ELSEVIER

Statistics & Probability Letters 45 (1999) 215–224

STATISTICS & PROBABILITY LETTERS

www.elsevier.nl/locate/stapro

A note on Bayesian estimation of process capability indices

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Received September 1998; received in revised form January 1999

Abstract

Process capability indices are useful for assessing the capability of manufacturing processes. Most traditional methods are obtained from the frequentist point of view. We view the problem from the Bayes and empirical Bayes approaches by using non-informative and conjugate priors, respectively.

MSC: 62N10; 62A15

Keywords: Process capability indices; Bayesian approach; Bayes estimators; Non-informative prior; Conjugate prior

1. Introduction

Process capability indices (PCIs), as a process performance measure, have become very popular in assessing the capability of manufacturing processes in practice during the past decade. More and more efforts have been devoted to studies and applications of PCIs. For example, Rado (1989) demonstrated how Imprimis Technology, Inc. used the PCIs for program planning and growth to enhance product development. The C_p and C_{pk} indices have been used in Japan and in the US automotive industry such as Ford Motor Company (see Kane, 1986a, b). For more information on PCIs, see Kotz and Johnson (1993), Kotz et al. (1993), and the references cited therein.

The usual practice is to estimate these PCIs from data and then judge the capability of the process by these estimates. Commonly used estimators are reviewed in Section 2. Most studies on PCIs are based on the traditional frequentist point of view. The main objective of this note is to provide both point and interval estimators of some popular PCIs from the Bayesian point of view. We believe this effort is well justified since Bayesian estimation has become one of popular approaches in estimation nowadays and resulting Bayes estimators in general have good theoretical properties, such as admissibility (Bernardo and Smith, 1993). In addition, the Bayesian approach has one great advantage over the traditional frequentist approach—the posterior distribution is very easy to derive and then credible intervals, which is the Bayesian analogue of

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0167-7152/99/$ - see front matter
PII: S0167-7152(99)00061-9

the classical confidence interval, can be easily obtained either by theoretical derivation or by Monte Carlo methods (Tanner, 1993). A simple estimate of the index is not very useful in making reasonable decision on the capability of a process. An interval estimate approach is more appropriate.

This paper is organized as follows. We give a brief review on the most popular PCIs, C_p, C_{pk}, and C_{pm} in Section 2. In Section 3, we derive Bayes estimators for C_p^2, C_{pm}^2 (with the process mean μ being the target value T) and C_{pk}^2 (with the process mean μ being the middle point m of the two specification limits), with respect to some chosen priors. In Sections 3.1 and 3.2, we consider the non-informative and the Gamma prior, respectively, and derive Bayes estimators and credible intervals for each prior. For the Gamma prior, the maximum-likelihood method in empirical Bayes approach is adopted for choosing the parameters in the prior. The derivation is given in the appendix. In Section 4, we propose a Bayesian procedure based on the credible intervals derived in Section 3. An example is given to demonstrate the application of the proposed Bayesian procedure. Finally, we conclude this note by a brief summary in Section 5.

Throughout this paper, it is assumed that the process measurements are independently and identically distributed from a normal distribution. In other words, the process is under statistical control. We remark that estimation of PCIs is meaningful only when the process is under statistical control.

2. A review of some process capability indices

The most popular PCIs are C_p, C_{pk}, C_{pm}, and C_{pmk}. The C_p index is defined as

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma},$$

where LSL and USL are the lower and upper specification limits, respectively, and σ is the process standard deviation. Note that C_p does not depend on the process mean. The C_{pk} is then introduced to reflect the impact of μ on the process capability indices. The C_{pk} index is defined as

$$C_{pk} = \min\left\{\frac{\text{USL} - \mu}{3\sigma}, \frac{\mu - \text{LSL}}{3\sigma}\right\}.$$

The C_{pm} index was introduced by Chan et al. (1988). This index takes into account the influence of the departure of the process mean μ from the process target T. The C_{pm} is defined as

$$C_{pm} = \frac{\text{USL} - \text{LSL}}{6\sqrt{\sigma^2 + (\mu - T)^2}}.$$

Combining the three indices, C_p, C_{pk}, and C_{pm}, Pearn et al. (1992) proposed the C_{pmk} index. This index is defined as

$$C_{pmk} = \min\left\{\frac{\text{USL} - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - \text{LSL}}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\}.$$

