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ON A PROBLEM OF DYNKIN

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ABSTRACT. Consider an (L, α)-superdiffusion X on $\mathbb{R}^d$, where L is an uniformly elliptic differential operator in $\mathbb{R}^d$, and $1 < \alpha \leq 2$. The $\mathbb{G}$-polar sets for X are subsets of $\mathbb{R}\times\mathbb{R}^d$ which have no intersection with the graph $\mathbb{G}$ of X, and they are related to the removable singularities for a corresponding nonlinear parabolic partial differential equation. Dynkin characterized the $\mathbb{G}$-polarity of a general analytic set $A \subset \mathbb{R} \times \mathbb{R}^d$ in term of the Bessel capacity of A, and Sheu in term of the restricted Hausdorff dimension. In this paper we study in particular the $\mathbb{G}$-polarity of sets of the form $E \times F$, where E and F are two Borel subsets of $\mathbb{R}$ and $\mathbb{R}^d$ respectively. We establish a relationship between the restricted Hausdorff dimension of $E \times F$ and the usual Hausdorff dimensions of E and F. As an application, we obtain a criterion for $\mathbb{G}$-polarity of $E \times F$ in terms of the Hausdorff dimensions of E and F, which also gives an answer to a problem proposed by Dynkin in the 1991 Wald Memorial Lectures.

1. Introduction

Suppose that L is an uniformly elliptic differential operator in $\mathbb{R}\times\mathbb{R}^d$ of the form

$$Lu(t,x) \ = \ \sum_{i,j} a_{ij}(t,x) \ \frac{\partial^2 u}{\partial x_i \partial x_j} \ + \ \sum_i \ b_i(t,x)\frac{\partial u}{\partial x_i}.$$

Here we assume that a_{ij} and b_i are bounded and smooth functions in $\mathbb{R} \times \mathbb{R}^d$. An (L, α)-superdiffusion, $1 < \alpha \leq 2$, is a branching measure-valued Markov process $X = (X_t, P_\mu)$ such that for every bounded positive Borel function f on $\mathbb{R}^d$, the function

$$v(r,x) \ = \ -\log P_{\delta_{r,x}} e^{-\langle f, X_t\rangle}$$

is a mild solution of the following problem:

$$\begin{cases} \frac{\partial v}{\partial t} \ + \ Lv \ = \ v^\alpha \ \text{ in } \ (-\infty, t) \times \mathbb{R}^d, \\ v(r,x) \quad \to \ f(x) \text{ as } r \uparrow t \text{ and } \ x \in \mathbb{R}^d. \end{cases}$$

(Here we write PY for the expected value of Y with respect to the probability measure P, and $\langle f, \mu\rangle$ for the integral of f with respect to the measure μ.) The graph of X is the minimal closed subset $\mathbb{G}$ of $\mathbb{R} \times \mathbb{R}^d$ such that, for every $t \in \mathbb{R}$,

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the measure X_t is concentrated on the t-section of $\mathbb{G}$ An analytic set A of $\mathbb{R} \times \mathbb{R}^d$ is called $\mathbb{G}$-polar if for every $r < t$ and $x \in \mathbb{R}^d$, we have

$$P_{\delta_{r,x}}(\mathbb{G} \cap A_t \ = \ \emptyset) \ = \ 1,$$

where $A_t \ = \ A \cap ([t, \infty) \times \mathbb{R}^d)$. In [2], Dynkin proved that the class of $\mathbb{G}$ -polar sets for the (L, α)-superdiffusion X is identical to the class of sets of Bessel capacity zero. Moreover he proved that a set A is $\mathbb{G}$-polar if and only if it is a removable singularity for the partial differential equation

$$\frac{\partial v}{\partial t} \ + \ Lv \ = \ v^\alpha. \tag{1}$$

(We say that $A \subset \mathbb{R} \times \mathbb{R}^d$ is a removable singularity for the equation (1) if 0 is the only nonnegative solution of the equation (1) in $(\mathbb{R} \times \mathbb{R}^d)\backslash A$.) In [7], Sheu demonstrated that the critical restricted Hausdorff dimension (to be introduced later) for the $\mathbb{G}$-polarity is $d - \frac{2}{\alpha - 1}$.

