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Christmas tree: A versatile 1-fault-tolerant design for token rings ☆

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Abstract

The token ring topology is required in token passing approach used in distributed operating systems. Fault tolerance is also required in the designs of distributed systems. Note that 1-fault-tolerant design for token rings is equivalent to design of 1-Hamiltonian graphs. This paper introduces a new family of graphs called *Christmas tree*, denoted by $CT(s)$. The graph $CT(s)$ is a 3-regular, planar, 1-Hamiltonian, and Hamiltonian-connected graph. The number of nodes in $CT(s)$ is $3 \cdot 2^s - 2$. Its diameter is 1 if $s = 1$, 3 if $s = 2$, and $2s$ if $s \geqslant 3$. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

There are many mutually conflicting requirements in designing the topology of computer networks. It is almost impossible to design a network that is optimal from all aspects. One has to design a suitable network depending on one's requirements. The Hamiltonian property is one of major requirements in designing the topology of networks. For example, as "*Token Passing*" approach is used in some distributed operating systems, interconnection networks require the presence of Hamiltonian cycles in the structure to apply this approach. Fault tolerance is also desirable in massive parallel systems that have a relatively high probability of failure. A number of fault-tolerant designs for specific multiprocessor architectures have been proposed based on graph theoretic models in which the processor-to-processor interconnection structure is represented by a graph.

Let $G = (V, E)$ be an undirected *graph*, where V is the *node set* and E is the *edge set* of G. A *path* is a sequence of nodes such that two consecutive nodes are adjacent. A path is represented by $\langle v_0, v_1, v_2, \ldots, v_{t-1}\rangle$. We also write the path $\langle v_0, v_1, v_2, \ldots, v_{t-1}\rangle$ as $\langle v_0 \to P_1 \to v_i, v_{i+1}, \ldots, v_j \to P_2 \to v_k, v_{k+1}, \ldots, v_{t-1}\rangle$, where $P_1 = \langle v_0, v_1, \ldots, v_i\rangle$ and $P_2 = \langle v_j, v_{j+1}, \ldots, v_k\rangle$. The length of a path is the number of nodes in this path. A *Hamiltonian path* is a path whose nodes are distinct and span V. A *cycle* is a path of at least three nodes such that the first node is the same as the last node. A cycle is called a *Hamiltonian cycle* if it traverses every node of V exactly once. If G has a Hamiltonian cycle, G is said to be *Hamiltonian*.

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Let $G - v$ be a subgraph of G induced by $V - v$ for $v \in V$ and $G - e$ be the subgraph with e removed from E for $e \in E$. We use $G - \{f\}$ for any $f \in V \cup E$ to denote the graph obtained by removing f from G. Given a path P in G, we use $G - P$ to denote the subgraph obtained by removing all nodes of P from G, i.e., the subgraph induced by $V - V(P)$ where $V(P)$ denotes the set of nodes in P. A graph G is 1-*Hamiltonian* if $G - \{f\}$ is Hamiltonian for any $f \in V \cup E$. We sometimes write $f \in V \cup E$ as $f \in G$ for convenience. Obviously, every 1-Hamiltonian graph is Hamiltonian and has at least 4 nodes. Moreover, the degree of every node in a 1-Hamiltonian graph is at least 3. A 1-Hamiltonian graph G^* is *optimal* if it contains the fewest edges among all 1-Hamiltonian graphs with the same number of nodes as G^*. The design of 1-Hamiltonian graphs is equivalent to 1-fault-tolerant designs for token rings. A graph G is *Hamiltonian-connected* if every two nodes of G can be joined by a Hamiltonian path. A Hamiltonian-connected graph G^* is *optimal* if it contains the fewest edges among all Hamiltonian-connected graphs with the same number of nodes as G^*. Moon [6] proved that every optimal Hamiltonian-connected graph with $n \geqslant 4$ nodes contains exactly $\lceil 3n/2 \rceil$ edges.

Mukhopadhyaya and Sinha [7] proposed a family of optimal 1-Hamiltonian graphs. The diameter of every graph with n nodes in the family is $\lfloor n/6 \rfloor + 2$ if n is even, and $\lfloor n/8 \rfloor + 3$ if n is odd. Harary and Hayes [2,3] also presented a family of optimal 1-Hamiltonian graphs of n nodes with diameter $\lfloor (n+1)/4 \rfloor$. Wang et al. [8] proposed another family of optimal 1-Hamiltonian graphs which have diameter $O(\sqrt{n})$. These three families of optimal 1-Hamiltonian graphs are planar. Furthermore, they can be shown to be optimal Hamiltonian-connected.

It is natural to ask if there are graphs having such good properties but with smaller diameter. This problem is related to the famous (n, d, D) problem in which we want to construct a graph of n nodes with maximum degree d such that the diameter D is minimized. When d and n are given, the lower bound on diameter D, called the *Moore bound*, is given by $D \geqslant \log_{d-1} n - 2/d$ [1]. In this paper, we propose a family of graphs called *Christmas trees*, denoted by $CT(s)$. The graph $CT(s)$ is 3-regular, planar, 1-Hamiltonian, and Hamiltonian-connected. The number of nodes in $CT(s)$ is $3 \cdot 2^s - 2$. The diameter is 1 if $s = 1$, 3 if $s = 2$, and $2s$ if $s \geqslant 3$. Thus the diameter is 2 times of the Moore bound.

2. Definitions and notation

To define Christmas tree, we first define slim tree. We write an sth *slim tree* $ST(s)$ as $ST(s) = (V, E, u, l, r)$, where V is the node set, E is the edge set, $u \in V$ is the *root node*, $l \in V$ is the *left node*, $r \in V$ is the *right node*, and $s \geqslant 2$ is an integer. The sth slim tree $ST(s)$ is recursively defined as follows:

(1) $ST(2)$ is the complete graph K_3 with its nodes labeled with u, l and r.

