33} = \mathbf{u}_i^{\langle r\rangle T} \hat{\mathbf{J}}_i^{\langle r\rangle} \mathbf{u}_i^{\langle r\rangle} \tag{36}$$

since $\mathbf{u}_i^{\langle r\rangle} = {}^i_r\mathbf{R}\mathbf{u}_i^{\langle i\rangle}$ is the third column of ${}^i_r\mathbf{R}$ or the third row of ${}^r_i\mathbf{R}$. Since the rotational joints in front of joint s are parallel to one another, $\mathbf{u}_i^{\langle r\rangle} = [0, 0, \pm 1]^T$ and then $(\hat{\mathbf{J}}_i^{\langle i\rangle})_{33} = (\hat{\mathbf{J}}_i^{\langle r\rangle})_{33}$ for rotational joint i, $i < s$.

It is then suggested to compute $\mathbf{V}_i^{\langle r\rangle}$, instead of $\mathbf{V}_i$, for $i < s$, where

$$\mathbf{V}_i^{\langle r\rangle} \equiv {}^i_r\mathbf{R}\mathbf{V}_i {}^i_r\mathbf{R}^T \tag{37}$$

The advantage is that the computations of the coordinate transformation of ${}^{i+1}_{\ \ i}\mathbf{R}\mathbf{V}_{i+1} {}^{i+1}_{\ \ i}\mathbf{R}^T$ in (34) and (35) are saved. This also entails that only the $(3,3)$th entry of $\mathbf{V}_i^{\langle r\rangle}$ is required to compute.

It follows from (37) that $(\mathbf{V}_i^{\langle r\rangle})_{33} = \mathbf{u}_r^{\langle i\rangle T}\mathbf{V}_i\mathbf{u}_r^{\langle i\rangle}$. Note that $\mathbf{u}_r^{\langle i\rangle}$ is a constant vector for $i < s$. If joint $i+1$ is joint s, then (34) can be reduced to

$$\begin{aligned}(\mathbf{V}_{s-1}^{\langle r\rangle})_{33} = {}& \mathbf{u}_r^{\langle s\rangle T}\left(\mathbf{V}_s + \begin{bmatrix}(\mathbf{U}_s)_{11} - (\mathbf{U}_s)_{22} & (\mathbf{U}_s)_{12} & (\mathbf{U}_s)_{13} \\ & 0 & (\mathbf{U}_s)_{23} \\ \text{Symmetry} & & (\mathbf{U}_s)_{33}\end{bmatrix} + \begin{bmatrix}2d_s(\boldsymbol{l}_s)_z & 0 & -d_s(\hat{\mathbf{c}}_s^{\langle s\rangle})_x \\ & 2d_s(\boldsymbol{l}_s)_z & -d_s(\hat{\mathbf{c}}_s^{\langle s\rangle})_y \\ \text{Symmetry} & & 0\end{bmatrix}\right)\mathbf{u}_r^{\langle s\rangle} \\ & + K_{s-1}^* 2b_s(\boldsymbol{l}_{s-1})_x + K_{s-1}2b_s\Big\{\Big[1 - (\mathbf{u}_r^{\langle s-1\rangle})_x^2\Big](\boldsymbol{l}_{s-1})_x - \Big[(\mathbf{u}_r^{\langle s-1\rangle})_x(\mathbf{u}_r^{\langle s-1\rangle})_y\Big](\boldsymbol{l}_{s-1})_y \\ & - \Big[(\mathbf{u}_r^{\langle s-1\rangle})_x(\mathbf{u}_r^{\langle s-1\rangle})_z\Big](\boldsymbol{l}_{s-1})_z\Big\}\end{aligned} \tag{38}$$

where

$$\mathbf{u}_r^{\langle s\rangle} = {}^{\ \ \ s}_{s-1}\mathbf{R}^T\mathbf{u}_r^{\langle s-1\rangle} = \begin{bmatrix}(\mathbf{u}_r^{\langle s-1\rangle})_x C\theta_s + \left[(\mathbf{u}_r^{\langle s-1\rangle})_y C\beta_s + (\mathbf{u}_r^{\langle s-1\rangle})_z S\beta_s\right]S\theta_s \\ -(\mathbf{u}_r^{\langle s-1\rangle})_x S\theta_s + \left[(\mathbf{u}_r^{\langle s-1\rangle})_y C\beta_s + (\mathbf{u}_r^{\langle s-1\rangle})_z S\beta_s\right]C\theta_s \\ -(\mathbf{u}_r^{\langle s-1\rangle})_y S\beta_s + (\mathbf{u}_r^{\langle s-1\rangle})_z C\beta_s\end{bmatrix} \tag{39}$$

which requires 4M and 2A, since $(\mathbf{u}_r^{\langle s\rangle})_z$ and $[(\mathbf{u}_r^{\langle s-1\rangle})_y C\beta_s + (\mathbf{u}_r^{\langle s-1\rangle})_z S\beta_s]$ are constants. However, when joint $s-1$ is a rotational joint, $\mathbf{u}_r^{\langle s-1\rangle} = [0, 0, \pm 1]^T$, so computing $\mathbf{u}_r^{\langle s\rangle}$ in (39) needs no arithmetic operations.

