which implies that quadratic stability of the unforced system of (1) because the last two terms in the inequality are positive-semi-definite. In order to establish the H_∞ disturbance attenuation property, we assume $x(0) = 0$ and need to show that

$$J := \sum_{k=0}^{\infty} [z^T(k)z(k) - \gamma^2 w^T(k)w(k)] < 0, \quad \text{whenever } w(k) \not\equiv 0. \tag{25}$$

The existence of the sum in (25) is guaranteed by the boundedness of $w(k)$ and the quadratic stability of the unforced system of (1). It is obvious that (25) holds if $x(k) = 0$ for all $k \geq 0$. Therefore, we assume $x(k) \not\equiv 0$ in the sequel.

Abbreviating $A + \Delta A(k)$ by A_Δ and defining

$$\Gamma = [P^{-1} + \gamma^2 B_1 B_1^T]^{-1} > 0 \tag{26}$$

it is straightforward to show by using the matrix inversion lemma that (26) and (22) imply

$$B_1^T \Gamma B_1 < \gamma^2 I \tag{27}$$

and

$$U(k) := A_\Delta^T \Gamma A_\Delta - \Gamma + \gamma^{-2} A_\Delta^T \Gamma B_1 [I - \gamma^{-2} B_1^T \Gamma B_1^T]^{-1} \cdot B_1^T \Gamma A_\Delta + C_1^T C_1 < 0. \tag{28}$$

Using $x(0) = 0$, we have

$$\sum_{k=0}^{N} [x^T(k+1)\Gamma x(k+1) - x^T(k)\Gamma x(k)] = x^T(N+1)\Gamma x(N+1) \geq 0.$$

Let

$$J(k) := z^T(k)z(k) - \gamma^2 w^T(k)w(k) + x^T(k+1)\Gamma x(k+1) - x^T(k)\Gamma x(k)$$

$$J_N := \sum_{k=0}^{N} [z^T(k)z(k) - \gamma^2 w^T(k)w(k)].$$

Then, we have

$$J_N = \sum_{k=0}^{N} J(k) - x^T(N+1)\Gamma x(N+1).$$

Using (27) and (28), it can be verified that

$$J(k) = x^T(k)U(k)x(k) - \gamma^2 V^T(k)[I - \gamma^{-2} B_1^T \Gamma B_1] V(k) \leq 0$$

where

$$V(k) := w(k) - \gamma^{-2}[I - \gamma^{-2} B_1^T \Gamma B_1]^{-1} B_1^T \Gamma A_\Delta x(k).$$

Since we assumed that $x(k) \not\equiv 0$, we must have $J(k) < 0$ for some $k \geq 0$. Hence, $J_N < 0$ for sufficiently large N, which implies $J < 0$. ▽ ▽ ▽

References

[1] J. C. Doyle, "Analysis of feedback systems with structured uncertainties," *IEE Proc., Part D*, vol. 129, pp. 242–250, Nov. 1982.

[2] ——, "Lecture notes on advances in multivariable control," presented at ONR/Honeywell Workshop, Minneapolis, MN, Oct. 1984.

[3] L. Xie and C. E. de Souza, "Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1188–1191, 1992.

[4] L. Xie, M. Fu, and C. E. de Souza, "H_∞ control and quadratic stabilization of systems with time-varying uncertainty," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1253–1256, 1992.

[5] ——, "H_∞ control of linear systems with time-varying parameter uncertainty," in *Control of Uncertain Dynamic Systems*, S. P. Bhattacharyya and L. H. Keel, Eds., pp. 63–75, Boca Raton, FL: CRC Press, 1991.

[6] A. Packard and J. C. Doyle, "Quadratic stability with real and complex perturbations," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 198–201, 1990.

[7] M. E. Magana and S. H. Zak, "Robust state feedback stabilization of discrete-time uncertain dynamical systems," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 887–891, 1988.

[8] ——, "Robust output feedback stabilization of discrete-time uncertain dynamical systems," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 1082–1085, 1988.

[9] B. R. Barmish, "Necessary and sufficient conditions for quadratic stabilizability of an uncertain system," *J. Opt. Theory Appl.*, vol. 46, no. 4, pp. 399–408, 1985.

[10] K. Furuta and S. Phoojaruenchanachai, "An algebraic approach to discrete-time H_∞ control problems," in *Proc. 1990 Amer. Contr. Conf.*, San Diego, CA, May 1990.