(2) The sth slim tree $ST(s)$, with $s \geqslant 3$, is composed of a root node u and two disjoint copies of $(s-1)$th slim trees as the left subtree and right subtree, denoted by $ST^l(s-1) = (V_1, E_1, u_1, l_1, r_1)$ and $ST^r(s-1) = (V_2, E_2, u_2, l_2, r_2)$, respectively, where in particular $u \notin V_1 \cup V_2$. To be specific, $ST(s) = (V, E, u, l, r)$ is given by $V = V_1 \cup V_2 \cup \{u\}$, $E = E_1 \cup E_2 \cup \{(u, u_1), (u, u_2), (r_1, l_2)\}$, $l = l_1$, $r = r_2$.

For example, $ST(3)$ and $ST(4)$ are illustrated in Fig. 1. By definition of $ST(s)$, the left subtree $ST^l(s-1)$ and the right subtree $ST^r(s-1)$ are isomorphic. This property is referred to as the *symmetry property* of $ST(s)$.

We can also define the slim tree $ST(s)$ from the *complete binary tree* $BT(s)$. An sth complete binary tree $BT(s)$ is a graph whose node set is $\{1, 2, \ldots, 2^s - 1\}$ and edge set is $\{(i, j) \mid \lfloor j/2 \rfloor = i\}$. The slim tree $ST(s)$ can be constructed by adding the set of edges

$$L = \left\{(i, i+1) \mid 2^{s-1} \leqslant i \leqslant 2^s - 2\right\}$$

to $BT(s)$ for $s \geqslant 2$. Each edge in L is called a *leaf edge*. Since $BT(s)$ is a spanning subgraph of $ST(s)$, we can apply the terms of complete binary trees to slim trees and define $ST(s)$ as follows. We define $ST(1)$ to be the $BT(1)$, where u, l, r of $ST(1)$ are identified with the only node of $BT(1)$. The *leaf nodes* of $ST(s)$ are nodes labeled i, where $2^{s-1} \leqslant i \leqslant 2^s - 1$. For $2^{s-1} \leqslant i \leqslant 2^s - 2$, the leaf node $i+1$ is the *right sibling* of the leaf node i and the node i is the *left sibling* of $i+1$. The nodes labeled j, where $j \leqslant 2^{s-1} - 1$, are *nonleaf nodes*. In particular, the node 1 is the root node of $ST(s)$, node 2^{s-1} the left node, and node $2^s - 1$ the right node.

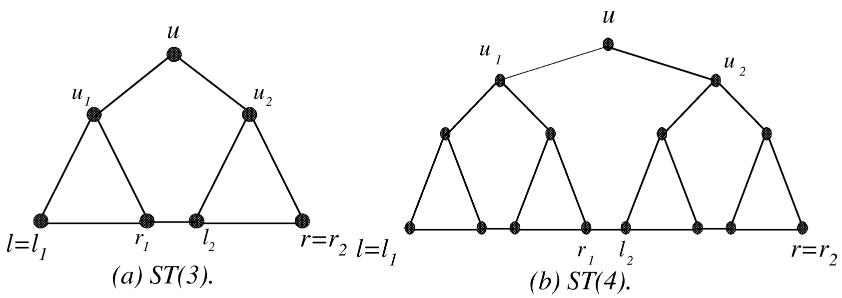

Fig. 1. The slim tree $ST(3)$ and $ST(4)$.

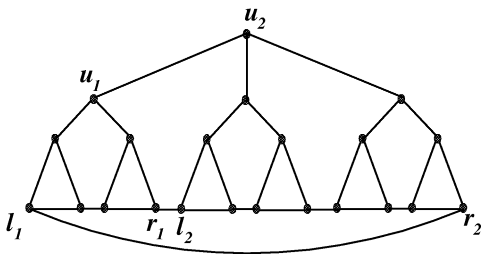

Fig. 2. The Christmas tree $CT(3)$.

The *Christmas tree* $CT(s)$ is composed of an sth slim tree

$$ST^l(s) = (V_1, E_1, u_1, l_1, r_1)$$

and an $(s+1)$th slim tree

$$ST^r(s+1) = (V_2, E_2, u_2, l_2, r_2).$$

To be specific, the node set of $CT(s)$ is $V_1 \cup V_2$ and the edge set of $CT(s)$ is $E_1 \cup E_2 \cup \{(u_1, u_2), (l_1, r_2), (l_2, r_1)\}$. For example, $CT(3)$ is shown in Fig. 2. The number of nodes of $CT(s)$ is $n = 2^s - 1 + 2^{s+1} - 1 = 3 \cdot 2^s - 2$. Every two nodes of $CT(s)$ can be treated as in an $(s+1)$th slim tree, so is each edge of $CT(s)$. Such property is referred to as the *symmetry property* of $CT(s)$.

Let $d_G(x, y)$ denote the distance between the nodes x and y in graph G and $D(G)$ denote the diameter of G. Let v be an arbitrary node in $ST(s)$. Note that the slim tree $ST(s)$ is formed from the complete binary tree $BT(s)$ by adding the leaf edges. We can distinguish two types of paths from v to the root u where type-1 paths use edges in $BT(s)$ only and type-2 paths use edges in $BT(s)$ and at least one leaf edge. It follows that type-2 paths have length at least $s+1$ which is larger than the length of type-1 paths. Consequently, the shortest path from v to u in $ST(s)$, denoted by $R(v, u)$, is of type-1 and the same as the shortest path from v to u in $BT(s)$. Thus $R(v, u)$ is unique.

Since $R(v, u)$ is unique, we then can define the *ancestors* of the node v as those nodes on $R(v, u)$. A node z is called a *common ancestor* of two nodes v_1 and v_2 if z is on both $R(v_1, u)$ and $R(v_2, u)$. A node z^* is called the *lowest common ancestor* of two nodes v_1 and v_2 if z^* is a common ancestor for v_1 and v_2 for which the distance $d_{ST(s)}(u, z^*)$ is maximum.

3. Diameter and Hamiltonian properties

In this section, we show the diameter of $CT(s)$ and prove that $CT(s)$ is optimal 1-Hamiltonian and optimal Hamiltonian-connected. It can be easily verified that $D(ST(2)) = 1$, $D(ST(3)) = 3$, $D(CT(1)) = 1$, and $D(CT(2)) = 3$.