We say that an analytic set F in $\mathbb{R}^d$ is $\mathbb{H}$-polar if $\{t\} \times F$ is $\mathbb{G}$-polar for every $t \in \mathbb{R}$. The notion of $\mathbb{H}$ -polarity is also related to solutions of the heat equation with initial measure value. (See Baras and Pierre [1].) Note that the critical Hausdorff dimension for the $\mathbb{H}$-polarity is $d - \frac{2}{\alpha - 1}$.

Our objective is to study the following problem proposed by Dynkin in the 1991 Wald Memorial Lectures.

Problem ([3], p. 1245)**.** Suppose F is $\mathbb{H}$-polar and $E \subset \mathbb{R}$ is a set of Lebesgue measure 0. Is $E \times F$ $\mathbb{G}$-polar?

In Section 2 we recall definitions of Hausdorff dimension, box dimension and the restricted Hausdorff dimension, and establish new relations between the restricted Hausdorff dimension of $E \times F$ and the Hausdorff dimensions of E and F. Namely, we prove that

$$\begin{aligned} 2H\text{-}\dim(E) \ + \ H\text{-}\dim(F) \ &\le \mathcal{R}\text{-}H\text{-}\dim(E \times F) \\ &\le 2H\text{-}\dim(E) \ + \ B\text{-}\dim_{-}(F). \end{aligned}$$

where H-dim means the Hausdorff dimension, B-$\dim_{-}$ the lower Hausdorff box dimension, and $\mathcal{R}$-H-dim the restricted Hausdorff dimension. (Our proofs are analogous (with some suitable modifications) to that of Falconer [4] and from there we also quote some interesting examples.) Under the assumption $H\text{-}\dim(F) \ = \ B\text{-}\dim_{-}(F)$, we obtain that if $2H\text{-}\dim(E) \ + \ H\text{-}\dim(F) < d - \frac{2}{\alpha - 1}$, then $E \times F$ is $\mathbb{G}$-polar, whereas, if $2H\text{-}\dim(E) \ + \ H\text{-}\dim(F) > d - \frac{2}{\alpha - 1}$, then $E \times F$ is not $\mathbb{G}$-polar. As an application, we give examples in Section 3 which show that the answer to Dynkin's problem is negative (see Theorem 6 for more details).

2. Hausdorff dimension, box dimension and the restricted Hausdorff dimension

Suppose that F is a subset of $\mathbb{R}^d$. First we recall a definition of the Haudorff dimension of F. For every $s > 0$ and $\epsilon > 0$, set

$$\wedge^s\text{-}m_\epsilon(F) \ = \ \inf \sum_i \ (\mathrm{diam}(B_i))^s$$

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where the infimum is taken over all countable coverings of F by open ball B_i with radius $r_i < \epsilon$. The Hausdorff measure with index s is defined by the formula

$$\wedge^s\text{-}m(F) \;=\; \lim_{\epsilon\downarrow 0} \wedge^s\text{-}m_\epsilon(F), \tag{2}$$

and the Hausdorff dimension $H\text{-}\dim(F)$ is the supremum s such that $\wedge^s\text{-}m(F) > 0$.

Let A be a subset of $\mathbb{R}\times\mathbb{R}^d$. In order to determine if A is polar relative to the heat equation

$$\sum_i \frac{\partial^2 u}{\partial x_i^2} \;=\; \frac{\partial u}{\partial t},$$

Taylor and Watson [8] introduced the notion of the restricted Hausdorff dimension of A. For any $s > 0$, the definition of the restricted Hausdorff measure with index s, denoted as $\mathcal{R}\text{-}\wedge^s\text{-}m(A)$, is the same as that for Hausdorff measure except that the balls for covering are replaced by sets of the form

$$P(t,x;r) \;=\; [t, t+r^2]\times[x_1, x_1+r]\times[x_2, x_r+r]\times\cdots\times[x_d, x_d+r] \tag{3}$$

where $t\in\mathbb{R}, r\geq 0$ and $x \;=\; (x_1, x_2,\cdots, x_d)$. The restricted Hausdorff dimension of A , denoted as $\mathcal{R}\text{-}H\text{-}\dim(A)$, is defined in terms of the restricted Hausdorff measure in the same way as the Hausdorff dimension is defined in terms of the Hausdorff measure.