Minimal Periodic Realizations of Transfer Matrices

Ching-An Lin and Chwan-Wen King

***Abstract*—Periodic controllers designed based on the so-called lifting technique are usually represented by transfer matrices. Real-time operations require that the controllers be implemented as periodic systems. We study the problem of realizing an $Nn_o \times Nn_i$ proper rational transfer matrix as an n_i-input n_o-output N-periodic discrete-time system. We propose an algorithm to obtain periodic realizations which have a minimal number of states. The result can also be used to remove any redundant states that exist in a periodic system.**

I. Introduction

It is reported in the literature that linear periodic controllers may be superior to the linear time-invariant ones for a large class of control problems [5], [2], [1], [3]. For discrete-time systems, Khargonekar, Poolla and Tannenbaum [5] proposed a framework for the design of linear periodic controllers for linear time-invariant plants. They show that to an n_i-input n_o-output linear N-periodic system there corresponds an Nn_i-input Nn_o-output linear time-invariant system and conversely to an Nn_i-input Nn_o-output linear time-invariant system there corresponds an n_i-input n_o-output linear N-periodic system. It is asserted [5] that from an input–output point of view, this correspondence is isomorphic in that both algebraic and analytic properties of systems are preserved and hence, the design of periodic controllers can be done using various LTI design techniques. However, the n_i-input n_o-output N-periodic controller so designed is "represented" as an Nn_i-input Nn_o-output time-invariant system, e.g., an $Nn_o \times Nn_i$ transfer matrix. Real-time operations require that the controller be realized as a periodic system. There are straightforward realizations of such a transfer matrix as an N-periodic system but usually with a large number of

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states, many of which are unnecessary and even undesirable. And it is not clear how to remove these extra states from the realization. In this note, we propose a method to obtain a minimal N-periodic realization of an $Nn_o \times Nn_i$ transfer matrix.

Consider the n_i-input n_o-output N-periodic linear causal discrete-time system described by

$$\begin{cases} x(k+1) = A_k x(k) + B_k u(k) \\ y(k) = C_k x(k) + D_k u(k) \end{cases} \quad (1.1)$$

where $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times n_i}$, $C_k \in \mathbb{R}^{n_o\times n}$ and $D_k \in \mathbb{R}^{n_o\times n_i}$ are N-periodic, i.e., $A_{k+N} = A_k$, $B_{k+N} = B_k$, $C_{k+N} = C_k$ and $D_{k+N} = D_k$ for all k. To the system (1.1), we associate an Nn_i-input Nn_o-output linear time-invariant system

$$\begin{cases} X(k+1) = \bar{A}X(k) + \bar{B}U(k) \\ Y(k) = \bar{C}X(k) + \bar{D}U(k) \end{cases} \quad (1.2)$$

where $\bar{A} \in \mathbb{R}^{n\times n}$, $\bar{B} \in \mathbb{R}^{n\times Nn_i}$, $\bar{C} \in \mathbb{R}^{Nn_o\times n}$ and $\bar{D} \in \mathbb{R}^{Nn_o\times Nn_i}$ are given by

$$\bar{A} = A_{N-1}A_{N-2}\cdots A_1A_0, \quad (1.3a)$$

$$\bar{B} = \begin{bmatrix} \bar{B}_0 & \bar{B}_1 & \cdots & \bar{B}_{N-1} \end{bmatrix}$$

$$\text{with } \bar{B}_i = \begin{cases} A_{N-1}\cdots A_{i+1}B_i, & i = 0,\cdots,N-2 \\ B_{N-1}, & i = N-1 \end{cases} \quad (1.3b)$$

$$\bar{C}^T = \begin{bmatrix} \bar{C}_0^T & \bar{C}_1^T & \cdots & \bar{C}_{N-1}^T \end{bmatrix}$$

$$\text{with } \bar{C}_i = \begin{cases} C_0, & i = 0 \\ C_iA_{i-1}\cdots A_0, & i = 1,\cdots,N-1 \end{cases} \quad (1.3c)$$

$$\bar{D} = \begin{bmatrix} \bar{D}_{0,0} & \cdots & \bar{D}_{0,N-1} \\ \vdots & \ddots & \vdots \\ \bar{D}_{N-1,0} & \cdots & \bar{D}_{N-1,N-1} \end{bmatrix}$$

$$\text{with } \bar{D}_{i,j} = \begin{cases} 0, & i < j \\ D_i, & i = j \\ C_iA_{i-1}\cdots A_{j+1}B_j, & i > j. \end{cases} \quad (1.3d)$$