Lemma 1. *$D(ST(s)) = 2s-2$ for $s \geqslant 4$ and $D(CT(s)) = 2s$ for $s \geqslant 3$.*

Proof. Let x and y be two arbitrary nodes in $ST(s)$. Since $d_{ST(s)}(v, u) \leqslant s-1$ for every node v in $ST(s)$, it follows that

$$d_{ST(s)}(x, y) \leqslant d_{ST(s)}(x, u) + d_{ST(s)}(u, y) \leqslant 2s-2.$$

Therefore, we have $D(ST(s)) \leqslant 2s-2$. Next, we want to show $d_{ST(s)}(l, r) = 2s-2$ for $s \geqslant 4$ by induction. It can be easily verified that $d_{ST(4)}(l, r) = 6$. Assume that $d_{ST(s-1)}(l, r) = 2(s-1)-2$ for $s \geqslant 5$. Obviously, the paths from l to r in $ST(s)$ have two types: type-1 paths contain the root u and type-2 paths contain the edge (r_1, l_2) that joins $ST^r(s-1)$ and $ST^l(s-1)$. The shortest path of type 1 has length

$$d_{ST(s)}(l,u) + d_{ST(s)}(u,r) = (s-1) + (s-1) = 2s-2.$$

The length of the shortest path of type 2 is

$$d_{ST^l(s-1)}(l_1, r_1) + d_{ST^r(s-1)}(l_2, r_2) + 1 = 2\big(2(s-1)-2\big) + 1$$

by induction hypotheses. Therefore, the shortest path from l to r is the shortest path of type 1 and we have $d_{ST(s)}(l,r) = 2s-2$ for $s \geqslant 4$. Hence $D(ST(s)) = 2s-2$ for $s \geqslant 4$.

It can be verified that $D(CT(3)) = 6$. Note that $CT(s)$ is composed of $ST^l(s) = (V_1, E_1, u_1, l_1, r_1)$ and $ST^r(s+1) = (V_2, E_2, u_2, l_2, r_2)$. Let x and y be arbitrary two nodes of $CT(s)$. Using similar arguments as in finding $D(ST(s))$, we can obtain that $D(CT(s)) \leqslant 2s$. Next, we want to show $D_{CT(s)}(l_2, r_2) = 2s$ for $s \geqslant 4$. We can distinguish three types of paths from l_2 to r_2 in $CT(s)$ as follows: type-1 paths contain edges in $ST^r(s+1)$ only; type-2 paths contain the edge (r_1, l_2) but not the edge (u_1, u_2); and type-3 paths contain (u_1, u_2). The shortest path of type 1 has length $2s$ since $D(ST(s+1)) = 2s$ for $s \geqslant 4$. The length of the shortest path of type 2 is $2 + d_{ST^l(s)}(l,r) = 2s$ for $s \geqslant 4$. The shortest path of type 3 is $2 + d_{ST^r(s+1)}(l_2, u_2) + d_{ST^l(s)}(l_1, u_1) = 2s+1$ for $s \geqslant 4$. Therefore, the shortest path in $CT(s)$ from l_2 to r_2 has length $d_{CT(s)}(l_2, r_2) = 2s$ for $s \geqslant 4$. Hence $D(CT(s)) = 2s$ for $s \geqslant 3$ and the lemma follows. □

To prove Hamiltonian properties of $CT(s)$, we first establish the following lemmas. Let $ST(s) = (V, E, u, l, r)$. Henceforth we use $P_{ST(s)}(x,y)$ to denote a Hamiltonian path from x to y in $ST(s)$. Given a fault $f \in V \cup E$, we use $P^f_{ST(s)}(x,y)$ to denote a Hamiltonian path from x to y in $ST(s) - \{f\}$.

Lemma 2. *In $ST(s)$, there is a Hamiltonian path $P_{ST(s)}(x,y)$ for any $x, y \in \{u, l, r\}$ and $s \geqslant 2$.*

Proof. Obviously, $ST(2)$ has three Hamiltonian paths as follows:

$$P_{ST(2)}(u,l) = \langle u, r, l\rangle,$$
$$P_{ST(2)}(l,r) = \langle l, u, r\rangle, \quad \text{and}$$
$$P_{ST(2)}(r,u) = \langle r, l, u\rangle.$$

Assume that this lemma is true for $ST(s-1)$ and $s \geqslant 3$. Let $ST^l(s-1) = (V_1, E_1, u_1, l_1, r_1)$ and $ST^r(s-1) = (V_2, E_2, u_2, l_2, r_2)$ be the left subtree and the right subtree of $ST(s)$, respectively. We can construct the following paths:

$$P_{ST(s)}(u,l) = \big\langle u, u_2 \to P_{ST^r(s-1)}(u_2, l_2) \to l_2, r_1 \to P_{ST^l(s-1)}(r_1, l_1) \to l_1 = l\big\rangle,$$
$$P_{ST(s)}(l,r) = \big\langle l = l_1 \to P_{ST^l(s-1)}(l_1, u_1) \to u_1, u, u_2 \to P_{ST^r(s-1)}(u_2, r_2) \to r_2 = r\big\rangle,$$

and

$$P_{ST(s)}(r,u) = \big\langle r = r_2 \to P_{ST^r(s-1)}(r_2, l_2) \to l_2, r_1 \to P_{ST^l(s-1)}(r_1, u_1) \to u_1, u\big\rangle.$$

Thus, this lemma is proved. □

Lemma 3. *For any fault $f \in V \cup E$, there is a Hamiltonian path $P^f_{ST(s)}(x,y)$ in $ST(s) - \{f\}$ for some $x, y \in \{u, l, r\}$.*

Proof. We prove this lemma by induction on s. Consider $s = 2$. The required Hamiltonian paths $P^f_{ST(2)}(x,y)$ are given as follows:

$$P^u_{ST(2)}(l,r) = \langle l, r\rangle, \qquad P^l_{ST(2)}(r,u) = \langle r, u\rangle,$$
$$P^r_{ST(2)}(u,l) = \langle u, l\rangle, \qquad P^{(u,l)}_{ST(2)}(u,l) = \langle u, r, l\rangle,$$
$$P^{(l,r)}_{ST(2)}(l,r) = \langle l, u, r\rangle, \qquad P^{(r,u)}_{ST(2)}(r,u) = \langle r, l, u\rangle.$$

(Note that $P^{(u,l)}_{ST(2)}(r,u)$ does not exist.) Assume that this lemma is true for $ST(s-1)$ and $s \geqslant 3$.