The natural and most commonly used estimators of C_p, C_{pk}, C_{pm}, and C_{pmk} are

$$\hat{C}_p = \frac{\text{USL} - \text{LSL}}{6s},$$

$$\hat{C}_{pk} = \min\left\{\frac{\text{USL} - \bar{x}}{3s}, \frac{\bar{x} - \text{LSL}}{3s}\right\},$$

$$\hat{C}_{pm} = \frac{\text{USL} - \text{LSL}}{6\sqrt{s^2 + (\bar{x} - T)^2}}$$

and

$$\hat{C}_{\mathrm{pmk}} = \min\left\{ \frac{\mathrm{USL} - \bar{x}}{3\sqrt{s^2 + (\bar{x} - T)^2}}, \frac{\bar{x} - \mathrm{LSL}}{3\sqrt{s^2 + (\bar{x} - T)^2}} \right\},$$

respectively, where $\bar{x}$ is the sample mean and s is the sample standard deviation.

3. Bayesian estimation for some PCIs

In this section, we derive the Bayes estimators for C_{p}^2, C_{pm}^2 with $\mu = T$, and C_{pk}^2 with $\mu = m$ with respect to some priors. Two prior distributions are considered. The first prior is the non-informative prior, and the second prior is Gamma(a, b). Reasons for choosing these priors are given at the beginning of the following two subsections, respectively. For the Gamma prior, the maximum-likelihood method in the empirical Bayes approach is adopted for choosing the parameters a and b in the prior.

Recall that the natural (most common) estimator of C_{p} is $\hat{C}_{\mathrm{p}} = (\mathrm{USL} - \mathrm{LSL})/(6s)$. Assuming that the process measurements follow a $\mathrm{N}(\mu, \sigma^2)$, Cheng and Spiring (1989) derived that the probability density function (p.d.f.) for $\hat{C}_{\mathrm{p}}$ is

$$f(y|C_{\mathrm{p}}) = 2\left(\Gamma\left(\frac{n-1}{2}\right)\right)^{-1} \left(\frac{(n-1)C_{\mathrm{p}}^2}{2}\right)^{(n-1)/2} y^{-n} \exp\left[-\frac{(n-1)C_{\mathrm{p}}^2}{2y^2}\right] \quad \text{for } 0 < y < \infty.$$

Let $W = \hat{C}_{\mathrm{p}}^2$, then the p.d.f. of W is

$$f(w|C_{\mathrm{p}}) = \left(\Gamma\left(\frac{n-1}{2}\right)\right)^{-1} \left(\frac{(n-1)C_{\mathrm{p}}^2}{2}\right)^{(n-1)/2} w^{-((n+1)/2)} \exp\left[-\frac{(n-1)C_{\mathrm{p}}^2}{2w}\right] \quad \text{for } 0 < w < \infty.$$

That is, $\hat{C}_{\mathrm{p}}^2$ follows an Inverse Gamma(ρ, τ) with parameters $\rho = (n-1)/2$ and $\tau = ((n-1)C_{\mathrm{p}}^2/2)^{-1}$.

Set the parameter $\theta = C_{\mathrm{p}}^2$. Them the likelihood function $L(\theta|w)$ of θ,

$$L(\theta|w) = \left(\Gamma\left(\frac{n-1}{2}\right)\right)^{-1} \left(\frac{(n-1)}{2}\right)^{(n-1)/2} \theta^{(n-1)/2} w^{-((n+1)/2)} \exp\left[-\frac{(n-1)\theta}{2w}\right].$$

We now derive the posterior distributions for θ under two different prior distributions.

3.1. Non-informative prior

For the choice of the prior, in this subsection, we consider the prior $\pi(\theta) = 1/\theta$, for $0 < \theta < \infty$. There are two reasons for choosing this prior. The first one is that the above prior can maximize the difference between the information (entropy) of the parameter provided by the prior and posterior distributions. In other words, the above prior allows the prior to provide information about the parameter as little as possible (see Bernardo and Smith, 1993). This prior is usually referred as a reference prior. The second reason is that with the above prior, the $p \times 100\%$ credible interval has coverage probability p up to the second order (in contrast to the first order for any other priors) in the frequentist sense (Welch and Peers, 1963). In other words, the credible interval obtained from the above prior has a more precise coverage probability than that obtained from any other priors.