The input–output relations of the systems (1.1) and (1.2) are related as follows. Assume that $x(0) = X(0) = 0$. Suppose $\{u(j)\}_{j=0}^{\infty}$ and $\{y(j)\}_{j=0}^{\infty}$ are respectively, the input and output of the system (1.1). If the input to the system (1.2) is, for each k

$$U(k) = \begin{bmatrix} u(Nk) \\ u(Nk+1) \\ \vdots \\ u(Nk+N-1) \end{bmatrix},$$

then the corresponding output, for each k, is given by

$$Y(k) = \begin{bmatrix} y(Nk) \\ y(Nk+1) \\ \vdots \\ y(Nk+N-1) \end{bmatrix}.$$

Conversely, suppose $\{U(k)\}_{k=0}^{\infty}$ and $\{Y(k)\}_{k=0}^{\infty}$ are respectively, the input and output of the system (1.2). If the input to the system (1.1) is

$$u(Nk+j) = U_j(k),$$

$$\text{for } j = 0,\cdots,N-1, k = 0,1,\cdots, \text{ where } U(k) = \begin{bmatrix} U_0(k) \\ \vdots \\ U_{N-1}(k) \end{bmatrix},$$

then the corresponding output is

$$y(Nk+j) = Y_j(k),$$

$$\text{for } j = 0,\cdots,N-1, k = 0,1,\cdots, \text{ where } Y(k) = \begin{bmatrix} Y_0(k) \\ \vdots \\ Y_{N-1}(k) \end{bmatrix}.$$

The transfer matrix of the system (1.2) is $G(z) = \bar{C}(zI - \bar{A})^{-1}\bar{B} + \bar{D} \in \mathbb{R}_p(z)^{Nn_o\times Nn_i}$, where $\mathbb{R}_p(z)$ is the set of proper rational functions in z with real coefficients. We say that the system (1.1), or equivalently $\{A_k, B_k, C_k, D_k\}$, is an *N-periodic realization* of the transfer matrix $G(z)$. It is claimed in [5] that if $G(z) \in \mathbb{R}_p(z)^{Nn_o\times Nn_i}$ with $G(\infty)$ $n_o \times n_i$-block lower triangular, then there exists a causal N-periodic realization for $G(z)$.

The problem we study is the following: Given $G(z) \in \mathbb{R}_p(z)^{Nn_o\times Nn_i}$ with $G(\infty)$ $n_o \times n_i$-block lower triangular, what is the minimal number of states required for an N-periodic realization of $G(z)$ and how to obtain a minimal N-periodic realization?

II. Minimal Periodic Realizations

Suppose $G(z) \in \mathbb{R}_p(z)^{Nn_o\times Nn_i}$ is given with $G(\infty)$ $n_o \times n_i$-block lower triangular and let $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ be a minimal realization of $G(z)$ with $\bar{A} \in \mathbb{R}^{n\times n}$. Consider (1.3) with $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ given. It is easy to see that if $\{A_k, B_k, C_k, D_k\}$ is a solution of (1.3), then (1.1) is a minimal N-periodic realization of $G(z)$. On the other hand, if (1.3) has no solution, then there is no N-periodic realization of $G(z)$ of dimension n, the McMillan degree of $G(z)$. In other words, any N-periodic realization of $G(z)$ must be of dimension larger than n. In this section, we give a procedure to obtain a minimal N-periodic realization of the transfer matrix $G(z)$. We start by deriving a necessary and sufficient condition under which $G(z)$ has an N-periodic realization with its dimension equal to the McMillan degree of $G(z)$. The condition is then used to determine the minimal number of states that are required for an N-periodic realization of $G(z)$.

Now since $G(\infty)$ is $n_o \times n_i$-block lower triangular, $\bar{D}$ is also $n_o \times n_i$-block lower triangular. From (1.3), we have

$$B_{N-1} = \bar{B}_{N-1}, C_0 = \bar{C}_0, \text{ and } D_k = \bar{D}_{kk}, \quad \text{for } k = 0,\cdots,N-1. \quad (2.1)$$

With (2.1), (1.3) reduces to

$$A_{N-1}A_{N-2}\cdots A_1A_0 = \bar{A} \quad (2.2a)$$

$$A_{N-1}A_{N-2}\cdots A_{i+1}B_i = \bar{B}_i, \quad \text{for } i = 0,\cdots,N-2 \quad (2.2b)$$

$$C_iA_{i-1}\cdots A_1A_0 = \bar{C}_i, \quad \text{for } i = 1,\cdots,N-1 \quad (2.2c)$$

$$C_iA_{i-1}\cdots A_{j+1}B_j = \bar{D}_{i,j}, \quad \text{for } i = 1,\cdots,N-1, j = 0,\cdots,i-1 \quad (2.2d)$$

where $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times n_i}$ and $C_i \in \mathbb{R}^{n_o\times n}$ are to be solved. Thus $G(z)$ has an N-periodic realization of dimension n if and only if (2.2) has a solution. The following lemma is needed in the proof of the main result.