Consider that $f \in ST^l(s-1) \cup ST^r(s-1)$. By the symmetry property of $ST(s)$, we can assume that $f \in ST^l(s-1)$. By induction hypotheses, $P^f_{ST^l(s-1)}(x,y)$ exists for some $x, y \in \{u_1, l_1, r_1\}$. Then we can construct $P^f_{ST(s)}(x,y)$ as follows:

$$P^f_{ST(s)}(l,r) = \big\langle l = l_1 \to P^f_{ST^l(s-1)}(l_1, u_1) \to u_1, u, u_2 \to P_{ST^r(s-1)}(u_2, r_2) \to r_2 = r\big\rangle \quad \text{if } P^f_{ST^l(s-1)}(l_1, u_1) \text{ exists},$$
$$P^f_{ST(s)}(u,l) = \big\langle u, u_2 \to P_{ST^r(s-1)}(u_2, l_2) \to l_2, r_1 \to P^f_{ST^l(s-1)}(r_1, l_1) \to l_1 = l\big\rangle \quad \text{if } P^f_{ST^l(s-1)}(r_1, l_1) \text{ exists},$$

$$P^f_{ST(s)}(r,u) = \langle r = r_2 \to P_{ST^r(s-1)}(r_2,l_2) \to l_2,\ r_1 \to P^f_{ST^l(s-1)}(r_1,u_1) \to u_1, u\rangle \quad \text{if } P^f_{ST^l(s-1)}(r_1,u_1) \text{ exists.}$$

It follows from the induction hypotheses that at least one of

$$P^f_{ST^l(s-1)}(l_1,u_1),$$
$$P^f_{ST^l(s-1)}(r_1,l_1), \quad \text{and}$$
$$P^f_{ST^l(s-1)}(r_1,u_1)$$

exists. Therefore, when $f \in ST^l(s-1) \cup ST^r(s-1)$, there is a $P^f_{ST(s)}(x,y)$ for some $x, y \in \{u,l,r\}$.

Consider that $f \notin ST^l(s-1) \cup ST^r(s-1)$, i.e., $f \in \{u, (u,u_1), (u,u_2), (r_1,l_2)\}$. Then we can construct $P^f_{ST(s)}(x,y)$ as follows:

$$P^u_{ST(s)}(r,l) = \langle r = r_2 \to P_{ST^r(s-1)}(r_2,l_2) \to l_2,\ r_1 \to P_{ST^l(s-1)}(r_1,l_1) \to l_1 = l\rangle,$$
$$P^{(u,u_1)}_{ST(s)}(u,l) = \langle u, u_2 \to P_{ST^r(s-1)}(u_2,l_2) \to l_2,\ r_1 \to P_{ST^l(s-1)}(r_1,l_1) \to l_1 = l\rangle,$$
$$P^{(u,u_2)}_{ST(s)}(r,u) = \langle r = r_2 \to P_{ST^r(s-1)}(r_2,l_2) \to l_2,\ r_1 \to P_{ST^l(s-1)}(r_1,u_1) \to u_1, u\rangle,$$
$$P^{(r_1,l_2)}_{ST(s)}(l,r) = \langle l = l_1 \to P_{ST^l(s-1)}(l_1,u_1) \to u_1,\ u, u_2 \to P_{ST^r(s-1)}(u_2,r_2) \to r_2 = r\rangle.$$

Therefore, there exists a Hamiltonian path $P^f_{ST(s)}(x,y)$ in $ST(s) - \{f\}$ for some $x, y \in \{u,l,r\}$. □

Theorem 1. *The Christmas tree $CT(s)$ is optimal 1-Hamiltonian for all $s \geqslant 1$.*

Proof. Since $CT(1) \cong K_4$, $CT(1)$ is 1-Hamiltonian. Now we consider the case that $s \geqslant 2$. By the symmetry property of $CT(s)$, we can assume that the fault f is in the slim tree $ST^r(s+1)$. Applying Lemma 3, we have a Hamiltonian path $P^f_{ST^r(s+1)}(x,y)$ in $ST^r(s+1) - \{f\}$ for some $x, y \in \{u_2, l_2, r_2\}$. Then we can construct a Hamiltonian cycle in $CT(s) - \{f\}$ as follows:

$$\langle l_2 \to P^f_{ST^r(s+1)}(l_2,u_2) \to u_2,\ u_1 \to P_{ST^l(s)}(u_1,r_1) \to r_1, l_2\rangle \quad \text{if } P^f_{ST^r(s+1)}(l_2,u_2) \text{ exists,}$$
$$\langle u_2 \to P^f_{ST^r(s+1)}(u_2,r_2) \to r_2,\ l_1 \to P_{ST^l(s)}(l_1,u_1) \to u_1, u_2\rangle \quad \text{if } P^f_{ST^r(s+1)}(u_2,r_2) \text{ exists,}$$
$$\langle l_2 \to P^f_{ST^r(s+1)}(l_2,r_2) \to r_2,\ l_1 \to P_{ST^l(s)}(l_1,r_1) \to r_1, l_2\rangle \quad \text{if } P^f_{ST^r(s+1)}(l_2,r_2) \text{ exists.}$$

Thus $CT(s)$ is 1-Hamiltonian. Since the degree of each node in $CT(s)$ is 3, $CT(s)$ is optimal 1-Hamiltonian. □

Let $\{ST(s_i) = (V_{s_i}, E_{s_i}, u_{s_i}, l_{s_i}, r_{s_i}) \mid s_1, s_2, \ldots, s_t$ are positive integers$\}$ be a set of disjoint slim trees. We can construct a *slim forest* $SF(s_1, s_2, \ldots, s_t)$ by adding edges $(r_{s_i}, l_{s_{i+1}})$ for all $1 \leqslant i \leqslant t-1$. For example, $SF(3,1,1,2)$ is shown in Fig. 3(a). Henceforth we use $P_{SF}(x,y)$ to denote a Hamiltonian path in $SF(s_1, s_2, \ldots, s_k)$ from x to y.

