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The correct diameter of trivalent Cayley graphs ☆

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Abstract

Let n be a positive integer with $n \geqslant 2$. The trivalent Cayley interconnection network, denoted by $TCIN(n)$, is proposed by Vadapalli and Srimani (1995). Later, Vadapalli and Srimani (1996) claimed that the diameter of $TCIN(n)$ is $2n-1$. In this paper, we argue that the above claim is not correct. Instead, we show that the diameter of $TCIN(n)$ is $2n-1$ only for $n=2$ and $2n-2$ for all other cases. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Most of the graph and interconnection network definitions used in this paper are standard (see, e.g., [1]). Let $G=(V,E)$ be a finite, undirected graph. Let u and v be two vertices of G. The *distance* between u and v, denoted by $d(u,v)$, is the length of the shortest path between them. The *diameter* of G, denoted by $D(G)$, is the maximum distance between every two vertices in G. Let n be a positive integer with $n \geqslant 2$. $TCIN(n)$ is the trivalent Cayley graph, proposed by Vadapalli and Srimani [2]. Each node corresponds to a circular permutation in lexicographic order of n symbols, $t_1, t_2, \ldots, t_n$, complemented or uncomplemented. Each edge is of the type $(v, \delta(v))$, where $\delta \in \{g, f, f^{-1}\}$, defined in the following way:

$$f\left(t_k^* t_{k+1}^* \ldots t_n^* t_1^* t_2^* \ldots t_{k-1}^*\right) = t_{k+1}^* t_{k+2}^* \ldots t_n^* t_1^* t_2^* \ldots t_{k-1}^* \bar{t}_k^*,$$

$$f^{-1}\left(t_k^* t_{k+1}^* \ldots t_n^* t_1^* t_2^* \ldots t_{k-1}^*\right) = \bar{t}_{k-1}^* t_k^* \ldots t_n^* t_1^* t_2^* \ldots t_{k-2}^*,$$

$$g\left(t_k^* t_{k+1}^* \ldots t_n^* t_1^* t_2^* \ldots t_{k-1}^*\right) = t_k^* t_{k+1}^* \ldots t_n^* t_1^* t_2^* \ldots \bar{t}_{k-1}^*,$$

where t_k, $1 \leqslant k \leqslant n$, denotes the kth symbol in the set of n symbols. The symbol t_k^* denotes either t_k or $\bar{t}_k$. We use English alphabets for symbols; thus for $n=4$, $t_1=a$, $t_2=b$, $t_3=c$ and $t_4=d$.

The graph $TCIN(n)$ is the trivalent Cayley graph with $n2^n$ vertices. Due to the nice structure of $TCIN(n)$, many studies have been on the investigation of its topological properties [2,3]. Vadapalli and Srimani [3] claimed that $D(TCIN(n))$ is $2n-1$. However, this result is not true except when $n=2$. For example, we consider the graph $TCIN(3)$ shown in Fig. 1 of

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Vadapalli and Srimani [3]. Since $TCIN(3)$ is vertex transitive, we can compute $D(TCIN(3))$ using the breadth-first search rooted at vertex abc. Obviously $D(TCIN(3))$ is 4. Thus the result obtained in [3] is incorrect. In this paper, we will show that $D(TCIN(n))$ is $2n-1$ only for $n=2$ and $2n-2$ for all other cases.

2. Diameter of $TCIN(n)$

Let I denote the *identity node* $t_1t_2\ldots t_n$ in $TCIN(n)$. For any node s of $TCIN(n)$, we use $d(s)$ to denote the distance between s and I. Since $TCIN(n)$ is vertex transitive,

$$D\big(TCIN(n)\big)=\max\big\{d(s)\mid s\in TCIN(n)\big\}.$$

As noted in the above section, $D(TCIN(2))=3$. In the following, we adopt all the notations that are used in [3] and assume that $n\geqslant 3$. In [3], we have

$$d(s)=\min\big\{d_L(s),\ d_R(s)\big\}.$$

Let $A(n)$, $B(n)$, $C(n)$, and $D(n)$ be the partition of a vertex set of $TCIN(n)$ where

$$A(n)=\big\{s\in V(TCIN(n))\mid j_L+m_1<k \text{ and } j_R+m_2\leqslant n\big\};$$
$$B(n)=\big\{s\in V(TCIN(n))\mid j_L+m_1<k \text{ and } j_R+m_2>n\big\};$$
$$C(n)=\big\{s\in V(TCIN(n))\mid j_L+m_1\geqslant k \text{ and } j_R+m_2\leqslant n\big\};$$
$$D(n)=\big\{s\in V(TCIN(n))\mid j_L+m_1\geqslant k \text{ and } j_R+m_2>n\big\}.$$