With this non-informative prior, we have the joint p.d.f. of (θ, w)

$$f(\theta,w) = f(w|\theta) \times \pi(\theta)$$
$$= (\Gamma(\alpha))^{-1}\left(\frac{n-1}{2}\right)^{(n-1)/2}\theta^{(\alpha-1)}w^{-((n+1)/2)}\exp\left(-\frac{\theta}{\beta}\right), \quad (1)$$

where $\alpha=(n-1)/2$ and $\beta=2w/(n-1)=2\hat{C}_{\mathrm{p}}^2/(n-1)$.

Hence the posterior distribution of θ given w is

$$f(\theta|w) = \frac{\theta^{(\alpha-1)}\exp(-\theta/\beta)}{\Gamma(\alpha)\beta^{\alpha}}.$$

That is, the posterior distribution of θ given w is a Gamma(α,β).

So, the posterior mean for $\theta = C_{\mathrm{p}}^2$ is

$$E(\theta|\hat{C}_{\mathrm{p}}^2) = \alpha\beta = \frac{n-1}{2}\cdot\frac{2\hat{C}_{\mathrm{p}}^2}{n-1} = \hat{C}_{\mathrm{p}}^2.$$

Therefore, $\hat{C}_{\mathrm{p}}^2$ is a Bayes estimator of C_{p}^2, and we have a nice Bayesian interpretation for the estimator $\hat{C}_{\mathrm{p}}^2$ of C_{p}^2.

In addition, it can be shown that the mode of $f(\theta|w)$ is $(\alpha-1)\beta=((n-3)/(n-1))\hat{C}_{\mathrm{p}}^2$, which is the Bayes estimator of C_{p}^2 is the sense of extended zero-one loss.

Next, consider C_{pm} under $\mu=T$ and C_{pk} under $\mu=m$. Recall that these two indices (with the special restrictions on μ) are both reduced to C_{p}. Chan et al. (1988) considered $\hat{C}_{\mathrm{pm}}=(\mathrm{USL}-\mathrm{LSL})/(6\hat{\sigma}')$, where $\hat{\sigma}'=((1/(n-1))\sum_{i=1}^{n}(x_i-T)^2)^{1/2}$. The p.d.f. of $\hat{C}_{\mathrm{pm}}$ when $\mu=T$, given by Theorem 8 in Chan et al. (1988), is

$$f(y|C_{\mathrm{pm}}) = 2\left(\Gamma\left(\frac{n}{2}\right)\right)^{-1}\left(\frac{(n-1)C_{\mathrm{pm}}^2}{2}\right)^{n/2} y^{-(n+1)}\exp\left(-\frac{(n-1)C_{\mathrm{pm}}^2}{2y^2}\right) \quad \text{for } 0<y<\infty.$$

By the same technique as that for C_{p}, we can obtain that the posterior p.d.f. of $\lambda=C_{\mathrm{pm}}^2$ given $\hat{C}_{\mathrm{pm}}$ is

$$f(\lambda|\hat{C}_{\mathrm{pm}}^2) = \frac{\lambda^{\tilde{\alpha}-1}\exp(-\lambda/\tilde{\beta})}{\Gamma(\tilde{\alpha})\tilde{\beta}^{\tilde{\alpha}}},$$

which is a Gamma$(\tilde{\alpha},\tilde{\beta})$ distribution with $\tilde{\alpha}=n/2$ and $\tilde{\beta}=2\hat{C}_{\mathrm{pm}}^2/(n-1)$. So, the posterior mean for C_{pm}^2 is

$$E(C_{\mathrm{pm}}^2|\hat{C}_{\mathrm{pm}}^2) = \tilde{\alpha}\tilde{\beta} = \frac{n}{2}\cdot\frac{2\hat{C}_{\mathrm{pm}}^2}{n-1} = \frac{n}{n-1}\hat{C}_{\mathrm{pm}}^2,$$

and thus a Bayes estimator for C_{pm}^2 is $(n/(n-1))\hat{C}_{\mathrm{pm}}^2$.