Lemma 4. *There are four Hamiltonian paths $P_{SF}(u_{s_1}, u_{s_k})$, $P_{SF}(u_{s_1}, r_{s_k})$, $P_{SF}(l_{s_1}, u_{s_k})$, and $P_{SF}(l_{s_1}, r_{s_k})$ in $SF(s_1, s_2, \ldots, s_k)$.*

Proof. It follows from Lemma 2 that there is a Hamiltonian path $P_{ST(s_i)}(l_{s_i}, r_{s_i})$ in each $ST(s_i)$. (When $s_i = 1$, the Hamiltonian path is simply a node.) We can define two paths Q_1 and Q_2 in $SF(s_1, s_2, \ldots, s_k)$ as follows:

$$Q_1 = \langle l_{s_2} \to P_{ST(s_2)}(l_{s_2}, r_{s_2}) \to r_{s_2}, l_{s_3} \to \cdots \to r_{s_{k-1}},\ l_{s_k} \to P_{ST(s_k)}(l_{s_k}, u_{s_k}) \to u_{s_k}\rangle,$$
$$Q_2 = \langle l_{s_2} \to P_{ST(s_2)}(l_{s_2}, r_{s_2}) \to r_{s_2}, l_{s_3} \to \cdots \to r_{s_{k-1}},\ l_{s_k} \to P_{ST(s_k)}(l_{s_k}, r_{s_k}) \to r_{s_k}\rangle.$$

Thus we can construct the four Hamiltonian paths as follows:

$$P_{SF}(u_{s_1}, u_{s_k}) = \langle u_{s_1} \to P_{ST(s_1)}(u_{s_1}, r_{s_1}) \to r_{s_1},\ l_{s_2} \to Q_1 \to u_{s_k}\rangle,$$
$$P_{SF}(u_{s_1}, r_{s_k}) = \langle u_{s_1} \to P_{ST(s_1)}(u_{s_1}, r_{s_1}) \to r_{s_1},\ l_{s_2} \to Q_2 \to r_{s_k}\rangle,$$
$$P_{SF}(l_{s_1}, u_{s_k}) = \langle l_{s_1} \to P_{ST(s_1)}(l_{s_1}, r_{s_1}) \to r_{s_1},\ l_{s_2} \to Q_1 \to u_{s_k}\rangle,$$
$$P_{SF}(l_{s_1}, r_{s_k}) = \langle l_{s_1} \to P_{ST(s_1)}(l_{s_1}, r_{s_1}) \to r_{s_1},\ l_{s_2} \to Q_2 \to r_{s_k}\rangle.$$

The lemma follows. □

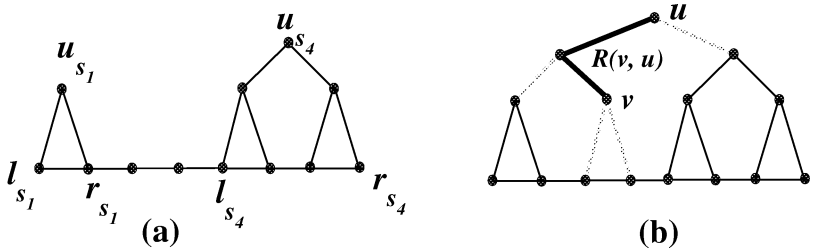

Fig. 3. (a) The slim forest $SF(3,1,1,2)$. (b) The graph $ST(4) - R(v,u)$.

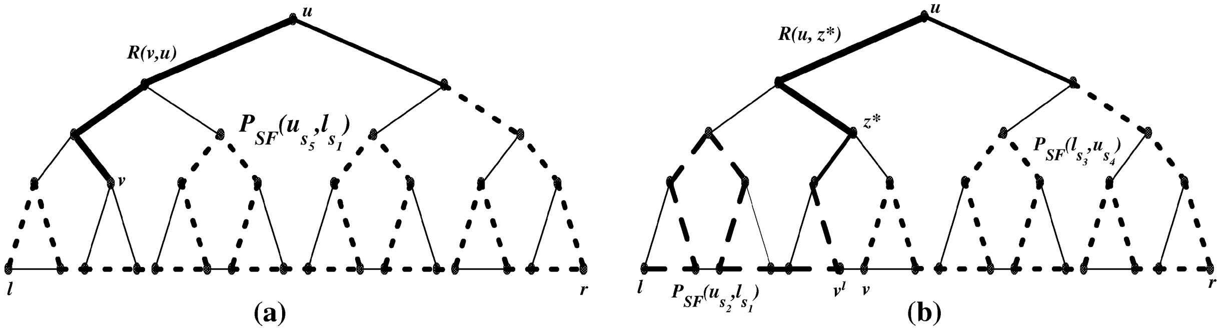

Fig. 4. $P_{ST(5)}(v,l)$ for a node v of the left subtree.

Lemma 5. *Let v be an arbitrary node in the left subtree of $ST(s) = (V, E, u, l, r)$ with $v \neq l$ and $s \geqslant 2$. Then there is a Hamiltonian path $P_{ST(s)}(v,l)$ in $ST(s)$ for all $s \geqslant 2$.*

Proof. First, consider that v is a non-leaf node. It is known that $BT(s) - R(v,u)$ is a set of disjoint complete binary trees. Since v is a non-leaf node, all of the leaf edges are contained in $ST(s) - R(v,u)$. Therefore, $ST(s) - R(v,u)$ is just $BT(s) - R(v,u)$ plus all of the leaf edges in $ST(s)$, which constitutes a set of slim trees and the leaf edges between slim trees. Therefore, $ST(s) - R(v,u)$ forms a slim forest $SF(s_1, s_2, \ldots, s_{k+1})$ for $1 \leqslant s_1, s_2, \ldots, s_{k+1} \leqslant s-1$, where k is the length of $R(v,u)$. (An example of $ST(4) - R(v,u)$ which yields a slim forest $SF(3,1,1,2)$ is illustrated in Fig. 3(b).) Applying Lemma 4, we can define

$$P_{ST(s)}(v,l) = \langle v \to R(v,u) \to u,\ u_{s_{k+1}} \to P_{SF}(u_{s_{k+1}}, l_{s_1}) \to l_{s_1} = l\rangle.$$

(An example of $P_{ST(5)}(v,l)$ is illustrated in Fig. 4(a) in which $P_{SF}(u_{s_{k+1}}, l_{s_1})$ is represented by dotted lines.)

