The Circular Chromatic Number of the Mycielskian of G_k^d

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Abstract: In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski developed a graph transformation that transforms a graph G into a new graph $\mu(G)$, we now call the Mycielskian of G, which has the same clique number as G and whose chromatic number equals $\chi(G)+1$. Chang, Huang, and Zhu [G. J. Chang, L. Huang, & X. Zhu, Discrete Math, to appear] have investigated circular chromatic numbers of Mycielskians for several classes of graphs. In this article, we study circular chromatic numbers of Mycielskians for another class of graphs G_k^d. The main result is that $\chi_c(\mu(G_k^d)) = \chi(\mu(G_k^d))$, which settles a problem raised in [G. J. Chang, L. Huang, & X. Zhu, Discrete Math, to appear, and X. Zhu, to appear]. As $\chi_c(G_k^d) = \frac{k}{d}$ and $\chi(G_k^d) = \lceil \frac{k}{d} \rceil$, consequently, there exist graphs G such that $\chi_c(G)$ is as close to $\chi(G)-1$ as you want, but $\chi_c(\mu(G)) = \chi(\mu(G))$.
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1. INTRODUCTION

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski [9] developed an interesting graph transformation as follows. For graph

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$G = (V, E)$ with vertex set V and edge set E, the *Mycielskian* of G is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$, and edge set $E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$. For $k \geq 2$, let $\mu^k(G) = \mu(\mu^{k-1}(G))$.

Mycielski showed that $\chi(\mu(G)) = \chi(G) + 1$ for any graph G, and $\omega(\mu(G)) = \omega(G)$ for any graph G with at least one edge. Hence, $\mu^k(K_2)$ is a triangle-free graph of chromatic number $k + 2$. Besides the interesting properties involving their chromatic numbers and cliques numbers, Mycielski's graphs also have some other parameters that behave in a predictable way. For example, it was shown by Larsen, Propp, and Ullman [8] that $\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$ for any graph G, where $\chi_f(G)$ is the fractional chromatic number of a graph. Mycielski's graphs were also used by Fisher [5] as examples of optimal fractional colorings that have large denominators. Many interesting properties for circular chromatic numbers of Mycielski's graphs were proved by Chang, Huang, and Zhu [4]. Yet more questions concerning this topic remain open. In this article, we investigate circular chromatic numbers of Mycielskians of the graphs G_k^d, to be defined below, and settle a problem raised in [4] (also see Problem 7.22 in [12]).

The circular chromatic number of a graph is a natural generalization of the chromatic number, introduced by Vince [10] under the name "star chromatic number." For a good survey, see [12]. Suppose k and d are positive integers such that $k \geq 2d$. A (k, d)-*coloring* of a graph $G = (V, E)$ is a mapping ϕ from V to $\{0, 1, \ldots, k-1\}$ such that $d \leq |\phi(x) - \phi(y)| \leq k - d$ for any edge xy in E. In the definition, we call $\phi(x)$ the *color* of x. The *circular chromatic number* $\chi_c(G)$ of G is the infimum of the ratios $\frac{k}{d}$ for which there exists a (k, d)-coloring of G. Note that Vince [10] proved that the infimum is attained for some $k \leq |V(G)|$.

Note that a $(k, 1)$-coloring of a graph G is just an ordinary k-coloring of G. It follows that $\chi_c(G) \leq \chi(G)$ for any graph G. On the other hand, it has been shown [1, 10, 11] that $\chi(G) - 1 < \chi_c(G)$. Therefore, $\chi(G) = \lceil \chi_c(G) \rceil$. However, two graphs with the same chromatic number may have different circular chromatic numbers. In some sense, $\chi_c(G)$ is a refinement of $\chi(G)$ and it contains more information about the graph.