For joint $i+1$, $i+1<s$, (34) and (35) are combined to obtain

$$\begin{aligned}(\mathbf{V}_i^{\langle r\rangle})_{33} = {} & K_{i+1}^*\big(\hat{\mathbf{J}}_{i+1}^{\langle i+1\rangle}\big)_{33} + K_i^* 2b_{i+1}(\boldsymbol{l}_i)_x \\ & + K_{i+1}\Big[(\mathbf{V}_{i+1}^{\langle r\rangle})_{33} + d_{i+1}\Big\{2\kappa_{2,i+1} \\ & + \Big[1-(\mathbf{u}_r^{\langle i+1\rangle})_z^2\Big][\hat{m}_{i+1}d_{i+1} + 2(\boldsymbol{l}_{i+1})_z] \\ & - \Big[(\mathbf{u}_r^{\langle i+1\rangle})_z(\mathbf{u}_r^{\langle i+1\rangle})_x\Big](\boldsymbol{l}_{i+1})_x \\ & + \Big[(\mathbf{u}_r^{\langle i+1\rangle})_z(\mathbf{u}_r^{\langle i+1\rangle})_y\Big](\boldsymbol{l}_{i+1})_y\Big\}\Big] \\ & + K_i 2b_{i+1}\Big\{\Big[1-(\mathbf{u}_r^{\langle i\rangle})_x^2\Big](\boldsymbol{l}_i)_x \\ & - \Big[(\mathbf{u}_r^{\langle i\rangle})_x(\mathbf{u}_r^{\langle i\rangle})_y\Big](\boldsymbol{l}_i)_y \\ & - \Big[(\mathbf{u}_r^{\langle i\rangle})_x(\mathbf{u}_r^{\langle i\rangle})_z\Big](\boldsymbol{l}_i)_z\Big\} \end{aligned} \tag{40}$$

by using (22), (28), and $K_j^*\mathbf{u}_r^{\langle j\rangle} = [0, 0, \pm 1]^T$ for $j<s$.

The algorithm for computing $\hat{\mathbf{J}}_i^{\langle i\rangle}$ and $\hat{\mathbf{c}}_i^{\langle i\rangle}$ is listed in Table VII and is divided into two parts: one is for $i \geq s-1$, the other for $i = r, \ldots, s-1$. However, no computation is required for $i=n$. The computations for $i=s-1$ are separated into these two parts, which will be described later. Computing $\boldsymbol{l}_i$ utilizes (24), while (22) is used to compute $\hat{\mathbf{c}}_i^{\langle i\rangle}$. However, it should be remarked that the z-component of $\hat{\mathbf{c}}_i^{\langle i\rangle}$ for any rotational joint i is redundant for computing the inertia matrix, and is not computed in the algorithm.

The computation of (34) and (35) is divided into three steps. Let $\mathbf{X}$ be the symmetrical matrix of the similarity transformation term on the right-hand side of (34) or (35), and $\mathbf{Y} = {}^{i+1}_{\ i}\mathbf{R}\mathbf{X}\,{}^{i+1}_{\ i}\mathbf{R}^T$. $\mathbf{V}_i$ is then the sum of $\mathbf{Y}$ and the last term on the right-hand side of (34) or (35). It should be remarked that only the computation of $\mathbf{X}$ is different for different type of joint in these three steps. An efficient computing technique for $\mathbf{Y} = {}^{i+1}_{\ i}\mathbf{R}\mathbf{X}\,{}^{i+1}_{\ i}\mathbf{R}^T$ is presented in the following, which is similar to that proposed by Kawasaki et al.[18]

Table VII. Algorithm of first moments and inertia tensors.

	Number of operations
• For $i = n-1, \ldots, s-1$:	
$\boldsymbol{l}_i = K_{i+1}^*\,{}^{i+1}_{\ i}\mathbf{R}\begin{bmatrix}(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_x \\ (\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_y \\ (\boldsymbol{l}_{i+1})_z\end{bmatrix} + K_{i+1}\,{}^{i+1}_{\ i}\mathbf{R}\left(\boldsymbol{l}_{i+1} + \begin{bmatrix}0 \\ 0 \\ \hat{m}_{i+1}d_{i+1}\end{bmatrix}\right)$	$\begin{cases}K_{i+1}^* 8\text{M}4\text{A} \\ K_{i+1} 9\text{M}5\text{A}\end{cases}$
$\begin{bmatrix}(\mathbf{X})_{11} \\ (\mathbf{X})_{22} \\ (\mathbf{X})_{33} \\ (\mathbf{X})_{12} \\ (\mathbf{X})_{13} \\ (\mathbf{X})_{23}\end{bmatrix} = K_{i+1}^*\begin{bmatrix}(\mathbf{V}_{i+1})_{11} + (\mathbf{U}_{i+1})_{11-22} + 2d_{i+1}(\boldsymbol{l}_{i+1})_z \\ (\mathbf{V}_{i+1})_{22} + 2d_{i+1}(\boldsymbol{l}_{i+1})_z \\ (\hat{\mathbf{J}}_{i+1}^{\langle i+1\rangle})_{33} \\ (\mathbf{V}_{i+1})_{12} + (\mathbf{U}_{i+1})_{12} \\ (\hat{\mathbf{J}}_{i+1}^{\langle i+1\rangle})_{13} - d_{i+1}(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_x \\ (\hat{\mathbf{J}}_{i+1}^{\langle i+1\rangle})_{23} - d_{i+1}(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_y\end{bmatrix}$	$K_{i+1}^* 3\text{M}6\text{A}$
$+K_{i+1}\begin{bmatrix}(\mathbf{V}_{i+1})_{11} + d_{i+1}[\hat{m}_{i+1}d_{i+1} + 2(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_z] \\ (\mathbf{V}_{i+1})_{22} + d_{i+1}[\hat{m}_{i+1}d_{i+1} + 2(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_z] \\ (\mathbf{V}_{i+1})_{33} \\ (\mathbf{V}_{i+1})_{12} \\ (\mathbf{V}_{i+1})_{13} - d_{i+1}(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_x \\ (\mathbf{V}_{i+1})_{23} - d_{i+1}(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_y\end{bmatrix}$	$K_{i+1} 4\text{M}5\text{A}$

Table VII. (*Continued*)