Next, consider that v is a leaf node. Since $v \neq l$, v has a left sibling v^l. Let z^* be the lowest common ancestor of v^l and v. Clearly, v^l and v is connected by an edge, denoted by (v, v^l). Since z^* is the lowest common ancestor of v^l and v, z^* is a non-leaf node. Thus $ST(s) - R(z^*, u)$ forms a slim forest $SF(s_1, s_2, \ldots, s_{k+1})$ for some $s_1, s_2, \ldots, s_{k+1}$, where k is the length of $R(z^*, u)$. Let T_1 denote the slim tree rooted at z^*. Since z^* is the lowest common ancestor of v^l and v, the node v^l must be in the left subtree of T_1 and v in the right subtree of T_1. Therefore, v and v^l are in different slim trees of $ST(s) - R(z^*, u)$ and the edge (v, v^l) is an edge between different slim trees of the slim forest. Removing (v, v^l) disconnects the slim forest. Consequently, the graph $ST(s) - R(z^*, u) - (v, v^l)$ forms two disjoint slim forests $SF(s_1, s_2, \ldots, s_j)$ and $SF(s_{j+1}, s_{j+2}, \ldots, s_{k+1})$. Applying Lemma 4, we can define

$$P_{ST(s)}(v,l) = \langle v = l_{s_{j+1}} \to P_{SF}(l_{s_{j+1}}, u_{s_{k+1}}) \to u_{s_{k+1}},\ u \to R(u, z^*) \to z^*,\ u_{s_j} \to P_{SF}(u_{s_j}, l_{s_1}) \to l_{s_1} = l\rangle,$$

where $P_{SF}(l_{s_{j+1}}, u_{s_{k+1}})$ and $P_{SF}(u_{s_j}, l_{s_1})$ are Hamiltonian paths with the specified source and destination in $SF(s_{j+1}, s_{j+2}, \ldots, s_{k+1})$ and $SF(s_1, s_2, \ldots, s_j)$, respectively. (An example of $P_{ST(5)}(v, l)$ is illustrated in Fig. 4(b), in which $ST(5) - R(z^*, u) - (v, v^l)$ forms the slim forests $SF(3, 2)$ and $SF(2, 4)$, and the corresponding Hamiltonian paths are represented by dashed lines and dotted lines, respectively.)

Hence this lemma is proved. □

It follows from the symmetry property of $ST(s)$ and Lemma 5 that there is a Hamiltonian path $P_{ST(s)}(v, r)$ for any node v with $v \neq r$ in the right subtree of $ST(s)$.

Lemma 6. *Let v be any node of $ST(s) = (V, E, u, l, r)$ with $s \geqslant 2$. Then there are two Hamiltonian paths $P_{ST(s)}(v, x_i)$ in $ST(s)$, where $x_i \in \{u, l, r\}$ for $i = 1, 2$ and $x_1 \neq x_2$.*

Proof. We prove this lemma by induction on s. Since $ST(2) \cong K_3$, the lemma is obviously true for $s = 2$. Assume that this lemma is true for $s - 1$ and $s \geqslant 3$. Let v be any node of $ST(s)$. When $v \in \{u, l, r\}$, the lemma is proved following from Lemma 2. Now, we consider $v \notin \{u, l, r\}$. By the symmetry property of $ST(s)$, we can assume that $v \in V(ST^l(s-1))$. Applying Lemma 5, we can construct a Hamiltonian path $P_{ST(s)}(v, l)$. Now, we need to construct the second Hamiltonian path $P_{ST(s)}(v, x)$ for $x = u$ or r. By induction hypotheses, there are two Hamiltonian paths $P_{ST^l(s-1)}(v, y_1)$ and $P_{ST^l(s-1)}(v, y_2)$ in $ST^l(s-1)$ for some $y_1, y_2 \in \{u_1, l_1, r_1\}$. Obviously, one of y_i, say y_1, is u_1 or r_1. We define $P_{ST(s)}(v, x)$ for $x \in \{r, u\}$ as follows:

$$P_{ST(s)}(v, r) = \bigl\langle v \to P_{ST^l(s-1)}(v, u_1) \to u_1, u, u_2 \to P_{ST^r(s-1)}(u_2, r_2) \to r_2 = r\bigr\rangle \quad \text{if } y_1 = u_1, \text{ and}$$

$$P_{ST(s)}(v, u) = \bigl\langle v \to P_{ST^l(s-1)}(v, r_1) \to r_1, l_2 \to P_{ST^r(s-1)}(l_2, u_2) \to u_2, u\bigr\rangle \quad \text{if } y_1 = r_1.$$

Thus we obtain two Hamiltonian paths from v to $x_i \in \{u, l, r\}$ for $i = 1, 2$ and $x_1 \neq x_2$, and the lemma follows. □

Lemma 7. *For any two different nodes v_1 and v_2 in $ST(s) = (V, E, u, l, r)$ with $s \geqslant 2$, there are two node-disjoint paths $P_1 = \langle v_1, \ldots, x_1\rangle$ and $P_2 = \langle v_2, \ldots, x_2\rangle$ such that $V(P_1) \cup V(P_2) = V$ and $x_1, x_2 \in \{u, l, r\}$.*

Proof. We prove this lemma by induction on s. The lemma is obviously true for $s = 2$. Assume that this lemma is true for $ST(s-1)$ and $s \geqslant 3$. We distinguish the following two cases.