We quote a result from Taylor and Watson [8].

Lemma 1. *Let A be a Borel subset of $\mathbb{R}\times\mathbb{R}^d$ and $s > 0$, c be two constants. If μ is a finite positive measure on $\mathbb{R}\times\mathbb{R}^d$ satisfying*

$$\limsup_{r\to 0} \frac{\mu(A\cap P(t,x;r))}{r^s} \leq c < \infty,$$

for all $(t,x)\in A$, then

$$\mathcal{R}\text{-}\wedge^s\text{-}m(A) \geq \frac{1}{c2^s}\,\mu(A).$$

Proposition 2. *Let E and F be two Borel subsets of $\mathbb{R}$ and $\mathbb{R}^d$ respectively. If $0 < \wedge^k\text{-}m(E) < \infty$ and $0 < \wedge^l\text{-}m(F) < \infty$ for some $k, l \geq 0$, then $\mathcal{R}\text{-}\wedge^{2k+l}\text{-}m(E\times F) > 0$.*

Proof. This is trivial in the case $k = l = 0$. We assume that $k + l > 0$. Since $0 < \wedge^k\text{-}m(E) < \infty$ and $0 < \wedge^l\text{-}m(F) < \infty$, it follows from Lemma 5.4 of Hayman and Kennedy [5] that there exist two measures μ_1 and μ_2 on $\mathbb{R}$ and $\mathbb{R}^d$ respectively satisfying the following two conditions:

(1) $0 < \mu_1(E) < \infty$ and $0 < \mu_2(F) < \infty$, and
(2) There exists a constant c such that for all $t\in E, x\in F$ and $0 < r\leq 1$, we have

$$\mu_1(B(t;r)) \leq cr^k$$

and

$$\mu_2(B(x;r)) \leq cr^l,$$

where $B(x;r)$ is the ball centered at x and radius r.

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Set $\mu = \mu_1 \times \mu_2$. For every $(t,x) \in E \times F$ and $0 < r \leq \frac{1}{\sqrt{d}}$, we have, by the condition (2), that

$$\begin{aligned}\mu((E\times F)\cap P(t,x;r)) &\leq \mu_1(E\cap[t,t+r^2])\mu_2(F\cap B(x;r\sqrt{d}))\\ &\leq c^2(\frac{r^2}{2})^k(r\sqrt{d})^l = \frac{c^2 d^{\frac{l}{2}}}{2^k}\, r^{2k+l}.\end{aligned}$$

Hence

$$\limsup_{r\to 0}\frac{\mu((E\times F)\cap P(t,x;r))}{r^{2k+l}} \leq \frac{c^2 d^{\frac{l}{2}}}{2^k} < \infty$$

for all $(t,x) \in E \times F$. It follows from Lemma 1 and condition (1) that

$$\mathcal{R}\text{-}\wedge^{2k+l}\text{-}m(E\times F) \geq \frac{1}{c^2 d^{\frac{l}{2}} 2^{k+l}}\mu(E\times F) = \frac{1}{c^2 d^{\frac{l}{2}} 2^{k+l}}\mu_1(E)\mu_2(F) > 0.$$

□

Corollary 3. *For every Borel sets $E \subset \mathbb{R}$ and $F \subset \mathbb{R}^d$, we have*

(4) $$\mathcal{R}\text{-}H\text{-}\dim(E\times F) \geq 2\, H\text{-}\dim(E) + H\text{-}\dim(F).$$