	Number of operations
$\mathbf{Y} = {}^{i+1}_{i}\mathbf{R}\mathbf{X}\,{}^{i+1}_{i}\mathbf{R}^T$, for $i \geq s$ [see (41)–(54)]	18M14A
$\begin{bmatrix}(\mathbf{V}_i)_{11}\\(\mathbf{V}_i)_{22}\\(\mathbf{V}_i)_{33}\\(\mathbf{V}_i)_{12}\\(\mathbf{V}_i)_{13}\\(\mathbf{V}_i)_{23}\end{bmatrix} = \begin{bmatrix}(\mathbf{Y})11\\(\mathbf{Y})22 + 2b_{i+1}(\boldsymbol{l}_i)_x\\(\mathbf{Y})33 + 2b_{i+1}(\boldsymbol{l}_i)_x\\(\mathbf{Y})12 - b_{i+1}(\boldsymbol{l}_i)_y\\(\mathbf{Y})13 - b_{i+1}(\boldsymbol{l}_i)_z\\(\mathbf{Y})23\end{bmatrix}$, for $i \geq s$	3M4A
$\hat{\mathbf{c}}_i^{\langle i\rangle} = \mathbf{k}_i + K_i^*\begin{bmatrix}(\boldsymbol{l}_i)_x\\(\boldsymbol{l}_i)_y\\0\end{bmatrix} + K_i\begin{bmatrix}(\boldsymbol{l}_i)_x\\(\boldsymbol{l}_i)_y\\(\boldsymbol{l}_i)_z\end{bmatrix}$, for $i \geq s$	$K_i^*2\text{A} + K_i3\text{A}$
$\begin{bmatrix}(\hat{\mathbf{J}}_i^{\langle i\rangle})_{13}\\(\hat{\mathbf{J}}_i^{\langle i\rangle})_{23}\\(\hat{\mathbf{J}}_i^{\langle i\rangle})_{33}\end{bmatrix} = K_i^*\begin{bmatrix}(\mathbf{U}_i)_{13} + (\mathbf{V}_i)_{13}\\(\mathbf{U}_i)_{23} + (\mathbf{V}_i)_{23}\\(\mathbf{U}_i)_{33} + (\mathbf{V}_i)_{33}\end{bmatrix}$, for $i \geq s$	$K_i^*3\text{A}$
$\mathbf{u}_r^{\langle s\rangle} = \begin{bmatrix}(\mathbf{u}_r^{\langle s-1\rangle})_x C\theta_s + \left[(\mathbf{u}_r^{\langle s-1\rangle})_y C\beta_s + (\mathbf{u}_r^{\langle s-1\rangle})_z S\beta_s\right] S\theta_s\\ -(\mathbf{u}_r^{\langle s-1\rangle})_x S\theta_s + \left[(\mathbf{u}_r^{\langle s-1\rangle})_y C\beta_s + (\mathbf{u}_r^{\langle s-1\rangle})_z S\beta_s\right] C\theta_s\\ (\mathbf{u}_r^{\langle s\rangle})_z\end{bmatrix}$, for $i = s-1$	$K_{s-1}4\text{M}2\text{A}$
$(\mathbf{Y})_{33} = \mathbf{u}_r^{\langle s\rangle T}\mathbf{X}\mathbf{u}_r^{\langle s\rangle}$, for $i = s-1$ [see (55)]	10M5A
• For $i = s-1, \ldots, r$:	
$\boldsymbol{l}_i = K_{i+1}^*\,{}^{i+1}_{i}\mathbf{R}\begin{bmatrix}(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_x\\(\hat{\mathbf{c}}_{i+1}^{\langle i+1\rangle})_y\\(\boldsymbol{l}_{i+1})_z\end{bmatrix} + K_{i+1}\,{}^{i+1}_{i}\mathbf{R}\left(\boldsymbol{l}_{i+1} + \begin{bmatrix}0\\0\\\hat{m}_{i+1}d_{i+1}\end{bmatrix}\right)$, for $i \leq s-2$	$\begin{cases}K_{i+1}^*8\text{M}4\text{A}\\K_{i+1}9\text{M}5\text{A}\end{cases}$
$(\mathbf{Y})_{33} = K_{i+1}^*(\hat{\mathbf{J}}_{i+1}^{\langle i+1\rangle})_{33}$ $+ K_{i+1}[(\mathbf{V}_{i+1}^{\langle r\rangle})_{33} + d_{i+1}\{2\kappa_{2,i+1} + [1 - (\mathbf{u}_r^{\langle i+1\rangle})_z^2][\hat{m}_{i+1}d_{i+1} + 2(\boldsymbol{l}_{i+1})_z]$ $- [(\mathbf{u}_r^{\langle i+1\rangle})_z(\mathbf{u}_r^{\langle i+1\rangle})_x](\boldsymbol{l}_{i+1})_x + [(\mathbf{u}_r^{\langle i+1\rangle})_z(\mathbf{u}_r^{\langle i+1\rangle})_y](\boldsymbol{l}_{i+1})_y\}]$, for $i \leq s-2$	$K_{i+1}5\text{M}5\text{A}$
$(\mathbf{V}_i^{\langle r\rangle})_{33} = (\mathbf{Y})_{33} + K_i^*2b_{i+1}(\boldsymbol{l}_i)_x$	$K_i^*1\text{M}1\text{A}$
$+ K_i 2b_{i+1}\{[1 - (\mathbf{u}_r^{\langle i\rangle})_x^2](\boldsymbol{l}_i)_x - [(\mathbf{u}_r^{\langle i\rangle})_x(\mathbf{u}_r^{\langle i\rangle})_y](\boldsymbol{l}_i)_y - [(\mathbf{u}_r^{\langle i\rangle})_x(\mathbf{u}_r^{\langle i\rangle})_z](\boldsymbol{l}_i)_z\}$	$K_i4\text{M}3\text{A}$
$\hat{\mathbf{c}}_i^{\langle i\rangle} = K_i^*\left(k_i + \begin{bmatrix}(\boldsymbol{l}_i)_x\\(\boldsymbol{l}_i)_y\\0\end{bmatrix}\right)$	$K_i^*2\text{A}$
$(\hat{\mathbf{J}}_i^{\langle i\rangle})_{33} = K_i^*[(\mathbf{U}_i)_{33} + (\mathbf{V}_i^{\langle r\rangle})_{33}]$	$K_i^*1\text{A}$
$(\mathbf{t}_{i,i}^{\langle r\rangle})_z = K_i^*(\mathbf{u}_r^{\langle i\rangle})_z(\hat{\mathbf{J}}_i^{\langle i\rangle})_{33} + K_i[\kappa_{1i} - (\mathbf{u}_r^{\langle i\rangle})_y(\boldsymbol{l}_i)_x + (\mathbf{u}_r^{\langle i\rangle})_x(\boldsymbol{l}_i)_y]$	$K_i2\text{M}2\text{A}$