It was shown [4] that $\chi_c(\mu(G)) = \chi(\mu(G))$ for several classes of graphs G, and also $\chi_c(\mu(H)) < \chi(\mu(H))$ for some classes of graphs H. However, it seems difficult to characterize those graphs G for which $\chi_c(\mu(G)) = \chi(\mu(G))$. For two positive integers k and d such that $k \geq 2d, G_k^d$ is the graph with vertex set $\{0, 1, \ldots, k-1\}$ in which ij is an edge if and only if $d \leq |i - j| \leq k - d$. It is easy to see (also [1]) that a graph G is (k, d)-colorable if and only if there exists a homomorphism from G to G_k^d. Therefore, in the study of circular chromatic numbers, the graphs G_k^d play the same role that complete graphs K_n do in the study of chromatic numbers. It was shown in [4] that $\chi_c(\mu(K_n)) = \chi(\mu(K_n))$ for any $n \geq 3$. We consider the analogous problem: does $\chi_c(\mu(G_k^d)) = \chi(\mu(G_k^d))$?

The main result of this article is to give a positive answer to the above problem. Also, since $\chi_c(G_k^d) = \frac{k}{d}$ and $\chi(G_k^d) = \lceil \frac{k}{d} \rceil$ (see Vince [10]), a consequence is that there exist graphs G such that $\chi_c(G)$ is as close to $\chi(G) - 1$ as you want, but $\chi_c(\mu(G)) = \chi(\mu(G))$. Although it is still not known what determines $\chi_c(\mu(G)) =$

$\chi(\mu(G))$, the result shows that the circular chromatic number of a graph G does not determine if $\chi_c(\mu(G)) = \chi(\mu(G))$.

2. CIRCULAR CHROMATIC NUMBER OF $\mu(G_k^d)$

The main result of this article is the following.

Theorem 1. *$\chi_c(\mu(G_k^d)) = \chi(\mu(G_k^d)) = \lceil \frac{k}{d} \rceil + 1$ for any positive integer $k > 2d$.*

Note that for $k = 2d$, we have $G_k^d \cong dK_2$ and $\mu(G_k^d)$ is the graph obtained from d copies of C_5 by identifying one vertex in each copy. Therefore, $\chi_c(\mu(G_k^d)) = 2.5 < 3 = \chi(\mu(G_k^d))$.

Also, since $\chi_c(G_k^d) = \frac{k}{d}$ and $\chi_c(\mu(G_k^d)) = \chi(\mu(G_k^d))$ for any positive integers $k > 2d$, we also have the following consequence.

Corollary 1. *There exists a graph G such that $\chi_c(G)$ is as close to $\chi(G) - 1$ as you want, but $\chi_c(\mu(G)) = \chi(\mu(G))$.*

In the remaining of this section, we shall prove Theorem 1. The following lemma was proved in [4], which takes care of a special case of the main theorem.

Lemma 1. *If $\chi(G) = 3$, then $\chi_c(\mu(G)) = \chi(\mu(G)) = 4$.*

For an n-coloring $c : V(G) \mapsto \{0, 1, \ldots, n-1\}$ of $G = (V, E)$, we denote by $D_c(G)$ the directed graph with vertex set V such that there exists an arc from x to y if and only if $xy \in E$ and $c(x) + 1 \equiv c(y) (\mathrm{mod} n)$. It was shown in [6] that an n-chromatic graph G satisfies $\chi_c(G) < n$ if and only if G has an n-coloring c for which $D_c(G)$ is acyclic. This result was refined [4] to the following lemma, which is useful for studying circular chromatic numbers of Mycielski's graphs.

Lemma 2. *If $\chi_c(\mu(G)) < \chi(\mu(G)) = n$, then there exists an n-coloring c of $\mu(G)$ such that $D_c(\mu(G))$ is acyclic, $c(u) = 1$, and $c(x') \notin \{0, 1\}$ for all $x' \in V'$. Moreover, for any such coloring c, there is an edge $xy \in E(G)$ such that $\{c(x), c(y)\} = \{0, 1\}$ and $c(x') = c(y')$.*

Write $k = dr + i$, where $d \geq 2$ and $1 \leq i \leq d$. Note that $\mu(G_{dr+1}^d)$ is a subgraph of $\mu(G_{dr+i}^d)$. If Theorem 1 holds for the special case when $i = 1$, then $r + 2 \leq \chi_c(\mu(G_{dr+1}^d)) \leq \chi_c(\mu(G_{dr+i}^d)) \leq \chi(\mu(G_{dr+i}^d)) = r + 2$ and so the general case follows. Hence, it remains to prove Theorem 1 for the special case when $k = dr + 1$.