Proof. Let $k = H\text{-}\dim(E)$ and $l = H\text{-}\dim(F)$. If $k' < k$ and $l' < l$, then $\wedge^{k'}\text{-}(E) = \infty$ and $\wedge^{l'}\text{-}(F) = \infty$. There exist two Borel subsets $E' \subset E, F' \subset F$ satisfying $0 < \wedge^{k'}\text{-}m(E') < \infty$ and $0 < \wedge^{l'}\text{-}m(F') < \infty$. By Proposition 2, we have

$$\mathcal{R}\text{-}\wedge^{2k'+l'}\text{-}m(E\times F) \geq \mathcal{R}\text{-}\wedge^{2k'+l'}\text{-}m(E'\times F') > 0.$$

By the definition of the restricted Hausdorff dimension, we get

$$\mathcal{R}\text{-}H\text{-}\dim(E\times F) \geq 2k' + l'.$$

Since this holds for every $k' < k$ and $l' < l$, we obtain that $\mathcal{R}\text{-}H\text{-}\dim(E\times F) \geq 2\, H\text{-}\dim(E) + H\text{-}\dim(F)$. □

Note that $P(t,x;r) \subset B((t,x); r\sqrt{d+r^2})$ for all $(t,x) \in \mathbb{R}^d$ and $r \geq 0$. It follows from the definitions that $\mathcal{R}\text{-}\wedge^s\text{-}m(A) \geq \wedge^s\text{-}m(A)$ for all $s \geq 0$ and all subsets $A \subset \mathbb{R}^{d+1}$. Hence

(5) $$\mathcal{R}\text{-}H\text{-}\dim(A) \geq H\text{-}\dim(A) \quad \text{for all } A \subset \mathbb{R}^{d+1}.$$

The following example is a modification of Example 7.8 in Falconer [4], and it shows that the equality in (4) does not hold for general Borel sets E and F.

Example 1. Let $0 = a_0 < a_1 < a_2 < \cdots$ be an increasing sequence of integers. Put

$$\begin{aligned}E = \{&r \in [0,1] | r = 0.r_1 r_2 \cdots r_i \cdots,\\ &\text{where } r_i = 0 \text{ whenever } a_{2k}+1 \leq i \leq a_{2k+1} \text{ for some integer } k\}\end{aligned}$$

and

$$\begin{aligned}F_1 = \{&r \in [0,1] | r = 0.r_1 r_2 \cdots r_i \cdots,\\ &\text{where } r_i = 0 \text{ whenever } a_{2k+1}+1 \leq i \leq a_{2k+2} \text{ for some integer } k\}.\end{aligned}$$

It was shown in Falconer [2] that if the a_i increase sufficiently rapidly, then $H\text{-}\dim(E \times F_1) \geq 1$ and $H\text{-}\dim(E) = H\text{-}\dim(F_1) = 0$. Set

$$F = \{(x_1, 0, 0, \cdots, 0) \in \mathbb{R}^d, |\ x_1 \in F_1\}.$$

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The formula for Hausdorff dimension of product sets implies that $H\text{-}\dim(F) = H\text{-}\dim(F_1) = 0$, and $H\text{-}\dim(E\times F) = H\text{-}\dim(E\times F_1) \geq 1$. It follows from (5) that

$$\mathcal{R}\text{-}H\text{-}\dim(E\times F) \geq H\text{-}\dim(E\times F) \geq 1 > 0 = 2\ H\text{-}\dim(E) + H\text{-}\dim(F).$$

To get a sufficient condition for the equality in (4) to hold, we recall a definition of box dimension for subset F of $\mathbb{R}^d$. For every $\epsilon > 0$, let $N_\epsilon(F)$ be the number of ϵ-mesh cubes that intersect F. Here an ϵ-mesh cube is one of the form

$$[m_1\epsilon, (m_1+1)\epsilon] \times \cdots \times [m_d\epsilon, (m_d+1)\epsilon]$$

where $m_1, ..., m_d$ are integers. The lower and upper box dimensions of F are defined as

$$B\text{-}\dim_-(F) = \liminf_{\epsilon\to 0} \frac{\log N_\epsilon(F)}{-\log \epsilon}$$

and

$$B\text{-}\dim_+(F) = \limsup_{\epsilon\to 0} \frac{\log N_\epsilon(F)}{-\log \epsilon}.$$