However, it seems more natural to consider the estimator $\hat{\sigma}''=((1/n)\sum_{i=1}^{n}(x_i-T)^2)^{1/2}$, since, under the assumption $\mu=T, \hat{\sigma}''^{\,2}$ is both an unbiased estimator and the maximum-likelihood estimator (MLE) of σ^2. Then, in this case, it is easily seen that the posterior mean of C_{pm}^2 is exactly $\hat{C}_{\mathrm{pm}}^2$. Also, by the same technique as that for C_{p}, the posterior mode for C_{pm}^2 is $(n/2-1)\cdot(2\hat{C}_{\mathrm{pm}}^2/n)=((n-2)/n)\hat{C}_{\mathrm{pm}}^2$. Likewise, if we let $\hat{C}_{\mathrm{pk}}=(\mathrm{USL}-\mathrm{LSL})/(6\hat{\sigma}^*)$ with $\hat{\sigma}^*=((1/n)\sum_{i=1}^{n}(x_i-m)^2)^{1/2}$, we can also obtain that $\hat{C}_{\mathrm{pk}}$ is the posterior mean of C_{pk}^2 and that the posterior mode is $((n-2)/n)\hat{C}_{\mathrm{pk}}^2$.

Next, we consider the interval estimation of these PCIs. Recall that the posterior distribution of $\theta=C_{\mathrm{p}}^2$ given $\hat{C}_{\mathrm{p}}$ is a Gamma$((n-1)/2,\beta)$, with $\beta=2\hat{C}_{\mathrm{p}}^2/(n-1)$. Then $2\theta/\beta$ is a χ^2 distribution with degrees of freedom

Table 1
Summary of the point and interval estimators for PCIs under non-informative prior

Index	Posterior mean	Postetrior mode	Credible interval
C_{p}^2	$\hat{C}_{\mathrm{p}}^2$	$\frac{n-3}{n-1}\hat{C}_{\mathrm{p}}^2$	$\left[\frac{\hat{C}_{\mathrm{p}}^2}{n-1}\chi^2_{n-1,1-p},\infty\right)$
C_{pm}^2	$\hat{C}_{\mathrm{pm}}^2$	$\frac{n-2}{n}\hat{C}_{\mathrm{pm}}^2$	$\left[\frac{\hat{C}_{\mathrm{pm}}^2}{n}\chi^2_{n,1-p},\infty\right)$
C_{pk}^2	$\hat{C}_{\mathrm{pk}}^2$	$\frac{n-2}{n}\hat{C}_{\mathrm{pk}}^2$	$\left[\frac{\hat{C}_{\mathrm{pk}}^2}{n}\chi^2_{n,1-p},\infty\right)$

$n-1$. Denote the $(1-p)\times 100$th percentile of a χ^2 distribution with degrees of freedom $n-1$ by $\chi^2_{n-1,1-p}$. Then a useful $p\times 100\%$ credible interval of C_{p}^2 is $[\theta_{\mathrm{p}},\infty)$, where $\theta_{\mathrm{p}}=(\beta/2)\chi^2_{n-1,1-p}=(\hat{C}_{\mathrm{p}}^2/(n-1))\chi^2_{n-1,1-p}$. Similarly, $[(\hat{C}_{\mathrm{pm}}^2/n)\chi^2_{n,1-p},\infty)$ is the corresponding $p\times 100\%$ credible interval of C_{pm}^2, when $\mu=T$; and $[(\hat{C}_{\mathrm{pk}}^2/n)\chi^2_{n,1-p},\infty)$ is the interval for C_{pk}^2, when $\mu=m$.

We summarize the results derived above in Table 1 for quick reference. Note that in this table $\hat{C}_{\mathrm{p}}=(\mathrm{USL}-\mathrm{LSL})/(6s)$, $\hat{C}_{\mathrm{pm}}=(\mathrm{USL}-\mathrm{LSL})/(6\hat{\sigma}'')$, and $\hat{C}_{\mathrm{pk}}=(\mathrm{USL}-\mathrm{LSL})/(6\hat{\sigma}^*)$.