Lemma 2.1 [7, page 25]: Consider matrix equations

$$UX = \bar{U} \quad (2.3a)$$

$$XV = \bar{V} \quad (2.3b)$$

where $U \in \mathbb{R}^{m_1\times n_1}$, $\bar{U} \in \mathbb{R}^{m_1\times n_2}$, $V \in \mathbb{R}^{n_2\times m_2}$ and $\bar{V} \in \mathbb{R}^{n_1\times m_2}$.

Equation (2.3) has a solution $X \in \mathbb{R}^{n_1 \times n_2}$ if and only if

$$\text{i)}\quad \mathscr{R}(\bar{U}) \subseteq \mathscr{R}(U) \quad\text{and}\quad \mathscr{R}(\bar{V}^T) \subseteq \mathscr{R}(V^T), \quad\text{and} \tag{2.4a}$$

$$\text{ii)}\quad U\bar{V} = \bar{U}V \tag{2.4b}$$

where $\mathscr{R}(A)$ is the range space of A. □

The following theorem is a necessary and sufficient condition for $G(z)$ to have an N-periodic realization of dimension n, the McMillan degree of $G(z)$.

Theorem 2.2: Equation (2.2) has a solution $\{A_k\}_{k=0}^{N-1}$, $\{B_k\}_{k=0}^{N-2}$ and $\{C_k\}_{k=1}^{N-1}$ if and only if for $i = 0,\cdots, N-2$

$$\rho(K_i) \le n \tag{2.5}$$

where $\rho(K_i)$ denotes the rank of K_i and

$$K_i = \begin{bmatrix} \bar{A} & \bar{B}_0 & \cdots & \bar{B}_i \\ \bar{C}_{N-1} & \bar{D}_{N-1,0} & \cdots & \bar{D}_{N-1,i} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{C}_{i+1} & \bar{D}_{i+1,0} & \cdots & \bar{D}_{i+1,i} \end{bmatrix}. \tag{2.6}$$

Comments: i) Thus, to check if $G(z)$ has N-periodic realization of dimension n, we only have to check the $N-1$ rank conditions in (2.5). ii) It is easy to check that condition (2.5) does not depend on any particular LTI minimal realization chosen for $G(z)$. iii) Condition (2.5) is very strong when $\bar{A}$ is nonsingular. The following corollary follows from the theorem and simple elementary operations [4, page 650].

Corollary 2.3: If $\bar{A}$ is nonsingular, then (2.2) has a solution if and only if

$$\bar{D}_{ij} = \bar{C}_i(\bar{A})^{-1}\bar{B}_j, \qquad \text{for all } i = 0,\cdots, N-1,\ j = 0,\cdots, i-1. \tag{2.7}$$

Proof of Theorem 2.2: ($\Rightarrow$) Let K_i be as defined in (2.6). Suppose $\{A_k\}_{k=0}^{N-1}$, $\{B_k\}_{k=0}^{N-2}$ and $\{C_k\}_{k=1}^{N-1}$ satisfy (2.2), then we must have for $i = 0, 1,\cdots, N-2$

$$K_i = \begin{bmatrix} A_{N-1} \cdots A_{i+1} \\ C_{N-1}A_{N-2} \cdots A_{i+1} \\ \vdots \\ C_{i+2}A_{i+1} \\ C_{i+1} \end{bmatrix} \cdot \begin{bmatrix} A_i \cdots A_0 & A_i \cdots A_1 B_0 & \cdots & A_i B_{i-1} & B_i \end{bmatrix}. \tag{2.8}$$

Since $A_i \in \mathbb{R}^{n \times n}$, it follows from (2.8) that $\rho(K_i) \le n$, for $i = 0, 1,\cdots, N-2$.