The box dimension of F is given by

$$B\text{-}\dim(F) = \lim_{\epsilon\to 0} \frac{\log N_\epsilon(F)}{-\log \epsilon}$$

(if this limit exists). Note that for every $\epsilon > 0$, the $N_\epsilon(F)$ number of ϵ-mesh cubes that intersect with F forms a covering for F. Hence for every $s > 0$, we have

$$\wedge^s\text{-}m_{\epsilon\sqrt{d}}(F) \leq (\epsilon\sqrt{d})^s N_\epsilon(F).$$

Taking the logarithm and then dividing by $-\log\epsilon$ (we assume that $\epsilon < 1$) on both sides gives

$$s + \frac{s}{2}\frac{\log d}{\log \epsilon} + \frac{\log(\wedge^s\text{-}m_{\epsilon\sqrt{d}}(F))}{-\log\epsilon} \leq \frac{\log\ N_\epsilon(F)}{-\log\epsilon}.$$

As $\epsilon \downarrow 0$, we get

$$s + \liminf_{\epsilon\to 0} \frac{\log(\wedge^s\text{-}m_{\epsilon\sqrt{d}}(F))}{-\log\epsilon} \leq B\text{-}\dim_-(F).$$

If $s < H\text{-}\dim(F)$, then $\lim_{\epsilon\to 0} \wedge^s\text{-}m_\epsilon(F) = \infty$ and hence $s \leq B\text{-}\dim_-(F)$. Since this holds for all $s < H\text{-}\dim(F)$, we observe that

$$B\text{-}\dim_-(F) \geq H\text{-}\dim(F). \tag{6}$$

Although there are many examples in which the above inequality is strict, many reasonably regular sets have the same Hausdorff and box dimension (see Falconer [4] for more details).

Example 2. Let m be a positive integer and $0 < \lambda < \frac{1}{m}$. Put $C_0 = [0,1]$. For $k \geq 0$, assume C_k consists of m^k closed intervals of lengths λ^k. Then each closed interval I in C_k is replaced by m equally spaced subintervals of length λ^{k+1}, the ends of the I coinciding with the ends of the extreme subintervals. The union of all these subintervals forms the set C_{k+1}. Put

$$C(m,\lambda) = \bigcap_k C_k.$$

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Note that $C(2, \frac{1}{3})$ is the middle third Cantor set. Clearly, for every m and $\lambda > 0, C(m, \lambda)$ is a set of Lebesgue measure zero. Moreover we have

$$H\text{-}\dim(C(m,\lambda)) \ = \ B\text{-}\dim(C(m,\lambda)) \ = \ \frac{\log m}{-\log \lambda}$$

(for a proof, see Falconer [4] Example 4.5)).

Proposition 4. *For any sets $E \subset \mathbb{R}$ and $F \subset \mathbb{R}^d$, we have*

$$\mathcal{R}\text{-}H\text{-}\dim(E \times F) \leq 2\ H\text{-}\dim(E)\ +\ B\text{-}\dim_{-}(F). \tag{7}$$

Proof. Let $k \ = \ H\text{-}\dim(E)$ and $l \ = \ B\text{-}\dim_{-}(F)$. Choose $k' > k$ and $l' > l$. By the definition of box dimension, there exists $\epsilon_0 > 0$ such that

$$N_\epsilon(F) \leq \epsilon^{-l'} \ \text{ for all } \epsilon < \epsilon_0. \tag{8}$$