3.2. Gamma prior

In the Bayesian literature, in addition to the non-informative prior, the conjugate prior is another important prior (Bernardo and Smith, 1993). The most important reason for using the conjugate prior is that, with the conjugate prior, the prior and posterior are in the same distribution family. That is, the prior and posterior distribution functions have the same mathematical form. Since $\hat{C}_{\mathrm{p}}^2$ follows an Inverse Gamma distribution, we know that the conjugate prior must be a Gamma prior. Assume that θ is distributed as Gamma(a,b) with p.d.f.

$$\pi(\theta)=(\Gamma(a)b^a)^{-1}\theta^{a-1}\exp\left(-\frac{\theta}{b}\right)\quad \text{for } 0<\theta<\infty,\ 0<a<\infty,\ 0<b<\infty.$$

Then, the joint p.d.f. for $\theta=C_{\mathrm{p}}^2$ and W is

$$f(\theta,w)=\frac{((n-1)/2)^{(n-1)/2}}{\Gamma(\alpha)\Gamma(a)b^a w^{\alpha+1}}\theta^{\alpha+a-1}\exp\left[-\theta\left(\frac{1}{\beta}+\frac{1}{b}\right)\right].$$

Thus, the posterior distribution becomes

$$f(\theta\,|\,w)=\frac{\theta^{a'-1}\exp(-\theta/b')}{\Gamma(a')b'^{a'}},$$

where $a'=\alpha+a=(n-1)/2+a$ and $b'=(1/\beta+1/b)^{-1}=((n-1)/2w+1/b)^{-1}$.

Note that $f(\theta|w)$ is a Gamma(a',b') density. Therefore, the posterior mean for C_{p}^2 is

$$E(C_{\mathrm{p}}^2|\hat{C}_{\mathrm{p}}^2)=a'b'=\left(\frac{n-1}{2}+a\right)\left(\frac{n-1}{2\hat{C}_{\mathrm{p}}^2}+\frac{1}{b}\right)^{-1}.$$

And the posterior mode for C_p^2 is

$$(a'-1)b' = \left(\frac{n-3}{2}+a\right)\left(\frac{n-1}{2\hat{C}_p^2}+\frac{1}{b}\right)^{-1}.$$

The parameters a and b in the prior distribution can be given either subjectively or objectively. To obtain the hyperparameters a and b objectively, we may adopt the maximum-likelihood method in the empirical Bayes approach (Bernardo and Smith, 1993). Consider for any fixed a and b,

$$\begin{aligned} f(w\,|\,a,b) &= \int_0^\infty f(\theta, w|a,b)\,\mathrm{d}\theta \\ &= \int_0^\infty \frac{((n-1)/2)^{(n-1)/2}}{\Gamma(\alpha)\Gamma(a)b^a w^{\alpha+1}}\theta^{\alpha+a-1}\exp\left[-\theta\left(\frac{1}{\beta}+\frac{1}{b}\right)\right]\mathrm{d}\theta \\ &= \frac{((n-1)/2)^{(n-1)/2}}{\Gamma(\alpha)\Gamma(a)b^a w^{\alpha+1}}\Gamma(a')b'^{a'}. \end{aligned} \tag{2}$$

If a is given, then, by maximizing (2) when w is fixed, we obtain that the maximum-likelihood estimator of b is $\hat{b} = \hat{C}_p^2/a$. The derivation is given in the appendix.

So, when $b = \hat{b}$, the posterior mean for C_p^2 is

$$\left(\frac{n-1}{2}+a\right)\left(\frac{n-1}{2\hat{C}_p^2}+\frac{1}{\hat{b}}\right)^{-1} = \hat{C}_p^2.$$

This shows that $\hat{C}_p^2$ is the Bayes estimator of C_p^2 in the sense of the empirical Bayes.

Also, the posterior mode for C_p^2 is

$$\left(\frac{n-3}{2}+a\right)\left(\frac{n-1}{2\hat{C}_p^2}+\frac{1}{\hat{b}}\right)^{-1} = \left(\frac{n-3}{2}+a\right)\left(\frac{n-1}{2}+a\right)^{-1}\hat{C}_p^2.$$

For Gamma(a,b) prior, again consider the estimator $\hat{\sigma}'' = ((1/n)\sum_{i=1}^n (x_i-T)^2)^{1/2}$ for $\hat{C}_{pm}$ under $\mu = T$, and the estimator $\hat{\sigma}^* = ((1/n)\sum_{i=1}^n (x_i-m)^2)^{1/2}$ for $\hat{C}_{pk}$ under $\mu = m$. Then, in these cases, it can be easily seen that the posterior distribution of $\lambda = C_{pm}^2$ is a gamma distribution with p.d.f.