$$(\mathbf{Y})_{11} = A_2 + (\mathbf{X})_{22} \tag{41}$$

$$(\mathbf{Y})_{22} = A_7 + (\mathbf{X})_{33} \tag{42}$$

$$(\mathbf{Y})_{33} = A_3 - A_7 \tag{43}$$

$$(\mathbf{Y})_{12} = C\beta_{i+1} A_4 - S\beta_{i+1} A_8 \tag{44}$$

$$(\mathbf{Y})_{13} = S\beta_{i+1} A_4 + C\beta_{i+1} A_8 \tag{45}$$

$$(\mathbf{Y})_{23} = A_5 + S\beta_{i+1} A_6 \tag{46}$$

where

$$A_1 = C\theta_{i+1}[(\mathbf{X})_{11} - (\mathbf{X})_{22}] - 2S\theta_{i+1}(\mathbf{X})_{12} \tag{47}$$

$$A_2 = C\theta_{i+1} A_1 \tag{48}$$

$$A_3 = -A_2 + (\mathbf{X})_{11} \tag{49}$$

$$A_4 = S\theta_{i+1} A_1 + (\mathbf{X})_{12} \tag{50}$$

$$A_5 = S\theta_{i+1}(\mathbf{X})_{13} + C\theta_{i+1}(\mathbf{X})_{23} \tag{51}$$

$$A_6 = [A_3 - (\mathbf{X})_{33}]C\beta_{i+1} - 2S\beta_{i+1} A_5 \tag{52}$$

$$A_7 = C\beta_{i+1} A_6 \tag{53}$$

$$A_8 = C\theta_{i+1}(\mathbf{X})_{13} - S\theta_{i+1}(\mathbf{X})_{23} \tag{54}$$

Similarly, $(\mathbf{V}_{s-1}^{\langle r \rangle})_{33}$ in (38) can also be divided into three steps. The matrix $\mathbf{X}$ is identical to that for (34). Let $(\mathbf{Y})_{33} = \mathbf{u}_r^{\langle s \rangle T} \mathbf{X} \mathbf{u}_r^{\langle s \rangle}$. Then

$$\begin{aligned}(\mathbf{Y})_{33} = {} & (\mathbf{u}_r^{\langle s \rangle})_x\left[(\mathbf{u}_r^{\langle s \rangle})_x(\mathbf{X})_{11} + 2(\mathbf{u}_r^{\langle s \rangle})_y(\mathbf{X})_{12}\right] \\ & + (\mathbf{u}_r^{\langle s \rangle})_y^2(\mathbf{X})_{22} \\ & + 2(\mathbf{u}_r^{\langle s \rangle})_z\left[(\mathbf{u}_r^{\langle s \rangle})_x(\mathbf{X})_{13} + (\mathbf{u}_r^{\langle s \rangle})_y(\mathbf{X})_{23}\right] \\ & + (\mathbf{u}_r^{\langle s \rangle})_z^2(\mathbf{X})_{33}\end{aligned} \tag{55}$$

which requires 10M and 5A, since $(\mathbf{u}_r^{\langle s \rangle})_z$ is constant [see (39)], so are then $2(\mathbf{u}_r^{\langle s \rangle})_z$ and $(\mathbf{u}_r^{\langle s \rangle})_z^2$. Finally, $(\mathbf{V}_{s-1}^{\langle r \rangle})_{33}$ is the sum of $(\mathbf{Y})_{33}$ and the other terms in (38). The first two steps are placed in the part of $i \geq s-1$, since the computation of $\mathbf{X}$ for $i = s-1$ is identical to that for $i \geq s$.

For $i < s-1$, we use (40) to compute $(\mathbf{V}_i^{\langle r \rangle})_{33}$. The terms associated with K_{i+1} and K_{i+1}^* in (40) are summed up to be $(\mathbf{Y})_{33}$. Then $(\mathbf{V}_i^{\langle r \rangle})_{33}$ is the sum of $(\mathbf{Y})_{33}$ and the rest terms. It is apparent that the second step for computing $(\mathbf{V}_i^{\langle r \rangle})_{33}$ is identical to the third steps for $(\mathbf{V}_{s-1}^{\langle r \rangle})_{33}$, so that the latter is also executed in the part of $r \leq i \leq s-1$. As was pointed out above, $(\mathbf{t}_{i,i}^{\langle r \rangle})_z$ in (29) for $r \leq s-1$ should be calculated out at the same time, since it uses $\boldsymbol{l}_i$ (see Table VII).

The computation requirements of the algorithm listed in Table VII for Type 1 and Type 2 of manipulators are

- Type 1: $(32n - 47)$M and $(33n - 57)$A
- Type 2: $(32n - 45)$M and $(33n - 59)$A

In the work of Kawasaki et al.,[18] the number of operations for computing $\hat{\mathbf{J}}_i^{\langle i \rangle}$ is 27M and 34A for a manipulator with only rotational joints and 30M and 37A for a general manipulator. Table VII shows that the present algorithm requires only 24M and 27A for the third column of $\hat{\mathbf{J}}_i^{\langle i \rangle}$, which is more efficient. The total computation requirements of Formulations 1 and 2 with considering the first moments and inertia tensors are summarized in Table VIII. Table VIII also shows that Formulation 1 is more efficient than Algorithms I and II of Kawasaki et al.[18] for a manipulator with only rotational joints, while Formulation 2 is also more efficient than both algorithms of Kawasaki et al. for $n < 8$.

Table VIII. Computation requirements for the inertia matrix with the consideration of the first moments and the inertia tensors.