Let E_j be any ϵ- cover of E by intervals with $\sum_j |E_j|^{k'} < 1$. For each j, let F_{jn} be the $\sqrt{|E_j|}$-mesh cubes that intersect with F. Then $E_j \times F_{jn}$ are sets of the form (3) and

$$\bigcup_j \bigcup_n E_j \times F_{jn}$$

is a covering of $E \times F$ with $\operatorname{diam}(E_j \times F_{jn}) \leq \sqrt{\epsilon(d+\epsilon)}$. Set $\epsilon' \ = \ \sqrt{\epsilon + d}$. For $\sqrt{\epsilon} < \epsilon_0$, we have

$$\begin{aligned}
\mathcal{R}\text{-} \wedge^{2k'+l'}\text{-}m_{\sqrt{\epsilon(d+\epsilon)}}(E \times F) &\leq \sum_j \sum_n \left[\sqrt{|E_j|(d+|E_j|)}\right]^{2k'+l'} \\
&\leq (\epsilon')^{2k'+l'} \sum_j |E_j|^{k'+\frac{l'}{2}} |E_j|^{-\frac{l'}{2}} \\
&\leq (\epsilon')^{2k'+l'} \sum_j |E_j|^{k'} < \infty,
\end{aligned}$$

which implies that $\mathcal{R}\text{-}H\text{-}\dim(E \times F) \geq 2k' \ + \ l'$. Since this holds for all $k' > k$ and $l' > l$, we obtain that $\mathcal{R}\text{-}H\text{-}\dim(E \times F) \geq 2H\text{-}\dim(E) \ + B\text{-}\dim_{-}(F)$. □

Combining Corollary 3 and Proposition 4 we get the following main theorem.

Theorem 5. *Let $E \subset \mathbb{R}$ and $F \subset \mathbb{R}^d$ be two Borel sets. If $H\text{-}\dim(F) \ = \ B\text{-}\dim(F)$, then we have*

$$\mathcal{R}\text{-}H\text{-}\dim(E \times F) \ = \ 2\ H\text{-}\dim(E) \ + \ H\text{-}\dim(F). \tag{9}$$

Remark. Consider the logarithmic Hausdorff dimension instead (for a definition see, e.g., Dynkin [3]) and define the box dimension of F as

$$\lim_{\epsilon \to 0} \frac{\log N_\epsilon(F)}{\log(-\log\ \epsilon)}.$$

Using the same approach as before, we prove that the restricted logarithmic Hausdorff dimension of $E \times F$ is the sum of the logarithmic Hausdorff dimensions of E and of F.

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3. Applications

Consider an (L, α)-superdiffusion X and assume that $d > \frac{2}{\alpha-1}$. (Note that there is no $\mathbb{G}$-polar set in the case $d < \frac{2}{\alpha-1}$.) Set $\gamma_0 = d - \frac{2}{\alpha-1}$. Sheu [7] showed that the critical restricted Hausdorff dimension for $\mathbb{G}$-polarity is γ_0. (This means that if $\mathcal{R}$-H-$\dim(A) < \gamma_0$, then A is $\mathbb{G}$-polar; whereas, if $\mathcal{R}$-H-$\dim(A) > \gamma_0$, then it is not $\mathbb{G}$-polar). In fact γ_0 is also the critical Hausdorff dimension for $\mathbb{H}$-polarity in $\mathbb{R}^d$(see, e.g., Dynkin [3] and Sheu [7]). Using these facts and Theorem 5, we obtain the following theorem (which give an answer to Dynkin's problem).

Theorem 6. *Assume that F is a Borel subset of $\mathbb{R}^d$ and satisfies the condition*

$$\gamma = H\text{-}\dim(F) = B\text{-}\dim(F) < \gamma_0.$$

Let E be a Borel subset of $\mathbb{R}$. Then

(1) *If $H\text{-}\dim(E) < \frac{1}{2}(\gamma_0 - \gamma)$, then $E \times F$ is a $\mathbb{G}$-polar set.*
(2) *If $H\text{-}\dim(E) > \frac{1}{2}(\gamma_0 - \gamma)$, then $E \times F$ is not $\mathbb{G}$-polar.*

Remark. (1) In [8], Taylor and Watson also considered polarity of sets of the form $T \times \{0\}, 0 \in \mathbb{R}^d$, $T \subset \mathbb{R}$, for the heat equation. They showed that $T \times \{0\}$ is polar if, and only if $C_{\frac{d}{2}}(T) = 0$, where C_β is the Riesz capacity of order β. (See also Kaufman and Wu [6].)