($\Leftarrow$) Suppose (2.5) holds. For each i, there exist $U_i \in \mathbb{R}^{(n+(N-i-1)n_o)\times n}$ and $V_i \in \mathbb{R}^{n\times(n+(i+1)n_i)}$ with $\rho(U_i) = \rho(V_i) = \rho(K_i)$ such that

$$K_i = U_i V_i. \tag{2.9}$$

Let us partition U_i and V_i as

$$U_i = \begin{bmatrix} U_{i,0} \\ U_{i,1} \\ \vdots \\ U_{i,N-i-1} \end{bmatrix}, \quad V_i = \begin{bmatrix} V_{i,0} & V_{i,1} & \cdots & V_{i,i+1} \end{bmatrix} \tag{2.10}$$

where $U_{i,0}, V_{i,0} \in \mathbb{R}^{n\times n}$, $U_{i,j} \in \mathbb{R}^{n_o \times n}$, $j = 1,\cdots, N-i-1$ and $V_{i,j} \in \mathbb{R}^{n \times n_i}$, $j = 1,\cdots, i+1$. From (2.6), we have, for $i = 0,\cdots, N-2$,

$$U_{i,0}V_{i,0} = \bar{A}, \tag{2.11}$$

$$U_{i,N-j}V_{i,0} = \bar{C}_j, \qquad j = i+1,\cdots, N-1, \tag{2.12}$$

$$U_{i,0}V_{i,l+1} = \bar{B}_l, \qquad l = 0,\cdots, i, \tag{2.13}$$

$$U_{i,N-j}V_{i,l+1} = \bar{D}_{j,l}, \qquad j = i+1,\cdots, N-1, l = 0,\cdots, i. \tag{2.14}$$

For $i = 0, 1,\cdots, N-2$, let

$$\bar{U}_i = \begin{bmatrix} U_{i,0} \\ \vdots \\ U_{i,N-i-2} \end{bmatrix} \quad\text{and}\quad \bar{V}_i = \begin{bmatrix} V_{i,0} & \cdots & V_{i,i} \end{bmatrix}. \tag{2.15}$$

Note that $\bar{U}_i$ and $\bar{V}_i$ are obtained respectively by deleting the last n_o rows from U_i and the last n_i columns from V_i. It follows from (2.11)–(2.15) that for $i = 1,\cdots, N-2$

$$U_i\bar{V}_i = \bar{U}_{i-1}V_{i-1} = \begin{bmatrix} \bar{A} & \bar{B}_0 \cdots & \bar{B}_{i-1} \\ \bar{C}_{N-1} & \bar{D}_{N-1,0} \cdots & \bar{D}_{N-1,i-1} \\ \vdots & \vdots \ \ddots & \vdots \\ \bar{C}_{i+1} & \bar{D}_{i+1,0} \cdots & \bar{D}_{i+1,i-1} \end{bmatrix} =: W_i. \tag{2.16}$$

From (2.16) and (2.5), we have

$$K_i = \left[\, W_i \;\middle|\; \begin{matrix} \bar{B}_i \\ \bar{D}_{N-1,i} \\ \vdots \\ \bar{D}_{i+1,i} \end{matrix} \right] \quad\text{and} \tag{2.17}$$

$$K_{i-1} = \left[\frac{W_i}{\bar{C}_i \quad \bar{D}_{i,0} \quad \cdots \quad \bar{D}_{i,i-1}} \right]. \tag{2.18}$$

From (2.9) and that $\rho(U_i) = \rho(K_i)$, we know that

$$\mathscr{R}(U_i) = \mathscr{R}(K_i). \tag{2.19}$$

Let us partition U_{i-1} as

$$U_{i-1} = \begin{bmatrix} \bar{U}_{i-1} \\ U_{i-1,N-i} \end{bmatrix}. \tag{2.20}$$

From (2.20), (2.18) and that $\mathscr{R}(U_{i-1}) = \mathscr{R}(K_{i-1})$, we have

$$\mathscr{R}(\bar{U}_{i-1}) = \mathscr{R}(W_i). \tag{2.21}$$

From (2.19), (2.21), and (2.17), it follows that

$$\mathscr{R}(\bar{U}_{i-1}) \subseteq \mathscr{R}(U_i). \tag{2.22}$$

By similar arguments, it can be shown that

$$\mathscr{R}(\bar{V}_i^T) \subseteq \mathscr{R}(V_{i-1}^T). \tag{2.23}$$

From (2.16), (2.22), (2.23), and Lemma 2.1, there exist A_i such that, for $i = 1,\cdots, N-2$

$$\begin{cases} U_i A_i = \bar{U}_{i-1} \\ A_i V_{i-1} = \bar{V}_i \end{cases} \tag{2.24}$$

or equivalently

$$U_{i,j}A_i = U_{i-1,j}, \qquad \text{for } j = 0,\cdots, N-i-1, \quad\text{and} \tag{2.25}$$