		Type 1				Type 2			
		n	4	6	8	n	4	6	8
Form. 1	M	$10n^2 + 14n - 37$	179	407	715	$10n^2 + 14n - 45$	171	399	707
	A	$6n^2 + 23n - 53$	135	301	515	$6n^2 + 23n - 59$	129	295	509
Form. 2	M	$11.5n^2 + 4.5n - 31$	171	410	741	$11.5n^2 - 8.5n + 15$	165	378	683
	A	$8n^2 + 15n - 47$	141	331	585	$8n^2 + 4n - 13$	131	299	531
Kawasaki	M	$11n^2 + 15n - 41$	194	445	783				
et al. I[18]	A	$7n^2 + 32n - 66$	174	378	638				
Kawasaki	M	$11n^2 + 9n - 35$	177	415	741				
et al. II[18]	A	$7n^2 + 23n - 57$	147	333	575				

5. CONCLUSION

In this paper, we deal with a type of composite body method for computing the inertia matrix of a manipulator and present five formulations based on this method. Four formulations are closely related to those in the earlier works, while Formulation 2 is novel. It is shown that Formulation 1 is the best efficient for some cases, while Formulation 2 is superior for the other cases. This fact still holds true when the formulations in the literature are also compared together.

The other salient feature of this paper is the algorithm for computing the first moments and inertia tensors of composite bodies. The algorithm uses a set of minimal linear combinations of inertia parameters (MLC's) instead of the natural inertia parameters, and is found superior to the others in the earlier works. The same set of MLC's was already used to reform the Newton–Euler recursive formulation of the inverse dynamics.[15] The resulting formulation was shown to be one of the most efficient inverse dynamics formulations. This paper shows again that using a set of MLC's can make the dynamics formulations more efficient.

APPENDIX A: DERIVATION OF THE COMPOSITE-BODY METHOD

The definition of a composite body is defined in section 2, and the inertia parameters of a composite, i.e., $\hat{m}_i$, $\hat{\mathbf{c}}_i^{\langle i\rangle}$, and $\hat{\mathbf{J}}_i^{\langle i\rangle}$, are defined in (3)–(5). The inertia force ($\mathbf{f}_{Ti}$) and torque ($\mathbf{t}_{Ti}$) of the composite body i can be obtained by using Newton–Euler equations as

$$\mathbf{f}_{Ti}^{\langle i\rangle} = -\hat{m}_i \ddot{\mathbf{r}}_i^{\langle i\rangle} \tag{A1}$$

$$\mathbf{t}_{Ti}^{\langle i\rangle} = -\hat{\mathbf{I}}_i^{\langle i\rangle} \dot{\boldsymbol{\omega}}_i^{\langle i\rangle} \tag{A2}$$

where $\boldsymbol{\omega}_i$ is the angular acceleration of joint i, $\mathbf{r}_i$ is the distance from the base to the center of mass of the composite body i, and $\hat{\mathbf{I}}_i$ is the inertia tensor of the composite body i about the center of mass of the composite body i, which can be expressed as

$$\mathbf{r}_i^{\langle i\rangle} = {}_0^i\mathbf{s}^{\langle i\rangle} + \frac{\hat{\mathbf{c}}_i^{\langle i\rangle}}{\hat{m}_i} \tag{A3}$$

$$\hat{\mathbf{I}}_j^{\langle j\rangle} = \hat{\mathbf{J}}_i^{\langle i\rangle} - \frac{1}{\hat{m}_i}\left[\hat{\mathbf{c}}_i^{\langle i\rangle}\times\right]\left[\hat{\mathbf{c}}_i^{\langle i\rangle}\times\right] \tag{A4}$$

where ${}_0^i\mathbf{s}$ is the distance from the base to the origin of frame E_i.

The acceleration ($\ddot{\mathbf{r}}_i$) of the center of mass and the angular acceleration ($\dot{\boldsymbol{\omega}}_i$) of the composite body i due to the motion of joint i only (i.e., the other joints are assumed stationary) are

$$\ddot{\mathbf{r}}_i^{\langle i\rangle} = \left[K_i^* \mathbf{u}_i^{\langle i\rangle} \times \left(\mathbf{r}_i^{\langle i\rangle} - {}_0^i\mathbf{s}^{\langle i\rangle}\right) + K_i \mathbf{u}_i^{\langle i\rangle}\right]\ddot{q}_i \tag{A5}$$

$$\dot{\boldsymbol{\omega}}_i^{\langle i\rangle} = K_i^* \mathbf{u}_i^{\langle i\rangle}\ddot{q}_i \tag{A6}$$

where $\mathbf{u}_i$ is the unit vector along joint i, q_i is the displacement of joint i. Then the joint force and torque of joint i applied on link i are, respectively,

$$\begin{aligned}\mathbf{f}_i^{\langle i\rangle} &= -\mathbf{f}_{Ti}^{\langle i\rangle}\\ &= \left[K_i^* \mathbf{u}_i^{\langle i\rangle} \times \hat{\mathbf{c}}_i^{\langle i\rangle} + K_i \hat{m}_i \mathbf{u}_i^{\langle i\rangle}\right]\ddot{q}_i \end{aligned}\tag{A7}$$

$$\begin{aligned}\mathbf{t}_i^{\langle i\rangle} &= -\mathbf{t}_{Ti}^{\langle i\rangle} - \frac{1}{\hat{m}_i}\hat{\mathbf{c}}_i^{\langle i\rangle} \times \mathbf{f}_{Ti}^{\langle i\rangle}\\ &= \left[K_i^* \hat{\mathbf{J}}_i^{\langle i\rangle}\mathbf{u}_i^{\langle i\rangle} + K_i \hat{\mathbf{c}}_i^{\langle i\rangle} \times \mathbf{u}_i^{\langle i\rangle}\right]\ddot{q}_i\end{aligned} \tag{A8}$$

Under the situation that only joint j moves and the gravity is neglected, the actuator force applied on joint j is to resist the component of the force or torque exerted on joint j by link j along the direction of joint j, which is an equivalent force or torque of the pair of $-\mathbf{f}_i$ and $-\mathbf{t}_i$. Thus, we obtain the actuator force of joint j due to $\ddot{q}_i$ as

$$\tau_{j,i} = \mathbf{u}_j^{\langle j\rangle} \cdot \left[K_j^*\left(\mathbf{t}_i^{\langle j\rangle} + {}_j^i\mathbf{s}^{\langle j\rangle} \times \mathbf{f}_i^{\langle j\rangle}\right) + K_j \mathbf{f}_i^{\langle j\rangle}\right], \quad j \le i. \tag{A9}$$

The definition of $\tau_{j,i}$ is equivalent to the product for the (j,i)th entry of the inertia matrix and $\ddot{q}_i$, i.e.,

$$\tau_{j,i} = D_{ji}\ddot{q}_i, \qquad j \le i \tag{A10}$$

where D_{ji} is the (j,i)th-entry of the inertia matrix $\mathbf{D}$. Since $\mathbf{D}$ is symmetrical, we just need to consider the upper triangular matrix, i.e., $j \le i$.