For clarity of notation, we consider G_{dr+1}^d as the graph with vertex set $V = \{x_0, x_1, \ldots, x_{dr}\}$ and edge set $E = \{x_i x_j : d \leq |i - j| \leq (dr + 1) - d\}$; and the Mycielskian $\mu(G_{dr+1}^d)$ as the graph with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x_i' : x_i \in V\}$, and edge set $E \cup \{x_i x_j' : x_i x_j \in E\} \cup \{x_j' u : x_j' \in V'\}$. Indices of the vertices x_i and x_i' are taken modulo $dr+1$, if arithmetic operations are performed on them. Let $V_{i,j} = \{x_i, x_{i+1}, \ldots, x_j\}$ and $V_{i,j}' = \{x_i', x_{i+1}', \ldots, x_j'\}$. Note that $V_{i,j}$ and $V_{i,j}'$ are of size $j - i + 1$ for $i \leq j$, and of size $dr + 1 + j - i + 1$ for $i > j$.

It is known that $\chi_c(G^d_{dr+1}) = r + \frac{1}{d}$ (see Vince [10]), $\chi(G^d_{dr+1}) = r + 1$, and $\chi(\mu(G^d_{dr+1})) = r + 2$. We now show the case of $k = dr + 1$ for Theorem 1.

Theorem 2. *$\chi_c(\mu(G^d_{dr+1})) = \chi(\mu(G^d_{dr+1})) = r + 2$ for any positive integer $r \geq 2$.*

Proof. Note that G^d_{2d+1} is, in fact, the odd cycle C_{2d+1}. According to Lemma 1, the theorem holds for $r = 2$. So, we may assume that $r \geq 3$. Let $G = (V, E)$ be the graph G^d_{dr+1}.

Suppose that the theorem does not hold, i.e., $\chi_c(\mu(G)) < \chi(\mu(G)) = r + 2$. Then, by Lemma 2, there exists an $(r+2)$-coloring c such that $D_c(\mu(G))$ is acyclic, $c(u) = 1$, and $c(x'_i) \notin \{0, 1\}$ for all $x'_i \in V'$. Note that, if x_i is a vertex of V such that $c(x_i) \notin \{0, 1\}$ and $c(x_i) \neq c(x'_i)$, then we can replace the color of x'_i with $c(x_i)$ and still preserve $D_c(\mu(G))$ being acyclic. Hence, we may assume that $c(x_i) = c(x'_i)$ for each $x_i \in V$ with $c(x_i) \notin \{0, 1\}$.

Moreover, by Lemma 2, there exists an edge $x_a x_b \in E$ such that $\{c(x_a), c(x_b)\} = \{0, 1\}$ and $c(x'_a) = c(x'_b) = t \notin \{0, 1\}$. It is clear that $|V_{a,b}| \geq d + 1$, since $x_a x_b$ is an edge. Without loss of generality, we may assume that $x_a x_b$ is chosen to satisfy the property that $|V_{a,b}|$ is minimum and, under this condition, $|c(V_{a,b})|$ is also minimum. Finally, we may also assume that $c(x_a) = 0$ and $0 = a < d \leq b \leq \frac{dr+1}{2}$.

Let $A_i = \{x_{i+dj} : 0 \leq j \leq r\}$ for $0 \leq i \leq dr$. It is clear that any two vertices of A_i, except the pair (x_i, x_{i-1}), are adjacent and, hence, have different colors. Note that $x_{i-1} = x_{i+dr}$.

