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Bilevel Hysteretic Service Rate Control For Bulk Arrival Queue

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ABSTRACT

This paper concerns $M^{[X]}/M/1$ queue. The number of customers in each arriving unit is a random variable. There is two control threshold values K and N, K is smaller than N. Service rate is switched from u to tu whenever the system size increases to N. The tu rate is switched to u when the system size drops to the value K. We derive the steady-state probabilities of the number of customers in system and the expected number of customers in system. A cost model is introduced for the service cost, queuing cost, and switching cost.

Keywords: Queue, Bulk arrival, Service control.

INTRODUCTION

Hysteretic service rate control policy has studied extensively in the literature. The control policy has been studied by number of authors (Gebhard, 1967; Crabill, Gross and Magazine, 1977; Lu and Serfozo, 1984; Teghem, 1986; Gray et al., 1992; Lee et al., 1998; Lin and Kumar, 1984; Wang, 1993). In this paper, we consider $M^{[X]}/M/1$ queuing system with unlimit size. The arrival stream forms a Poisson process in which the number of customers in each arriving unit is a random variable X, with probability density c_x. There is two control threshold values K and N, K is smaller than N. Service rate is switched from u to tu whenever the system size increases to N. The tu rate is switched to u when the system size drops to the value K. We derive the steady-state probabilities of the number of customers in system and the expected number of customers in system. A cost model is introduced for the service cost, queuing cost, and switching cost.

THE MAIN RESULTS

This model may be analyzed by continuous time parameter Markov chain. We divide the state of the system into two classes. $\Gamma_1 = \{(n,1); n = 0, \mathrm{K}, N\}$ be the state in which n customers in system and the service rate is u. $\Gamma_2 = \{(n,2); n = K+1, \mathrm{K}\}$ be the state in which n customers in system and the service rate is tu. Let $\pi(n,1)$ denotes the steady-state probability of the state $(n, 1)$, $\pi(n,2)$ denotes the steady-state probability of the state $(n, 2)$. $\pi_0, \pi(0,1)$ denotes the probability of empty state (0,1)

The steady-state equations are given as follows:

$$0 = -\lambda\pi(0, 1) + u\pi(1, 1), \tag{1}$$

$$0 = -(\lambda + u)\pi(n, 1) + \sum_{i=1}^{n} \lambda c_i \pi(n-i, 1) + u\pi(n+1, 1), 1 \le n < N, \ n \ne K, \tag{2}$$

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$$0=-(\lambda+u)\pi(K,\ 1)+\sum_{i=1}^{K}\lambda c_i\pi(K-i,\ 1)+tu\pi(K+1,\ 2)+u\pi(K+1,\ 1), \tag{3}$$

$$0=-(\lambda+u)\pi(N,\ 1)+\sum_{i=1}^{N}\lambda c_i\pi(N-i,\ 1), \tag{4}$$

$$0=-(\lambda+tu)\pi(K+1,\ 2)+tu\pi(K+2,\ 2), \tag{5}$$

$$0=-(\lambda+tu)\pi(n,\ 2)+\sum_{i=1}^{n-K-1}\lambda c_i\pi(n-i,\ 2)+tu\pi(n+1,\ 2),\quad n=K+2,\ \mathrm{K}\ ,N, \tag{6}$$

$$0=-(\lambda+tu)\pi(n,\ 2)+\sum_{i=1}^{n}\lambda c_i\pi(n-i,\ 1)+\sum_{i=1}^{n-K-1}\lambda c_i\pi(n-i,\ 2)+tu\pi(n+1,\ 2),\qquad n=N+1,\ \mathrm{K}\ , \tag{7}$$

Multiply equations (1)~(7) with appropriate z^n, and take summation.

Let $C(z)=\sum_{n=1}^{\infty}c_n z^n$, $\pi_1(z)=\sum_{i=0}^{N}\pi(i,1)z^i$ and $\pi_2(z)=\sum_{i=K+1}^{\infty}\pi(i,2)z^i$. We obtain,

$$\pi_1(z)[u-(\lambda+u)z+\lambda zC(z)]+\pi_2(z)[tu-(\lambda+tu)z+\lambda zC(z)]=u\pi_0(1-z) \tag{8}$$

Let $\pi(n,\ 1)=\pi_n\pi_0,\ 1\le n\le N$, and let $\pi(n,\ 2)=\psi_n\pi_0,\ n\ge K+1$. Let π'_n $(0\le n\le N)$ be the coefficient of the probability of the empty state in typical $M^{[X]}/M/1$ queue with service rate u.