Let $\mathbf{f}_{i,i}^{\langle i\rangle}$ and $\mathbf{t}_{i,i}^{\langle i\rangle}$ be, respectively, $\mathbf{f}_i^{\langle i\rangle}$ and $\mathbf{t}_i^{\langle i\rangle}$ when $\ddot{q}_i = 1$. Substituting (A7) and (A8) into (A9) again, we finally obtain the recursive formulations (7)–(11), which are similar to that of Walker and Orin[1] in structure, but use the first moments and the inertia tensors of composite bodies about the origin of the local frame (instead of the center of mass).

APPENDIX B: FORMULATION OF KAWASAKI ET AL.

This section is to show that the formulation proposed by Kawasaki et al.[18] is also a variant form of the composite body method, although it is in terms of MLC's.

Atkeson et al.[22] indicated that the dynamic model of a manipulator is still preserved if the values of individual inertia parameters are so changed that the values of a set of minimal linear combinations of inertia parameters (MLC's) are retained. Since the set of MLC's discovered by Khalil et al.[8–10] and Mayer et al.[11,12] is obtained by regrouping the inertia parameters [masses (m_i), and first moments ($m_i \mathbf{c}_i^{\langle i \rangle}$) and inertia tensors ($\mathbf{J}_i^{\langle i \rangle}$) about the origin of the local frame] of same dynamic effects to the members of MLC's, the natural inertia parameters, $m_i, m_i \mathbf{c}_i^{\langle i \rangle}$, and $\mathbf{J}_i^{\langle i \rangle}$, can be replaced with their counterparts in the MLC's if the counterparts are not redundant or with zero if they are not in the set of MLC's. This allows us to repeat the formulation of Kawasaki et al.[18] in terms of the natural parameters instead of the MLC's without toil. Equations (25) and (44) in ref. 18 are repeated by using the notation of this article (i.e., M_i, $\mathbf{MS}_i$, $\mathbf{J}_i$, ν_i, and ${}^{i-1}\mathbf{p}_i$ in ref. 18 are replaced by m_i, $m_i \mathbf{c}_i^{\langle i \rangle}$, $\mathbf{J}_i^{\langle i \rangle}$, $\hat{m}_i$, and ${}_{i-1}^{\;\;i}\mathbf{s}^{\langle i-1 \rangle}$, respectively) as

$$ {}^{m}\mathbf{h}_i^j = \sum_{k=i}^{n} {}_m^i\mathbf{R}\left(m_k \mathbf{c}_k^{\langle i \rangle} + m_{k\,j-1}{}^{k}\mathbf{s}^{\langle i \rangle} \right), \qquad j \le i \quad \text{(B1)} $$

$$ \begin{aligned} \mathbf{L}_i = \sum_{k=i}^{n} {}_i^k\mathbf{R}\Big(& \mathbf{J}_k^{\langle k \rangle} - \left[{}_{\;k}^{k+1}\mathbf{s}^{\langle k \rangle} \times\right]\left[{}^{k}\mathbf{h}_{k+1}^{k+1} \times\right] \\ & - \left[{}^{k}\mathbf{h}_{k+1}^{k+1} \times\right]\left[{}_{\;k}^{k+1}\mathbf{s}^{\langle k \rangle} \times\right] \\ & + \hat{m}_{k+1}\left[{}_{\;k}^{k+1}\mathbf{s}^{\langle k \rangle} \times\right]\left[{}_{\;k}^{k+1}\mathbf{s}^{\langle k \rangle} \times\right]\Big) {}_i^k\mathbf{R}^T \end{aligned} \quad \text{(B2)} $$

It is then apparent that

$$ {}^{m}\mathbf{h}_i^j = {}_m^i\mathbf{R}\hat{\mathbf{c}}_i^{\langle i \rangle} + \hat{m}_{i\,j-1}{}^{i}\mathbf{s}^{\langle m \rangle} \quad \text{(B3)} $$

$$ \mathbf{L}_i = \hat{\mathbf{J}}_i^{\langle i \rangle} \quad \text{(B4)} $$

which correspond to the first moment and the inertia tensor of the composite body i.

Steps 4.1 and 4.2 in Table II in ref. 18 can be combined together in the following compact form (${}^j\mathbf{k}_i$ and ${}^j\mathbf{w}_i$ in ref. 18 are replaced by $\mathbf{t}_{j,i}^{\langle j \rangle}$ and $\mathbf{f}_{j,i}^{\langle j \rangle}$, respectively):

$$ {}^{i}\mathbf{k}_i = K_i^* \hat{\mathbf{J}}_i^{\langle i \rangle} \mathbf{u}_i^{\langle i \rangle}, \qquad i = 1, \ldots, n \quad \text{(B5)} $$

$$ D_{ii} = K_i^* \mathbf{u}_i^{\langle i \rangle}\, {}^{i}\mathbf{k}_i + K_i \hat{m}_i, \qquad i = 1, \ldots, n \quad \text{(B6)} $$