(2) Example 1 shows that there exist Borel sets $E \subset \mathbb{R}$ and $F \subset \mathbb{R}^d$ satisfying $\mathcal{R}$-H-$\dim(E \times F) \geq 1$ and $H\text{-}\dim(E) = H\text{-}\dim(F) = 0$. Clearly $E \times F$ is not $\mathbb{G}$-polar if $d - \frac{2}{\alpha-1} < 1$. Therefore the condition that $H\text{-}\dim(F) = B\text{-}\dim(F)$ is crucial.

Example 3. Let $F = \{0\}$ be the origin point of $\mathbb{R}^d$ and $E = C(m, \lambda)$, where m is an integer, $0 < \lambda < \frac{1}{m}$ and $C(m, \lambda)$ is defined as in Example 2. Clearly $H\text{-}\dim(F) = B\text{-}\dim(F) = 0$. By Theorem 5 and (6), we have

$$\mathcal{R}\text{-}H\text{-}\dim(C(m, \lambda) \times \{0\}) = 2 \ H\text{-}\dim(C(m, \lambda)) = \frac{2 \ \log m}{-\log \lambda}.$$

Theorem 6 says that if $m < \lambda^{-\frac{\gamma_0}{2}}$, then $C(m, \lambda) \times \{0\}$ is $\mathbb{G}$-polar; whereas, if $m > \lambda^{-\frac{\gamma_0}{2}}$, then it is not $\mathbb{G}$-polar.

Example 4. Take $E = C(m_1, \lambda_1)$ and F the d copies of $C(m_2, \lambda_2)$. Here m_1, m_2 are two integers and $0 < \lambda_i < \frac{1}{m_i}, i = 1, 2$. By induction and (1.5), (7.8)-(7.9) of Falconer [4], we have

$$H\text{-}\dim(F) = B\text{-}\dim(F) = \frac{d \ \log m_2}{-\log \lambda_2}.$$

Theorem 5 implies that

$$\mathcal{R}\text{-}H\text{-}\dim(E \times F) = \frac{2 \ \log m_1}{-\log \lambda_1} + \frac{d \ \log m_2}{-\lambda_2}. \tag{9}$$

As in Taylor and Watson [8], we assume that

$$\lambda_1 = {m_1}^{-\frac{1}{a_1}} \text{ and } \lambda_2 = {m_2}^{-\frac{1}{a_2}}$$

for some $0 < a_1, a_2 < 1$. Then (9) becomes

$$\mathcal{R}\text{-}H\text{-}\dim(E \times F) = 2 \ a_1 + d \ a_2.$$

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(A similar result was also obtained by Taylor and Watson for the case $m_1 = m_2 = 2$.) Under the assumption that $0 < a_2 < 1-\frac{2}{d(\alpha-1)}$, we restate Theorem 6 as follows: If $0 < a_1 < \frac{1}{2}[d(1-a_2)-\frac{2}{\alpha-1}]$, then $E\times F$ is $\mathbb{G}$-polar, and if $a_1 > \frac{1}{2}[d(1-a_2)-\frac{2}{\alpha-1}]$, then it is not $\mathbb{G}$-polar.

Remark. Assume that $d = \frac{2}{\alpha-1}$. Then the critical logarithmic Hausdorff dimension for $\mathbb{H}$-polarity is $\frac{1}{\alpha-1}$, and it is equal to the critical restricted logarithmic Hausdorff dimension of $\mathbb{G}$-polarity (see,e.g., Dynkin [3], Sheu [7]). Using these facts and the remark to Theorem 5, we have results similar to Theorem 6.

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