Lemma 1. *For all $s\in A(n)$, $d(s)\leqslant 2n-2$. Moreover, $d(s)=2n-2$ if and only if n is even and $s=\bar t_{n/2+1}\bar t_{n/2+2}\ldots\bar t_n\bar t_1\bar t_2\ldots\bar t_{n/2}$.*

Proof. Since $d(s)=\min\{d_L(s),d_R(s)\}$,

$$\begin{aligned}d(s)&=\min\big\{2(n-1-m_1)+c_1-c_2,\\&\qquad 2(n-1-m_2)+c_2-c_1\big\}\\&=2n-2-\max\big\{c_2-c_1+2m_1,\\&\qquad c_1-c_2+2m_2\big\}.\end{aligned}$$

Since $|c_1-c_2|\geqslant 0$, $m_1\geqslant 0$, and $m_2\geqslant 0$, $d(s)\leqslant 2n-2$. Moreover, $d(s)=2n-2$ if and only if $c_1=c_2$ and $m_1=m_2=0$. In other words, n is even and $s=\bar t_{n/2+1}\bar t_{n/2+2}\ldots\bar t_n\bar t_1\bar t_2\ldots\bar t_{n/2}$. □

Lemma 2. *For all $s\in B(n)$, $d(s)\leqslant 2n-2$. Moreover, $d(s)=2n-2$ if and only if n is odd and $s=\bar t_{(n+1)/2+1}\bar t_{(n+1)/2+2}\ldots\bar t_n\bar t_1\bar t_2\ldots\bar t_{(n+1)/2-1}t_{(n+1)/2}$.*

Proof. Since $s\in B(n)$, $m_2\geqslant 1$. Since $d(s)=\min\{d_L(s),d_R(s)\}$,

$$\begin{aligned}d(s)&=\min\big\{2(n-1-m_1)+c_1-c_2,\\&\qquad 2(n-m_2)+c_2-c_1\big\}\\&=2n-2-\max\big\{c_2-c_1+2m_1,\\&\qquad c_1-c_2+2m_2-2\big\}.\end{aligned}$$

Since $|c_1-c_2|\geqslant 0$, $m_1\geqslant 0$, and $m_2\geqslant 1$, $d(s)\leqslant 2n-2$. Moreover, $d(s)=2n-2$ if and only if $c_1=c_2$, $m_1=0$ and $m_2=1$. In other words, n is odd and $s=\bar t_{(n+1)/2+1}\bar t_{(n+1)/2+2}\ldots\bar t_n\bar t_1\bar t_2\ldots\bar t_{(n+1)/2-1}t_{(n+1)/2}$. □

Lemma 3. *For all $s=a_1a_2\ldots a_n\in C(n)$, $d(s)\leqslant 2n-2$. Moreover, $d(s)=2n-2$ if and only if $s=\bar t_{(n-1)/2+1}\bar t_{(n-1)/2+2}\ldots\bar t_{n-1}t_n\bar t_1\bar t_2\ldots\bar t_{(n-1)/2}$ where n is an odd integer or $s=t_1\bar t_2\bar t_3$, or $\bar t_1t_2\bar t_3$ for $n=3$; $s=\bar t_1t_2\bar t_3t_4$, $t_1\bar t_2t_3\bar t_4$, or $t_1\bar t_2\bar t_3t_4$ for $n=4$; and $s=t_1\bar t_2t_3\bar t_4t_5$ for $n=5$.*

Proof. Since $d(s)=\min\{d_L(s),d_R(s)\}$,

$$\begin{aligned}d(s)&=\min\big\{2\cdot(n-m_1)+c_1-c_2,\\&\qquad 2(n-1-m_2)+c_2-c_1\big\}\\&=2n-2\\&\quad-\max\big\{c_2-c_1+2m_1-2,\ c_1-c_2+2m_2\big\}.\end{aligned}$$

Case 1. $a_1=t_1^*$. Hence $m_1=0$ and $c_1=0$. Thus $d(s)=2n-2-\max\{c_2-2,2m_2-c_2\}$. Since $c_2\geqslant 0$ and $m_2\geqslant 0$, $d(s)\leqslant 2n-2$. Moreover, $d(s)=2n-2$ if and only if $c_2=2$ and $m_2=1$. In other words, $s=t_1\bar t_2\bar t_3$, $\bar t_1t_2\bar t_3$ if $n=3$; $s=\bar t_1t_2\bar t_3t_4$, $t_1\bar t_2t_3\bar t_4$, $t_1\bar t_2\bar t_3t_4$ if $n=4$; and $s=t_1\bar t_2t_3\bar t_4t_5$ if $n=5$.