$$A_i V_{i-1,j} = V_{i,j}, \qquad \text{for } j = 0,\cdots, i. \tag{2.26}$$

Let

$$B_i = V_{i,i+1}, \qquad i = 0,\cdots, N-2 \tag{2.27}$$

$$C_i = U_{i-1,N-i}, \qquad i = 1,\cdots, N-1 \tag{2.28}$$

$$A_{N-1} = U_{N-2,0} \quad\text{and}\quad A_0 = V_{0,0}. \tag{2.29}$$

We now show that the A_i's, B_i's, and C_i's defined in (2.24), (2.27)–(2.29) satisfy (2.2). From (2.29), (2.25), and (2.26), we get

$$\begin{aligned} A_{N-1}A_{N-2}\cdots A_1A_0 &= U_{N-2,0}A_{N-2}\cdots A_1V_{0,0} \\ &= U_{N-3,0}\cdots A_{i+1}A_i\cdots V_{1,0} \\ &\vdots \\ &= U_{i,0}V_{i,0} \end{aligned} \tag{2.30}$$

where i can be any integer between 0 and $N-2$. It follows from (2.11) and (2.30) that these $\{A_k\}_{k=0}^{N-1}$ satisfy (2.2a).

The verification that the $\{A_k\}_{k=0}^{N-1}$, $\{B_k\}_{k=0}^{N-2}$, and $\{C_k\}_{k=1}^{N-1}$ so defined satisfy (2.2b), (2.2c), and (2.2d) is similar and straightforward and hence is omitted. □

If condition (2.5) is not satisfied, that is, $\rho(K_i) > n$ for some i, then any N-periodic realization of $G(z)$ must be of dimension larger than n. To find the minimum number of states that are required for an N-periodic realization, we note that the system

$$\tilde{A} = \begin{bmatrix} O_1 & O_2 \\ O_2^T & \bar{A} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} O_3 \\ \bar{B} \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} O_4 & \bar{C} \end{bmatrix}, \quad \tilde{D} = \bar{D} \tag{2.31}$$

where O_1 is a $\Delta n \times \Delta n$ zero matrix, O_2, O_3, and O_4 are zero matrices with compatible dimensions, has the same transfer matrix $G(z)$ as the system $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ and the matrix $\tilde{K}_i$, similarly defined through (2.6) with $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$ replacing $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$, has the same rank as K_i does. Note that (2.31) amounts to adding Δn *uncontrollable and unobservable* hidden modes at $z = 0$ to the original realization $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ of $G(z)$.

Suppose Δn is the smallest positive integer such that

$$n + \Delta n \geq \rho(\tilde{K}_i) = \rho(K_i), \qquad i = 0, \cdots, N-2.$$

Then any minimal N-periodic realization of $G(z)$ must be of dimension equal to $n + \Delta n$. Since it is impossible to obtain an N-periodic realization of $G(z)$ with dimension less than n, the McMillan degree of $G(z)$, we have that the dimension of the minimal N-periodic realization of $G(z)$ is

$$\bar{n} = \max\{[\max_{0 \leq i \leq N-2} \rho(K_i)], n\}. \tag{2.32}$$

Based on Theorem 2.2 and the analysis above, we propose the following algorithm for obtaining a minimal N-periodic realization of a given transfer matrix $G(z) \in \mathbb{R}_p(z)^{Nn_o \times Nn_i}$.

Algorithm 2.4:

Data: $G(z) \in \mathbb{R}_p(z)^{Nn_o \times Nn_i}$ with $G(\infty)$ $n_o \times n_i$-block lower triangular.

Step 1: Find a minimal realization $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ of $G(z)$, where $\bar{A} \in \mathbb{R}^{n \times n}$.

Step 2: For $i = 0, \cdots, N-2$, form the matrix K_i as defined in (2.6).

Step 3: Compute $\bar{n}$ as defined in (2.32).

Step 4: If $\bar{n} \leq n$, then go to Step 5; else,

$$\bar{A} := \begin{bmatrix} O_1 & O_2 \\ O_2^T & \bar{A} \end{bmatrix}, \quad \bar{B} := \begin{bmatrix} O_3 \\ \bar{B} \end{bmatrix}, \quad \bar{C} := \begin{bmatrix} O_4 & \bar{C} \end{bmatrix}$$

and form the corresponding K_i for $i = 0, \cdots, N-2$.

Step 5: Obtain B_{N-1}, C_0, and $\{D_i\}_{i=0}^{N-1}$ from (2.1).