$$f(\lambda|w) = \frac{\lambda^{a''-1}\exp(-\lambda/b'')}{\Gamma(a'')b''^{a''}},$$

where $a'' = \tilde{\alpha}+a = n/2+a$ and $b'' = (n/(2\hat{C}_{pm}^2)+1/b)^{-1}$. Then we obtain that the posterior mean for C_{pm}^2 is $a''b'' = (n/2+a)(n/(2\hat{C}_{pm}^2)+1/b)^{-1}$ and the posterior mode for C_{pm}^2 is $(a''-1)b'' = ((n-2)/2+a)(n/(2\hat{C}_{pm}^2)+1/b)^{-1}$. Similarly, the posterior mean for C_{pk}^2 is $(n/2+a)(n/(2\hat{C}_{pk}^2)+1/b)^{-1}$ and the posterior mode for C_{pk}^2 is $((n-2)/2+a)(n/(2\hat{C}_{pk}^2)+1/b)^{-1}$.

Assume a is given. To estimate b empirically for C_{pm} under $\mu = T$ and C_{pk} under $\mu = m$, we adopt the maximum-likelihood method. Similar results as that for C_p hold for these two cases. That is, for C_{pm}^2, $\hat{b} = \hat{C}_{pm}^2/a$, the posterior mean is $\hat{C}_{pm}^2$, and the posterior mode is $((n-2)/2+a)(n/2+a)^{-1}\hat{C}_{pm}^2$ $(=((n-2+2a)/(n+2a))\hat{C}_{pm}^2)$ in the sense of the empirical Bayes. Similarly, for C_{pk}^2, $\hat{b} = \hat{C}_{pk}^2/a$, the posterior mean is $\hat{C}_{pk}^2$, and the posterior mode is $((n-2)/2+a)(n/2+a)^{-1}\hat{C}_{pk}^2$ $(=((n-2+2a)/(n+2a))\hat{C}_{pk}^2)$ in the sense of the empirical Bayes.

Table 2
Summary of the point and interval estimators for PCIs under gamma prior when a is given and b is estimated by the maximum-likelihood method

Index	Posterior mean	Postetrior mode	Credible interval
C_{p}^2	$\hat{C}_{\mathrm{p}}^2$	$\frac{n-3+2a}{n-1+2a}\hat{C}_{\mathrm{p}}^2$	$\left[\frac{\hat{C}_{\mathrm{p}}^2}{n-1+2a}\chi^2_{n-1,1-p},\infty\right)$
C_{pm}^2	$\hat{C}_{\mathrm{pm}}^2$	$\frac{n-2+2a}{n+2a}\hat{C}_{\mathrm{pm}}^2$	$\left[\frac{\hat{C}_{\mathrm{pm}}^2}{n+2a}\chi^2_{n,1-p},\infty\right)$
C_{pk}^2	$\hat{C}_{\mathrm{pk}}^2$	$\frac{n-2+2a}{n+2a}\hat{C}_{\mathrm{pk}}^2$	$\left[\frac{\hat{C}_{\mathrm{pk}}^2}{n+2a}\chi^2_{n,1-p},\infty\right)$

Next, we consider the interval estimation. Again assume that a is given. If $2a$ is an integer, then a $p\times 100\%$ credible interval for C_{p}^2 is $[((n-1)/\hat{C}_{\mathrm{p}}^2+2/\hat{b})^{-1}\chi^2_{n-1+2a,1-p},\infty)$, which can be simplified to $[(\hat{C}_{\mathrm{p}}^2/(n-1+2a))\chi^2_{n-1+2a,1-p},\infty)$ when $\hat{b}=\hat{C}_{\mathrm{p}}^2/a$. For C_{pm}^2 and C_{pk}^2 under the special cases, the $p\times 100\%$ credible interval are $[(n/\hat{C}_{\mathrm{pm}}^2+2/\hat{b}))^{-1}\chi^2_{n+2a,1-p},\infty)$, and $[(n/\hat{C}_{\mathrm{pk}}^2+2/\hat{b})^{-1}\chi^2_{n+2a,1-p},\infty)$, respectively. These two intervals can also be simplified to $[(\hat{C}_{\mathrm{pm}}^2/(n+2a))\chi^2_{n+2a,1-p},\infty)$, when $\hat{b}=\hat{C}_{\mathrm{p}}^2/a$ and $[(\hat{C}_{\mathrm{pk}}^2/(n+2a))\chi^2_{n+2a,1-p},\infty)$, when $\hat{b}=\hat{C}_{\mathrm{pk}}^2/a$, respectively. If $2a$ is not an integer, then we may approximate $\chi^2_{n+2a,1-p}$ by interpolating values of $\chi^2_{n+\lceil 2a\rceil,1-p}$ and $\chi^2_{n+\lceil 2a\rceil+1,1-p}$, where $\lceil x\rceil$ denotes the largest integer less than or equal to x.