$$ \mathbf{f}_{i-1,i}^{\langle i-1 \rangle} = -K_i^* \left({}^{i-1}\mathbf{h}_i^i - \hat{m}_{i\,i-1}{}^{i}\mathbf{s}^{\langle i-1 \rangle} \right) \times \mathbf{u}_i^{\langle i-1 \rangle} + K_i \hat{m}_i \mathbf{u}_i^{\langle i-1 \rangle}, \qquad i = 2, \ldots, n \quad \text{(B7)} $$

$$ \mathbf{t}_{i-1,i}^{\langle i-1 \rangle} = K_i^* \left\{ -\mathbf{f}_{i-1,i}^{\langle i-1 \rangle} \times {}_{i-1}^{\;\;i}\mathbf{s}^{\langle i-1 \rangle} + {}_{i-1}^{\;\;i}\mathbf{R}\, {}^{i}\mathbf{k}_i \right\} + K_i\, {}^{i-1}\mathbf{h}_i^i \times \mathbf{u}_i^{\langle i-1 \rangle}, \qquad i = 2, \ldots, n \quad \text{(B8)} $$

$$ \mathbf{f}_{j,i}^{\langle j \rangle} = {}_{\;j}^{j+1}\mathbf{R}\mathbf{f}_{j+1,i}^{\langle j+1 \rangle}, \qquad 1 \le j \le i-2 \quad \text{(B9)} $$

$$ \mathbf{t}_{j,i}^{\langle j \rangle} = {}_{\;j}^{j+1}\mathbf{R}\mathbf{t}_{j+1,i}^{\langle j+1 \rangle} + {}_{\;j}^{j+1}\mathbf{s}^{\langle j \rangle} \times \mathbf{f}_{j,i}^{\langle j \rangle}, \qquad 1 \le j \le i-2 \quad \text{(B10)} $$

$$ D_{ij} = \mathbf{u}_j^{\langle j \rangle} \cdot \left(K_i^* \mathbf{t}_{j,i}^{\langle j \rangle} + K_i \mathbf{f}_{j,i}^{\langle j \rangle} \right), \qquad 1 \le j \le i-1 \quad \text{(B11)} $$

It should be remarked that there was an erratum in (32) in ref. 18, which should be corrected to

$$ \mathbf{u}_i^{\langle i \rangle} \cdot \mathbf{f}_i^{\langle j \rangle} = \mathbf{u}_j^{\langle j \rangle} \cdot \left[K_j^* \left({}^{j}\mathbf{h}_i^{j+1} \times \mathbf{u}_i^{\langle j \rangle} \right) + K_j \hat{m}_i \mathbf{u}_i^{\langle j \rangle} \right] \quad \text{(B12)} $$

So we modify the computation of ${}^{j}\mathbf{h}_i^{j+1}$ in Step 4.2 in Table II of ref. 18 to be that of ${}^{j}\mathbf{h}_i^{j+1} \times \mathbf{u}_i^{\langle j \rangle}$. And ${}^{j}\mathbf{h}_i^{j+1} \times \mathbf{u}_i^{\langle j \rangle} = {}^{j}\mathbf{h}_i^{i} \times \mathbf{u}_i^{\langle j \rangle} + \hat{m}_{i\,j}{}^{i}\mathbf{s}^{\langle j \rangle} \times \mathbf{u}_i^{\langle j \rangle}$ for translational joint i is embedded in (B8) and (B10). For a manipulator with only rotational joints, the masses of the composite bodies, $\hat{m}_i$, are all redundant for the manipulator dynamics and can be set to be zero. In such a case, ${}^{i-1}\mathbf{h}_i^i = \hat{\mathbf{c}}_i^{\langle i-1 \rangle}$ and then the formulation of Kawasaki et al. (B5)–(B11) is reduced to Formulation 1 [i.e., Eqs. (7)–(11)]. This indicates that Algorithm II in ref. 18 is a special form of Formulation 1.

APPENDIX C: INERTIA CONSTANTS OF COMPOSITE BODIES

The inertia constants of the composite body i ($\mathbf{k}_i$ and $\mathbf{U}_i$) in (22) and (23) are defined in the following:

$$ \mathbf{k}_i = m_i \mathbf{c}_i^{\langle i \rangle} + K_{i+1}^* \left(\hat{m}_{i+1}\, {}_{\;i}^{i+1}\mathbf{s}^{\langle i \rangle} + {}_{\;i}^{i+1}\mathbf{R} \begin{bmatrix} 0 \\ 0 \\ (\mathbf{k}_{i+1})_z \end{bmatrix} \right) + K_{i+1} \left(\hat{m}_{i+1} \begin{bmatrix} \left({}_{\;i}^{i+1}\mathbf{s}^{\langle i \rangle} \right)_x \\ 0 \\ 0 \end{bmatrix} + {}_{\;i}^{i+1}\mathbf{R}\mathbf{k}_{i+1} \right) \quad \text{(C1)} $$

If joint $i+1$ is a rotational joint, then

$$\begin{aligned}\mathbf{U}_i = \mathbf{I}_i^{\langle i\rangle} - m_i\left[\mathbf{c}_i^{\langle i\rangle}\times\right]\left[\mathbf{c}_i^{\langle i\rangle}\times\right] \\ - \hat{m}_{i+1}\left[{}^{i+1}_{\ \ i}\mathbf{s}^{\langle i\rangle}\times\right]\left[{}^{i+1}_{\ \ i}\mathbf{s}^{\langle i\rangle}\times\right] \\ + {}^{i+1}_{\ \ i}\mathbf{R}\begin{bmatrix}(\mathbf{U}_{i+1})_{22} & 0 & 0\\ 0 & (\mathbf{U}_{i+1})_{22} & 0\\ 0 & 0 & 0\end{bmatrix}{}^{i+1}_{\ \ i}\mathbf{R}^T \\ - \left[{}^{i+1}_{\ \ i}\mathbf{s}^{\langle i\rangle}\times\right]\left[\left({}^{i+1}_{\ \ i}\mathbf{R}_b(\mathbf{k}_{i+1})_z\right)\times\right] \\ - \left[\left({}^{i+1}_{\ \ i}\mathbf{R}_b(\mathbf{k}_{i+1})_z\right)\times\right]\left[{}^{i+1}_{\ \ i}\mathbf{s}^{\langle i\rangle}\times\right] \qquad \text{(C2)}\end{aligned}$$