Claim 1. If x_i and $x_j \in A_i \setminus \{x_i, x_{i-1}\}$ are vertices such that $2 \leq |c(x_i) - c(x_{i-1})| \leq r$ and $2 \leq |c(x_j) - c(x'_j)| \leq r$, then $A_i \cup \{x'_j\}$ induces a directed cycle in $D_c(\mu(G))$.

Proof of Claim 1. Since $x_j \in A_i \setminus \{x_i, x_{i-1}\}$, any two vertices of $A_i \cup \{x'_j\}$ are adjacent, except the two pairs (x_i, x_{i-1}) and (x_j, x'_j). Also, $c(x_i) \neq c(x_{i-1})$ and $c(x_j) \neq c(x'_j)$ imply $|c(A_i \cup \{x'_j\})| = r + 2$. Since any two vertices of $A_i \cup \{x'_j\}$ with consecutive colors are adjacent, $A_i \cup \{x'_j\}$ induces a directed cycle in $D_c(\mu(G))$. ∎

For any color $k \in c(V)$, let $V_{f(k),e(k)}$ be the $V_{i,j}$ of smallest size including $c^{-1}(k)$ as a subset. It is clear that $c(x_{f(k)}) = c(x_{e(k)}) = k$, but $c(x_{f(k)-1}) \neq k$ and $c(x_{e(k)+1}) \neq k$. Also, $V_{f(k),e(k)}$ has a size at most d, since any two vertices in it are nonadjacent. Moreover, $V = \cup_{k \in c(V)} V_{f(k),e(k)}$ implies that $r+1 \leq |c(V)| \leq r+2$.

Claim 2. $|c(V)| = r + 2$.

Proof of Claim 2. Suppose to the contrary that $|c(V)| = r + 1$, say $p \notin c(V)$ for some p with $2 \leq p \leq r + 1$. Let

$$S = \{x_i \in V : 2 \leq |c(x_i) - c(x_{i-1})| \leq r \text{ or } 2 \leq |c(x_i) - c(x_{i+1})| \leq r\}.$$

Suppose that $p-1 \notin c(S)$. Then, $c(x_{f(p-1)}) = c(x_{e(p-1)}) = p-1$ imply $x_{f(p-1)} \notin S$ and $x_{e(p-1)} \notin S$. Therefore, $c(x_{f(p-1)-1})$ and $c(x_{e(p-1)+1})$ are in $\{p-2, p-1\}$.

However, $c(x_{f(p-1)-1}) \neq p-1$ and $c(x_{e(p-1)+1}) \neq p-1$ by the definition of $V_{f(p-1),e(p-1)}$. Hence, $c(x_{f(p-1)-1}) = c(x_{e(p-1)+1}) = p-2$, and so $V_{f(p-1),e(p-1)}$ is a subset of $V_{f(p-2),e(p-2)}$. Then, V is the union of r sets $V_{f(k),e(k)}$, each of size at most d, for $k \in \{0,1,\ldots,r+1\} \setminus \{p-1,p\}$, a contradiction to the fact that $|V| = dr+1$. This proves that $p-1 \in c(S)$. Similarly, $p+1 \in c(S)$.

We then choose a vertex $x_i \in S$ such that $c(x_i) = 1$, when $p=2$ and $c(x_i) = p+1$ otherwise. For the case of $2 \leq |c(x_i) - c(x_{i-1})| \leq r, c(A_i) = c(V)$ and $\{c(x_i), c(x_{i-1})\} \neq \{0,1\}$. Then, there exists a vertex $x_j \in A_i \setminus \{x_i, x_{i-1}\}$ of color 0 or 1. Since x'_j is adjacent to all vertices of $A_i \setminus \{x_j\}$, the color of x'_j must be $c(x_j)$ or p. However, a vertex in V' cannot be colored by 0 or 1, hence we have $c(x'_j) = p$. Recall that $2 \leq p \leq r+1$ and $c(x_j) \in \{0,1\}$. If $|c(x_j) - c(x'_j)| \in \{0,1,r+1\}$, then either $c(x_j) = 0, p = r+1$, or $c(x_j) = 1, p = 2$. Both cases lead to $c(x_j) = c(x_i)$, a contradiction to the fact that x_i and x_j are adjacent. Hence, $2 \leq |c(x_j) - c(x'_j)| \leq r$. According to Claim 1, $A_i \cup \{x'_j\}$ induces a directed cycle in $D_c(\mu(G))$, a contradiction. Similar arguments also lead to a contradiction for the case of $2 \leq |c(x_i) - c(x_{i+1})| \leq r$. Therefore, $|c(V)| = r+2$. ∎