If both a and b need to be estimated, there is no explicit form for $\hat{a}$, the MLE of a. $\hat{a}$ can only be obtained numerically.

We now summarize the above results in Table 2. As in Table 1, here we use $\hat{C}_{\mathrm{p}}=(\mathrm{USL}-\mathrm{LSL})/(6s)$, $\hat{C}_{\mathrm{pm}}=(\mathrm{USL}-\mathrm{LSL})/(6\hat{\sigma}'')$, and $\hat{C}_{\mathrm{pk}}=(\mathrm{USL}-\mathrm{LSL})/(6\hat{\sigma}^*)$.

Note that Table 2 is reduced to Table 1 when $a=0$. This is not surprising since the non-informative prior considered in Section 3.1 is the limiting case of Gamma$(a,\hat{b})$ when a goes to 0.

When using these point and interval estimates in practice, all the quantities in Tables 1 and 2 should be square-rooted for better interpretation.

4. A Bayesian procedure and an example

In this section, we describe how to use the estimators described in the previous section in real-life applications. Point estimates can give some assessment on the process capability. However, as mentioned before, it is more appropriate to use interval estimates when it comes to determine whether the process is capable or not. With these interval estimates at hand, we now describe a Bayesian procedure in the following.

A $p\times 100\%$ credible interval means the posterior probability that the true PCI lies in this interval is p. Let p be a high probability, say, 0.95. Suppose for this particular process under consideration to be capable, the process index must reach at least a certain level C^*, say, 1.33. Now, from the process data, we compute the lower bound of the credible interval for the index (not for the squared index) and denote it by C. The Bayesian procedure is very simple — if $C>C^*$, then we say that the process is capable in a Bayesian sense.

We use the data given in Table 6–1 of Montgomery (1990, p. 207) to demonstrate this Bayesian procedure. This example is about a manufacturing process which produced piston rings for an automotive engine. The

Table 3
Point estimates of the three indices obtained by the posterior mean and posterior mode under the non-informative prior

Index	Estimate by posterior mean	Estimate by posterior mode
C_p	1.6551	1.6417
C_{pm}	1.6439	1.6307
C_{pk}	1.6162	1.6032

Table 4
The lower bound of the interval estimates of the three indices under the non-informative prior obtained with the posterior probability being 0.9, 0.95, 0.99, and 0.999

	Posterior probability p			
Index	0.9	0.95	0.99	0.999
C_p	1.5179	1.4810	1.4126	1.3374
C_{pm}	1.5082	1.4717	1.4040	1.3296
C_{pk}	1.4827	1.4468	1.3803	1.3071

Table 5
The lower bound of the credible intervals C of the three indices for various a and p

		a					
Index	p	0.01	0.1	1	10	50	100
C_p	0.9	1.5179	1.5180	1.5190	1.5279	1.5535	1.5708
	0.95	1.4810	1.4811	1.4824	1.4936	1.5257	1.5476
	0.99	1.4126	1.4128	1.4145	1.4299	1.4742	1.5045
	0.999	1.3374	1.3376	1.3340	1.3597	1.4172	1.4567
C_{pm}	0.9	1.5082	1.5083	1.5094	1.5195	1.5478	1.5668
	0.95	1.4717	1.4718	1.4732	1.4854	1.5203	1.5438
	0.99	1.4040	1.4042	1.4060	1.4223	1.4690	1.5008
	0.999	1.3296	1.3299	1.3322	1.3528	1.4124	1.4532
C_{pk}	0.9	1.4828	1.4829	1.4844	1.4972	1.5331	1.5564
	0.95	1.4468	1.4470	1.4487	1.4637	1.5058	1.5335
	0.99	1.3803	1.3805	1.3827	1.4015	1.4550	1.4909
	0.999	1.3072	1.3074	1.3100	1.3330	1.3989	1.4436