Case 2. $a_1\neq t_1^*$. Hence $m_1\geqslant 1$. Thus $d(s)=2n-2-\max\{c_2-c_1+2m_1-2,\ c_1-c_2+2m_2\}\leqslant 2n-2$ because $|c_1-c_2|\geqslant 0$, $m_1\geqslant 1$, and $m_2\geqslant 0$. Therefore, $d(s)=2n-2$ if and only if $c_1=c_2$, $m_1=1$, and $m_2=0$. In other words, n is odd and $s=\bar t_{(n-1)/2+1}\bar t_{(n-1)/2+2}\ldots\bar t_{n-1}t_n\bar t_1\bar t_2\ldots\bar t_{(n-1)/2}$. □

Lemma 4. *For all $s=a_1a_2\ldots a_n\in D(n)$, $d(s)\leqslant 2n-2$. Moreover, $d(s)=2n-2$ if and only if $s=$*

$\bar{t}_{n/2+1}\bar{t}_{n/2+2}\dots\bar{t}_{n-1}t_n\bar{t}_1\bar{t}_2\dots\bar{t}_{n/2-1}t_{n/2}$ *where* n *is an even integer or* $s=\bar{t}_1\bar{t}_2t_3$ *for* $n=3$; $s=\bar{t}_1\bar{t}_2t_3t_4$ *for* $n=4$; $s=\bar{t}_1t_2\bar{t}_3t_4t_5$ *or* $t_1\bar{t}_2\bar{t}_3t_4t_5$ *for* $n=5$; *and* $s=t_1\bar{t}_2t_3\bar{t}_4t_5t_6$ *for* $n=6$.

Proof. Since $s\in D(n)$, $m_2\geqslant 1$. Since $d(s)=\min\{d_L(s),d_R(s)\}$,

$$\begin{aligned}d(s)&=\min\bigl\{2(n-m_1)+c_1-c_2,\\&\qquad 2(n-m_2)+c_2-c_1\bigr\}\\&=2n-2-\max\bigl\{c_2-c_1+2m_1-2,\\&\qquad c_1-c_2+2m_2-2\bigr\}.\end{aligned}$$

Case 1. $a_1=t_1^*$. Hence $m_1=0$ and $c_1=0$. Thus $d(s)=2n-2-\max\{c_2-2,2m_2-c_2-2\}$. Suppose that $m_2\geqslant 2$. Obviously, $d(s)\leqslant 2n-2$. Suppose that $m_2=1$. By definition, $c_2=n-1$. Since $n\geqslant 3$, $d(s)\leqslant 2n-2$. Moreover, $d(s)=2n-2$ if and only if $c_2=2$, $m_2=2$ or $m_2=1$, $c_2=2$. In other words, $s=\bar{t}_1\bar{t}_2t_3$ if $n=3$; $s=\bar{t}_1\bar{t}_2t_3t_4$ if $n=4$; $s=\bar{t}_1t_2\bar{t}_3t_4t_5$, $t_1\bar{t}_2\bar{t}_3t_4t_5$ if $n=5$; and $s=t_1\bar{t}_2t_3\bar{t}_4t_5t_6$ if $n=6$.

Case 2. $a_1\neq t_1^*$. Hence $m_1\geqslant 1$. Thus $d(s)=2n-2-\max\{c_2-c_1+2m_1-2,\ c_1-c_2+2m_2-2\}\leqslant 2n-2$ because $|c_1-c_2|\geqslant 0$, $m_1\geqslant 1$, and $m_2\geqslant 1$. Therefore $d(s)=2n-2$ if and only if $c_1=c_2$ and $m_1=m_2=1$. In other words, n is even and $s=\bar{t}_{n/2+1}\bar{t}_{n/2+2}\dots\bar{t}_{n-1}t_n\bar{t}_1\bar{t}_2\dots\bar{t}_{n/2-1}t_{n/2}$. □

From the above discussion, we have the following theorem.

Theorem 1. $D(TCIN(n))=2n-1$ *if* $n=2$ *and* $2n-2$ *if* $n\geqslant 3$.

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