Case 1. $u \in \{v_1, v_2\}$. Without loss of generality, we can assume that $v_1 = u$ and $v_2 \in V(ST^l(s-1))$. Applying Lemma 6, we know that there are two Hamiltonian paths $P_{ST^l(s-1)}(v_2, x_i)$ in $ST^l(s-1)$ with $x_i \in \{u_1, l_1, r_1\}$ and $i = 1, 2$. If $l_1 \in \{x_1, x_2\}$, then $P_1 = \langle v_1 = u, u_2 \to P_{ST^r(s-1)}(u_2, r_2) \to r_2 = r\rangle$ and $P_2 = \langle v_2 \to P_{ST^l(s-1)}(v_2, l_1) \to l_1 = l\rangle$ are the desired paths, as illustrated in Fig. 5(a). Otherwise, $r_1 \in \{x_1, x_2\}$; then $P_1 = \langle v_1 = u\rangle$ and $P_2 = \langle v_2 \to P_{ST^l(s-1)}(v_2, r_1) \to r_1, l_2 \to P_{ST^r(s-1)}(l_2, r_2) \to r_2 = r\rangle$ are the desired paths.

Case 2. $u \notin \{v_1, v_2\}$. We first consider that v_1 and v_2 are in the same subtree of $ST(s)$. With the symmetry property of $ST(s)$, we can assume that v_1, v_2 are in the left subtree $ST^l(s-1) = (V_1, E_1, u_1, l_1, r_1)$. By induction hypotheses, there are two node-disjoint paths $P_1' = \langle v_1, \ldots, y_1\rangle$ and $P_2' = \langle v_2, \ldots, y_2\rangle$ with $V(P_1') \cup V(P_2') = V_1$ for some $y_1, y_2 \in \{u_1, l_1, r_1\}$. When $y_1, y_2 \in \{u_1, l_1\}$, we can assume without loss of generality that $y_1 = u_1$ and $y_2 = l_1$. Then $P_1 = \langle v_1 \to P_1' \to u_1, u, u_2 \to P_{ST^r(s-1)}(u_2, r_2) \to r_2 = r\rangle$ and $P_2 = P_2'$ are the desired paths, as illustrated in Fig. 5(b). When $y_1, y_2 \in \{u_1, r_1\}$, we can assume without loss of generality that $y_1 = u_1$ and $y_2 = r_1$. Then $P_1 = \langle v_1 \to P_1' \to u_1, u\rangle$ and $P_2 = \langle v_2 \to P_2' \to r_1, l_2 \to P_{ST^r(s-1)}(l_2, r_2) \to r_2 = r\rangle$ are the desired paths. When $y_1, y_2 \in \{l_1, r_1\}$, we can assume without loss of generality that $y_1 = l_1$ and $y_2 = r_1$. Then $P_1 = P_1'$ and $P_2 = \langle v_2 \to P_2' \to r_1, l_2 \to P_{ST^r(s-1)}(l_2, u_2) \to u_2, u\rangle$ are the desired paths.

Next, we consider that v_1 and v_2 are in different subtrees of $ST(s)$. Without loss of generality, we can assume that v_1 is in $ST^l(s-1)$ and v_2 is in $ST^r(s-1)$. Applying Lemma 6, we can obtain Hamiltonian paths $P_{ST^l(s-1)}(v_1, y_i)$ and $P_{ST^r(s-1)}(v_2, w_i)$ for $y_i \in \{u_1, l_1, r_1\}$ and $w_i \in \{u_2, l_2, r_2\}$ with $i = 1, 2$. We distinguish all of the possible conditions in the following subcases.

Case 2.1. $u_1 \in \{y_1, y_2\}$ and $r_2 \in \{w_1, w_2\}$. $P_1 = \langle v_1 \to P_{ST^l(s-1)}(v_1, u_1) \to u_1, u\rangle$ and $P_2 = \langle v_2 \to P_{ST^r(s-1)}(v_2, r_2) \to r_2 = r\rangle$ are the desired paths.

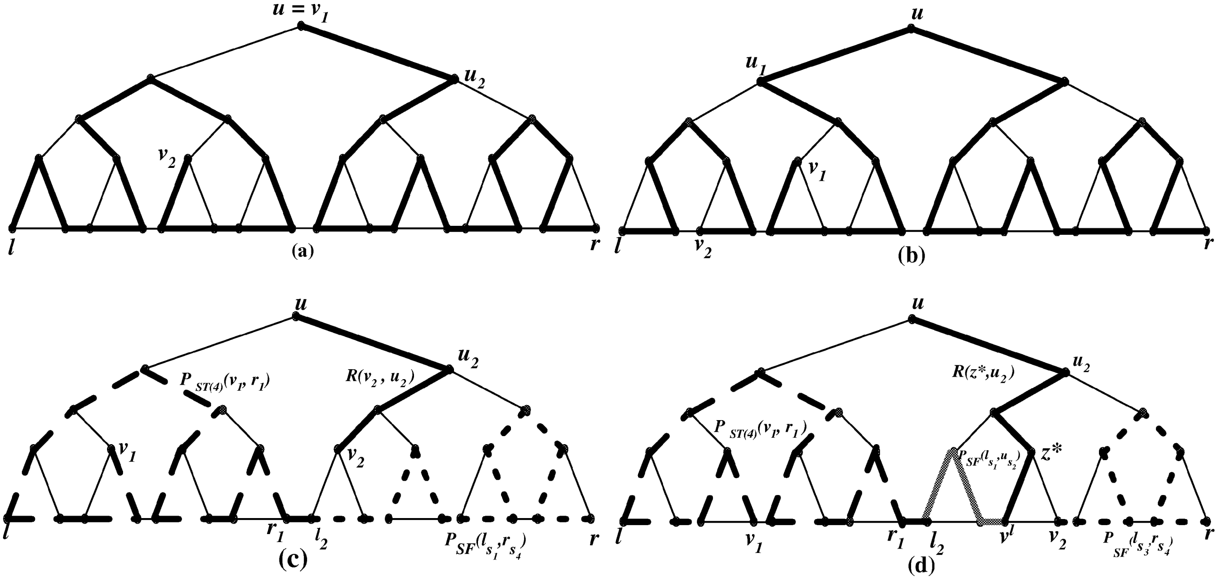

Fig. 5. Two node-disjoint paths of $ST(5)$.