Property 1

π'_n, $n\ge 0$, that satisfy the following relation

$$\pi'_0=1,\pi'_1=\phi\ \text{, where } \phi=\lambda/u \tag{9a}$$

$$\pi'_{i+1}=[(1+\phi)\pi'_i-\phi\sum_{j=1}^{i}c_j\pi'_{i-j}] \tag{9b}$$

Property 2

$\pi_n=\pi'_n, 0\le n\le K$,

$$\pi_{K+i}=\pi'_{K+i}+h_i\psi_{K+i},\ i=1,\ \mathrm{K}\ ,N-K, \tag{10a}$$

Where

$$h_1=-t, h_i=(1+\phi)h_{i-1}-\phi\sum_{j=1}^{i-2}c_j h_{i-j-1},\ i=2,\mathrm{K}\ ,N-K, \tag{10b}$$

$$\psi_{K+1}=\frac{\phi\sum_{i=1}^{N}c_i\pi'_{N-i}-(1+\phi)\pi'_n}{(1+\phi)h_{N-K}-\phi\sum_{i=1}^{N-K-1}c_i h_{N-K-i}} \tag{11}$$

Define $\hat{\pi}_i(z)=\pi_i(z)/\pi_0,\ i=1,\ 2,$

By the boundary condition, since $\pi_1(1)+\pi_2(1)=1$, that is, $\pi_1(1)+\pi_2(1)=\pi_0[\hat{\pi}_1(1)+\hat{\pi}_2(1)]$

Therefore,

$$\pi_0=1/[\hat{\pi}_1(1)+\hat{\pi}_2(1)] \tag{12}$$

Furthermore,

$$\pi_1(z) = \sum_{i=0}^{N} \pi_i \pi_0 z^i = \pi_0 [\sum_{i=0}^{N} \pi_i' z^i + \psi_{K+1} \sum_{i=K+1}^{N} h_{i-K} z^i]$$

Divide above equation by π_0, we obtain,

$$\hat{\pi}_1(z) = \sum_{i=0}^{N} \pi_i' z^i + \psi_{K+1} \sum_{i=K+1}^{N} h_{i-K} z^i = \sum_{i=0}^{N} \pi_i' z^i + \psi_{K+1} \sum_{i=1}^{N-K} h_i z^{K+i}, \tag{13}$$

Let $S = \hat{\pi}_1(1)$, we obtain,

$$S = \sum_{i=0}^{N} \pi_i' + \psi_{K+1} \sum_{i=1}^{N-K} h_i, \tag{14}$$

Take the first derivative of equation (13) with respect to z

$$\frac{d}{dz}\hat{\pi}_1(z) = \sum_{i=1}^{N} i\pi_i' z^{i-1} + \psi_{K+1} \sum_{i=1}^{N-K} (K+i) h_i z^{K+i-1},$$

Let $T = \frac{d}{dz}\hat{\pi}_1(1)$, we obtain,

$$T = \sum_{i=1}^{N} i\pi_i' + \psi_{K+1} \sum_{i=1}^{N-K} (K+i) h_i, \tag{15}$$

Divide equation (8) by π_0, we obtain,

$$\hat{\pi}_1(z)[u-(\lambda+u)z+\lambda z C(z)] + \pi_2(z)[tu-(\lambda+tu)z+\lambda z C(z)] = u(1-z) \tag{16}$$

Take first derivative of (16) with respect to z and evaluate at z = 1, we obtain,

$$\hat{\pi}_1(1)[-u+\lambda E(x)] + \hat{\pi}_2(1)[-tu+\lambda E(x)] = -u$$

Let $U = \hat{\pi}_2(1)$, $\rho_1 = \phi E(X)$, the first traffic intensity. $\rho_2 = \phi E(X)/t$, the second traffic intensity. Assume $\rho_2 < 1$. We get

$$U = \frac{-u - \hat{\pi}_1(1)[-u+\lambda E(X)]}{-tu+\lambda E(X)} = \frac{\rho_2[-1+S(1-\rho_1)}{\rho_1(\rho_2-1)} \tag{17}$$

From (12)–(17), we obtain,

$$\pi_0 = \frac{1}{\hat{\pi}_1(1)+\hat{\pi}_2(1)} = \frac{1}{S+U} = \frac{\rho_1(1-\rho_2)}{\rho_2+S(\rho_1-\rho_2)} \tag{18}$$

Take second derivative of equation (16) with respect to z, and evaluate at z = 1, we obtain,

$$2\frac{d}{dz}\hat{\pi}_2(1)[-tu+\lambda E(X)] = -2\frac{d}{dz}\hat{\pi}_1(1)[-u+\lambda E(X)] - \{2\lambda E(X) + \lambda E[X(X-1)]\}[\hat{\pi}_1(1)+\hat{\pi}_2(1)]$$

Let $V = \frac{d}{dz}\hat{\pi}_2(1)$, then,

$$V = \frac{2T(\rho_1-\rho_2) + \rho_2\{2\rho_1 + \phi E[X(X-1)]\}(S+U)}{2\rho_1(1-\rho_2)} \tag{19}$$

From (12)–(19), we can obtain the expected number of customers in system L,

$$L = \frac{T(\rho_1-\rho_2)}{\rho_2+S(1-\rho_2)} + \frac{\rho_2\{2\rho_1+\phi E[X(X-1)]\}}{2\rho_1(1-\rho_2)} \tag{20}$$

SPECIAL CASES

It is interesting that for various combination of (N, K) the model generalize several models.

(a) K=N=0, Pr(X=1)=1. The model is reduced to typical $M/M/1$ queuing model with service rate tu.

(b) K=$N \neq 0$, $Pr(X=1)$=1. The model reduce to $M/M/1$ queuing model with state dependent

(c) K=N=0. The model reduce to regular $M^{[X]}/M/1$ queuing model with service rate tu

(d) K=$N \neq 0$. The model is reduced to $M^{[X]}/M/1$ queuing model with state dependent.

(e) $Pr(X=1)$=1. The model reduce to $M/M/1$ with bilevel hysteretic service rate control (Gebhard, 1967)

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