Step 6: For $i = 0, \cdots, N-2$, decompose K_i into U_i, V_i with $\rho(U_i) = \rho(V_i) = \rho(K_i)$ as in (2.9).

Step 7: Obtain A_{N-1}, A_0, $\{B_i\}_{i=0}^{N-2}$, and $\{C_i\}_{i=1}^{N-1}$ defined in (2.27)–(2.29).

Step 8: For $i = 1, \cdots, N-2$, form $\bar{U}_{i-1}$ and $\bar{V}_i$ as in (2.15) and solve (2.24) for A_i. □

We give an example to illustrate the proposed algorithm.

Example: Let

$$G(z) = \frac{1}{z-1}\begin{bmatrix} z+2 & 4 & 1 \\ 6z & 3z+5 & 2 \\ 9z & z+11 & z+2 \end{bmatrix}.$$

We wish to obtain a 3-periodic minimal realization for $G(z)$. Compute a minimal LTI realization as

$$\bar{A} = 1, \quad \bar{B} = \begin{bmatrix} 3 & 4 & 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 3 & 0 \\ 9 & 1 & 1 \end{bmatrix}.$$

By Step 2 and Step 3, form K_0 and K_1. By computations, $\rho(K_0) = 1$ and $\rho(K_1) = 2$. Thus $\bar{n} = 2 > 1$. By Step 4, let

$$\bar{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 4 & 1 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 3 & 0 \\ 9 & 1 & 1 \end{bmatrix}. \tag{2.33}$$

Based on (2.33), form the corresponding K_0 and K_1. By Step 5, $D_0 = 1$, $D_1 = 3$, $D_2 = 1$, $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}$. By Step 6, decompose K_0, K_1 as

$$K_0 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and}$$

$$K_1 = \begin{bmatrix} 0 & 0 \\ 1 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By Step 7,

$$A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 2 & 4 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 3 & 1 \end{bmatrix}.$$

By Step 8, obtain $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

The minimal 3-periodic realization is then given by $\{A_k, B_k, C_k, D_k\}_{k=0}^2$. □

Remarks: i) Formula (2.31) is not the only way to expand the dimension of $\bar{A}$ such that Theorem 2.2 can still be used for realization, however, it is the most straightforward. In this fashion, the hidden modes introduced are both uncontrollable and unobservable. One can also add uncontrollable or unobservable hidden modes. ii) The result we obtained so far can also be used to determine whether a given periodic system contains redundant states. More precisely, suppose we are given an N-periodic system $\{\tilde{A}_k, \tilde{B}_k, \tilde{C}_k, \tilde{D}_k\}$, we can use (1.3) to obtain the corresponding $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ and the transfer matrix $G(z) = \bar{C}(zI - \bar{A})^{-1}\bar{B} + \bar{D}$. Following Algorithm 2.4, we can decide the minimal number of states required for the periodic system and obtain a new N-periodic realization which is minimal. Note that the new N-periodic realization has exactly the same input–output relation as the original one.

III. Conclusion

In this note, we develop a procedure to obtain a minimal N-periodic realization of a given transfer matrix $G(z) \in \mathbb{R}_p(z)^{Nn_o \times Nn_i}$. The result is useful in implementing periodic controllers designed by the so-called lifting technique; it can also be used to remove redundant states in a given periodic system.

REFERENCES

[1] B. D. O. Anderson and J. B. Moore, "Time-varying feedback laws for decentralized control," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 1133–1139, 1981.

[2] B. A. Francis and T. T. Georgiou, "Stability theory for linear time-invariant plants with periodic digital controllers," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 820–832, 1988.

[3] I. Horowitz and O. Yaniv, "Quantitative design for SISO nonminimum-phase unstable plants by the singular-G method," *Int. J. Contr.*, vol. 46, no. 1, pp. 281–294, 1987.

[4] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[5] P. P. Khargonekar, K. Poolla, and A. Tannenbaum, "Robust control of linear time-invariant plants using periodic compensation," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 1088–1096, 1985.

[6] B. Noble and J. W. Daniel, *Applied Linear Algebra*. Englewood Cliffs, NJ: Prentice-Hall, 1977.

[7] C. R. Rao and S. K. Mitra, *Generalized Inverse of Matrices and its Applications*. New York: Wiley, 1971.