Case 2.2. $l_1 \in \{y_1, y_2\}$ and $u_2 \in \{w_1, w_2\}$. We construct the two paths $P_1 = \langle v_1 \to P_{ST^l(s-1)}(v_1, l_1) \to l_1 = l\rangle$ and $P_2 = \langle v_2 \to P_{ST^r(s-1)}(v_2, u_2) \to u_2, u\rangle$.

Case 2.3. $r_1 \in \{y_1, y_2\}$ and $l_2 \in \{w_1, w_2\}$. Consider that v_2 is a non-leaf node. It follows from the proof in Lemma 5 that $ST^r(s-1) - R(v_2, u_2)$ forms a slim forest $SF(s_1, s_2, \ldots, s_{k+1})$ where k is the length of $R(v_2, u_2)$. Applying Lemma 4, we can obtain that $P_1 = \langle v_1 \to P_{ST^l(s-1)}(v_1, r_1) \to r_1, l_2 = l_{s_1} \to P_{SF}(l_{s_1}, r_{s_{k+1}}) \to r_{s_{k+1}} = r_2 = r\rangle$ and $P_2 = \langle v_2 \to R(v_2, u_2) \to u_2, u\rangle$ are the desired paths, as illustrated in Fig. 5(c).

Next, consider that v_2 is a leaf node. Since $l_2 \in \{w_1, w_2\}$, it follows that $v_2 \neq l_2$. Let v^l be the left sibling of v_2 and (v_2, v^l) be the edge linking v_2 and v^l. Let z^* be the lowest common ancestor of v_2 and v^l. The graph $ST^r(s-1) - R(z^*, u_2) - (v_2, v^l)$ forms two slim forests $SF(s_1, s_2, \ldots, s_j)$ and $SF(s_{j+1}, s_{j+2}, \ldots, s_{k+1})$ where k is the length of $R(z^*, u_2)$. Applying Lemma 4, we obtain two desired paths $P_1 = \langle v_1 \to P_{ST^l(s-1)}(v_1, r_1) \to r_1, l_2 = l_{s_1} \to P_{SF}(l_{s_1}, u_{s_j}) \to u_{s_j}, z^* \to R(z^*, u_2) \to u_2, u\rangle$ and $P_2 = \langle v_2 = l_{s_{j+1}} \to P_{SF}(l_{s_{j+1}}, r_{s_{k+1}}) \to r_{s_{k+1}} = r_2 = r\rangle$, where $P_{SF}(l_{s_1}, u_{s_j})$ and $P_{SF}(l_{s_{j+1}}, r_{s_{k+1}})$ are two Hamiltonian paths in $SF(s_1, s_2, \ldots, s_j)$ and $SF(s_{j+1}, s_{j+2}, \ldots, s_{k+1})$, respectively, as illustrated in Fig. 5(d). Thus, this lemma is proved. □

Theorem 2. *The Christmas tree $CT(s)$ is optimal Hamiltonian-connected.*

Proof. Since $CT(1) \cong K_4$, $CT(1)$ is Hamiltonian-connected. Now we consider the case that $s \geqslant 2$. By the symmetry property of $CT(s)$, we can assume that any two nodes of $CT(s)$ are in the $(s+1)$th slim tree $ST^r(s+1)$. It follows from Lemma 7 that for any two different nodes v_1, v_2 in $ST^r(s+1)$, there exist two node-disjoint paths $P_1 = \langle v_1, \ldots, x_1\rangle$ and $P_2 = \langle x_2, \ldots, v_2\rangle$ such that $V(P_1) \cup V(P_2) = V_2$ and $x_1, x_2 \in \{u_2, l_2, r_2\}$. When $\{x_1, x_2\} = \{u_2, l_2\}$, we can assume without loss of generality $x_1 = u_2$ and $x_2 = l_2$. Then $\langle v_1 \to P_1 \to u_2, u_1 \to P_{ST^l(s)}(u_1, r_1) \to r_1, l_2 \to P_2 \to v_2\rangle$ is a Hamiltonian path. When $\{x_1, x_2\} = \{l_2, r_2\}$, we can assume without loss of generality that $x_1 = l_2$ and $x_2 = r_2$. Then $\langle v_1 \to P_1 \to l_2, r_1 \to P_{ST^l(s)}(r_1, l_1) \to l_1, r_2 \to P_2 \to v_2\rangle$ is a Hamiltonian path. When $\{x_1, x_2\} = \{r_2, u_2\}$, we can assume without loss of generality that $x_1 = r_2$ and $x_2 = u_2$. Then $\langle v_1 \to P_1 \to r_2, l_1 \to P_{ST^l(s)}(l_1, u_1) \to u_1, u_2 \to P_2 \to v_2\rangle$ is a Hamiltonian path. Thus, $CT(s)$ is Hamiltonian-connected.

Since the number of edges of $CT(s)$ is $3n/2$, $CT(s)$ is optimal Hamiltonian-connected following from the Moon's result [6]. This theorem follows. □

4. Concluding remarks

In this paper, we propose a new family of interconnection networks called Christmas tree, denoted by $CT(s)$. We prove that $CT(s)$ is optimal 1-Hamiltonian and Hamiltonian-connected. Obviously, $CT(s)$ is a planar graph. We also prove that the diameter of $CT(s)$ is $2s$, that is, 2 times of Moore bound.

We also note that the Christmas tree is constructed from the complete binary tree. The complete binary tree is one of the most important architectures for interconnection networks [4]. Another important architecture called *fat tree* which is used in CM5 machine [5] can be constructed from a slim tree $ST(s) = (V, E, u, l, r)$ by adding an edge (l, r). Since the Christmas tree has a number of nice properties and a very similar topology as the complete binary tree and the fat tree, we believe that the Christmas tree is another candidate for interconnection networks.

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