measurements are the inside diameter of the rings manufactured in this process. 125 measurements were taken from the process when the process was in control. The upper specification limit $\mathrm{USL}=74.05$ and the lower specification limit $\mathrm{LSL}=73.95$. The target value $T=74$. From the process data, we obtain that sample mean $\bar{x}=74.00118$ and the sample standard deviation $s=0.01006997$. $\hat{C}_p=(\mathrm{USL}-\mathrm{LSL})/(6s)=1.655086$, $\hat{C}_{pm}=(\mathrm{USL}-\mathrm{LSL})/(6\hat{\sigma}'')=1.643914$, and $\hat{C}_{pk}=(\mathrm{USL}-\mathrm{LSL})/(6\hat{\sigma}^*)=1.616159$.

Table 3 reports the point estimates and Table 4 reports the interval estimates under the non-informative prior with $p=0.9$, 0.95, 0.99, 0.999 for the piston ring example. Numbers given in Table 4 are the lower bound C of the credible interval for the indices (i.e., not squared). For $C^* = 1.33$, these C values indicate that the process is capable in the Bayesian sense, except for the two cases when the "confidence level" is very high (0.999). We also notice that these estimates are not much different for the three indices. This can be explained by the fact that the three values $\bar{x}, T$, and m are very close in this example.

To demonstrate the importance of the interval estimate, we now turn to a hypothetical example. Suppose that the index $\hat{C}_p = 1.4$, which is greater than the presumed level of capability 1.33. The C value obtained in this case is 1.2840, which is below 1.33. So we cannot conclude that the process is capable. Point estimate does not give us clue on how big the estimation error is, while a credible interval estimate can provide us a statement about the true index based on the posterior probability.

Now, under the Gamma prior, Table 5 gives the C values for various a and p. It is noticed from this table that the prior parameter a seems not affecting the C values much. Again, for $p = 0.9$, 0.95, 0.99, the conclusions are all the same—the process is capable in the Bayesian sense.

5. Summary

PCIs are getting more and more popular in the efforts of quality and productivity improvement. In this paper, we provide both point and interval estimators by the Bayesian approach. We derive the posterior distributions for C_p^2, C_{pm}^2 with $\mu = T$, and C_{pk}^2 with $\mu = m$ with respect to the two priors. We then derive the Bayes estimators, posterior mean and mode, for each of these PCIs. We remark that in the Gamma(a, b) prior with a fixed, the parameter b is estimated by the maximum-likelihood method in the empirical Bayes approach. It is found that these Bayes estimators either are the traditional estimators themselves or just differ from the traditional estimators by a constant multiplier that converges to 1 as the sample size goes to infinity. In addition, Bayesian credible interval estimate are obtained analytically. Based on these interval estimates, a simple Bayesian procedure for determining if the process is capable is proposed for practitioners to use.

For the special cases that we consider in this paper, interval estimators can also be easily derived from the frequentist approach. We do not intend to replace the frequentist approach on PCIs. We simply provide a Bayesian alternative. However, when the distribution of the PCI estimators are very complicated, as they often are for some favorable estimators, such as C_{pm} with no restriction on the process mean, then our approach becomes very valuable in obtaining an interval estimate of the index. Results for this particular problem is given in a subsequent paper.

Acknowledgements

The authors would like to express their gratitude to a referee for the valuable comments, which greatly improved the quality of the paper. This research work was partially supported by the National Science Council of the Republic of China. Grant No. NSC87-2118-M-009-004 and NSC87-2118-M-009-003.

Appendix

In this appendix, we derive the MLE of b by maximizing (2) given in Section 3.2. Let $f(b) = b^{-a}(1/\beta + 1/b)^{-(\alpha+a)} = e^{l(b)}$, where $l(b) = -a \log b - (\alpha + a)\log(1/\beta + 1/b)$. Then we have

$$\frac{dl(b)}{db} = -\frac{a}{b} + (\alpha + a)\frac{\beta}{b^2 + b\beta} = \frac{-a(b - \alpha\beta/a)}{b^2 + b\beta}.$$

Since a, b, and β are positive numbers, we have $dl(b)/db > 0$ if $b < \alpha\beta/a$; $dl(b)/db = 0$ if $b = \alpha\beta/a$; and $dl(b)/db < 0$ if $b > \alpha\beta/a$. Hence $\hat{b} = \alpha\beta/a = \hat{C}_p^2/a$ is the MLE for b.

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