Singular Perturbation of the Time-Optimal Soft-Constrained Cheap-Control Problem

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Abstract—We consider the solution of a time-optimal soft-constrained control problem with linear dynamics. The cost function has no penalty on the integral of the state. The solution is formulated in terms of the controllability Grammian and is obtained as the solution of a system of linear algebraic equations and a nonlinear scalar algebraic equation. As the state approaches the origin, or equivalently, as the control becomes cheap, the optimal final time becomes small. This introduces a highly degenerate hierarchy of amplitude scales. A new approach, solely based on expanding the controllability Grammian, is developed to obtain an asymptotic solution of the problem without resorting to boundary-layer theory.

I. INTRODUCTION

We are concerned with the class of time-optimal soft-constrained control problems that minimize the cost function

$$J = t_f + \frac{1}{2}\int_0^{t_f}[x^T(t)Qx(t) + \epsilon^2 u^T(t)Ru(t)]\,dt \qquad (1)$$

subject to

$$\dot{x}(t) = Ax(t) + bu(t) \qquad (2)$$

$$x(0) \text{ specified}, \qquad \text{and } x(t_f) = 0. \qquad (3)$$

The conventional "cheap"-control problem [1] is obatined by letting $\epsilon \to 0$ with the final time t_f being held fixed, in which case the control action becomes increasingly cheap. In the case where $t_f \to \infty$, the cheap-control linear quadratic regulator (LQR) problem yields a high-gain linear state feedback control policy with many desirable properties, such as disturbance rejection, insensitivity to parameter errors and distortions, improved tracking, and minimum steady-state error [2]. The case of finite final time is considered in [3] and [4].

A typical solution of the cheap-control problem proceeds by transforming the problem into a singularly perturbed system [2], [5]. The analysis of such a system is well understood if it is written in the so-called standard form [6], [7]. The method of matched asymptotic expansions can then be used to determine an approximate solution consisting of an outer expansion and a boundary-layer correction (inner expansion) [6], [8], [9]. Impulsive controls are needed to get onto and away from the singular arcs [10], [11]. If, an addition, the control is constrained by inequalities, then both bang-bang and singular arcs exist [3].

We consider the time-optimal soft-constrained control problem. Moreover, we take $Q = 0$. This case is not included in the classification introduced in [4]. The solution of this case is derived [12] in terms of the controllability Grammian matrix [13]. It consists of the solution of a system of linear algebraic equations parametrized on the final time, coupled with a nonlinear scalar algebraic equation that yields the optimal final time. When the control becomes increasingly cheap, a very fast maneuver is expected, and the optimal final time becomes small even for large initial conditions. In this note, we obtain an asymptotically valid solution for the soft-constrained time-optimal cheap-control problem without resorting to the method of matched asymptotic expansions.

II. PROBLEM FORMULATION

The open-loop, soft-constrained, time-optimal control problem is to find a measurable function $u(t)$ that drives the state $x(0)$ into the target point $x(t_f) = 0$ while minimizing (1) with $Q = 0$. The solution was described in [12] in terms of the controllability Grammian matrix

$$W(0,t_f) = \int_0^{t_f} v(\tau)v^T(\tau)\,d\tau, \qquad \text{where } v(t) = e^{-At}b. \quad (4)$$

The costate is given by $\lambda(t) = e^{-A^T t}\mu$, where $\mu = \lambda(0)$ is the initial costate and

$$W(0,t_f)\mu = \xi \qquad (5)$$

where $\xi = \epsilon^2 x(0)$. The control function can be written as $u(t) = -\epsilon^{-2}\lambda^T(0)v(t)$. For $t = 0$, we have $u(0) = -x^T(0)y(t_f)$, where $y(t_f) = [W(0,t_f)]^{-1}b$. Furthermore, the cost function is given by $J = t_f + (1/2)x^T(0)\lambda(0)$ or

$$J = t_f + \frac{1}{2}\epsilon^2 x^T(0)[W(0,t_f)]^{-1}x(0). \qquad (6)$$

The optimal final time has to satisfy the transversality condition

$$2 = \left[z^T(t_f)\epsilon x(0)\right]^2, \qquad \text{where } z(t_f) = [W(0,t_f)]^{-1}v(t_f). \qquad (7)$$

We note that the mapping $t_f \mapsto W(0,t_f)$ and $t_f \mapsto v(t_f)$ depend only on the problem data (specifically A and b). Then, with fixed initial data $x(0)$, (7) is a scalar equation in t_f. The roots of (7) specify all the candidates for the minimum t_f. If many roots $t_f^1, t_f^2, \cdots$, exist, the corresponding costs can be computed via (6). Direct comparison of the costs can then be used to find the

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