whereas for translational joint $i+1$,

$$\begin{aligned}\mathbf{U}_i = \mathbf{I}_i^{\langle i\rangle} - m_i\left[\mathbf{c}_i^{\langle i\rangle}\times\right]\left[\mathbf{c}_i^{\langle i\rangle}\times\right] + {}^{i+1}_{\ \ i}\mathbf{R}\mathbf{U}_{i+1}{}^{i+1}_{\ \ i}\mathbf{R}^T \\ - \hat{m}_{i+1}\left[\mathbf{b}_{i+1}^{\langle i\rangle}\times\right]\left[\mathbf{b}_{i+1}^{\langle i\rangle}\times\right] \\ - \left[\mathbf{b}_{i+1}^{\langle i\rangle}\times\right]\left[\left({}^{i+1}_{\ \ i}\mathbf{R}\mathbf{k}_{i+1}\right)\times\right] \\ - \left[\left({}^{i+1}_{\ \ i}\mathbf{R}\mathbf{k}_{i+1}\right)\times\right]\left[\mathbf{b}_{i+1}^{\langle i\rangle}\times\right] \qquad \text{(C3)}\end{aligned}$$

Note that ${}^{i+1}_{\ \ i}\mathbf{R}_b$ is the third column of ${}^{i+1}_{\ \ i}\mathbf{R}$ (i.e., ${}^{i+1}_{\ \ i}\mathbf{R}_b = [0, -S\beta_{i+1}, C\beta_{i+1}]^T$), $\mathbf{b}_{i+1}^{\langle i\rangle} = [b_{i+1}, 0, 0]^T$ and $\mathbf{d}_{i+1}^{\langle i+1\rangle} = [0, 0, d_{i+1}]^T$ (i.e., ${}^{i+1}_{\ \ i}\mathbf{s}^{\langle i\rangle} = \mathbf{b}_{i+1}^{\langle i\rangle} + \mathbf{d}_{i+1}^{\langle i\rangle}$).

This article was supported in part by the National Science Council, Taiwan, under Grant No. NSC87-2212-E-009-001.

REFERENCES

1. M.W. Walker and D.E. Orin, Efficient dynamic computer simulation of robotic mechanisms, ASME J Dynam Syst Meas Control 104 (1982), 205–211.
2. M. Renaud, ''An efficient iterative analytical procedure for obtaining a robot manipulator dynamic model,'' Robotics Research, M. Brady and R. Paul (Editors), MIT Press, Cambridge, 1984, pp. 749–764.
3. M. Renaud, ''A near minimum iterative analytical procedure for obtaining a robot manipulator dynamic model,'' Dynamics of Multibody Systems, G. Bianchi and W. Schiehlen (Editors), Springer-Verlag, Berlin, 1986, pp. 201–212.
4. M. Vukobratovic, S.G. Li and N. Kircanski, An efficient procedure for generating dynamic manipulator models, Robotica 3 (1985), 147–152.
5. A. Fijany and A.K. Bejczy, An efficient method for computation of the manipulator inertia matrix, J Robotic Syst 7 (1990), 57–80.
6. K.W. Lilly and D.E. Orin, Alternate formulations for the manipulator inertia matrix, Int J Robotics Res 10 (1991), 64–74.
7. S.K. Lin, A new composite body method for manipulator dynamics, J Robotic Syst 8 (1991), 197–219.
8. M. Gautier and W. Khalil, Direct Calculation of minimum set of inertial parameters of serial robots, IEEE Trans Robotics Automat 6 (1990), 368–373.
9. W. Khalil and F. Bennis, Comments on Direct Calculation of minimum set of inertial parameters of serial robots, IEEE Trans Robotics Automat 10 (1994), 78–79.
10. M. Gautier, F. Bennis, and W. Khalil, The use of generalized links to determine the minimum inertial parameters of robots, J Robotic Syst 7 (1990), 225–242.
11. H. Mayeda, K. Yoshida, and K. Osuka, Base parameters of manipulator dynamics models, IEEE Trans Robotics Automat 6 (1990), 312–321.
12. H. Mayeda and K. Ohashi, Base parameters of dynamic models for general open loop kinematic chains, Proc 5th Int Symp Robotics Res 1990, pp. 271–278.
13. S.K. Lin, An identification method for estimating the inertia parameters of a manipulator, J Robotic Syst 9 (1992), 505–528.
14. S.K. Lin, An approach to the identifiable parameters of a manipulator, J Robotic Syst 11 (1994), 641–656.
15. S.K. Lin, Minimal linear combinations of the inertia parameters of a manipulator, IEEE Trans Robotics Automat 11 (1995), 360–373.
16. W. Khalil and J.F. Kleinfinger, Minimum operations and minimum parameters of the dynamic models of tree structure robots, IEEE J Robotics Automat 3 (1987), 517–526.
17. M. Gautier and W. Khalil, A direct determination of minimum inertial parameters of robots, Proc 1988 IEEE Conf Robotics Automat, 1988, pp. 1682–1687.
18. H. Kawasaki, A. Murata, and K. Kanzaki, An efficient algorithm for generating manipulator inertia matrix using the minimum set of dynamics parameters, J Robotic Syst 13 (1996), 262–273.
19. S.K. Lin, Microprocessor implementation of the inverse dynamical system for industrial robot control, Proc 10th IFAC World Congress Automatic Control, Munich, FRG, 1987, vol. 4, pp. 332–339.
20. J.J. Craig, ''Introduction to robotics: Mechanics and control,'' Addison-Wesley, Reading, MA, 1986.
21. J. Wittenburg, ''Dynamics of systems of rigid bodies,'' B.G. Teubner, Stuttgart, 1977, p. 36.
22. C.G. Atkeson, C.H. An, and J.M. Hollerbach, Estimation of inertial parameters of manipulator loads and links, Int J Robotics Res 5 (1986), no. 3, 101–119.