According to Claim 2, $t \in c(V)$. Since $c(x'_0) = c(x'_b) = t$, we have $V_{f(t),e(t)} \subseteq V_{-d+1,d-1} \cap V_{b-d+1,b+d-1}$, and so $d \leq b \leq 2d-2$ and $V_{f(t),e(t)} \subseteq V_{b-d+1,d-1} \subset V_{0,b}$. Therefore, $\{0,1,t\} \subseteq c(V_{0,b})$.

Claim 3. For any vertex $x_j \in V_{0,b}$ colored by $t, \{c(x_{j-1}), c(x_{j+1})\} \subseteq \{0,1,t\}$.

Proof of Claim 3. Suppose to the contrary that there exist vertices $x_i, x_{i+1} \in V_{0,b}$ such that $\{c(x_i), c(x_{i+1})\} = \{t,q\}$ for some $q \notin \{0,1,t\}$. Then $b \leq i+d$; and both $c(x'_{i-d}) = \ell$ and $c(x'_{i+1+d}) = m$ are not in $\{t,q\}$. If $r \geq 4$, we must have $\ell \neq m$. Otherwise, since every vertex of V is adjacent to at least one of x'_{i-d}, x'_{i+1+d}, we would have $\ell \notin c(V)$, contrary to Claim 2. Hence, none of the vertices of $V_{i+1+2d,i-2d}$ can be colored by $0,1,t,q,\ell,m$. This implies that the $dr+1-4d$ vertices of $V_{i+1+2d,i-2d}$ must be colored by only $r-4$ colors, a contradiction (see Fig. 1). If $r=3$, then $|V_{i+1+d,i-d}| = dr+1-2d = d+1$ and $\ell = m$ as there are only five colors to be used. Therefore, $c(x_{i+1+d}) = 1, c(x_{i-d}) = 0$, and $c(V_{i+1+d,i-d}) = \{0,1,\ell\}$. It follows that $x_{i+1+d}x_{i-d}$ is an edge with $\{c(x_{i+1+d}), c(x_{i-d})\} = \{0,1\}$ and $c(x'_{i+1+d}) = c(x'_{i-d})$, but $|V_{i+1+d,i-d}| = d+1 \leq |V_{0,b}|$ and $|c(V_{i+1+d,i-d})| < |c(V_{0,b})|$, a contradiction to the choice of x_0x_b. Hence, Claim 3 holds. ∎

Claim 4. $c(V_{0,b}) = \{0,1,t\}$.

Proof of Claim 4. Suppose to the contrary that there exists a color $p \in c(V_{0,b}) \setminus \{0,1,t\}$. Choose any $x_j \in V_{0,b}$ such that $c(x_j) = t$. By Claim 3, $c(x_{j-1}) \neq p$ and $c(x_{j+1}) \neq p$. If there exists some vertex in $V_{1,j-2}$ colored by p, then choose the largest integer $i \leq j-1$ such that $c(x_i) \in \{0,p\}$ and $c(x'_i) = p$. We now consider the following three cases according to the color of x_{i+1}.

Case 1:. $c(x_{i+1}) = 0$.

By the choice of i, we have $c(x'_{i+1}) \neq p$. Suppose that $c(x_{i+1+d}) \neq 1$. Then $c(x_{i+1+d}) = q$ for some $q \notin \{0,1,p,t\}$, since x_{i+1+d} is adjacent to x_0, x'_0, and x'_i.

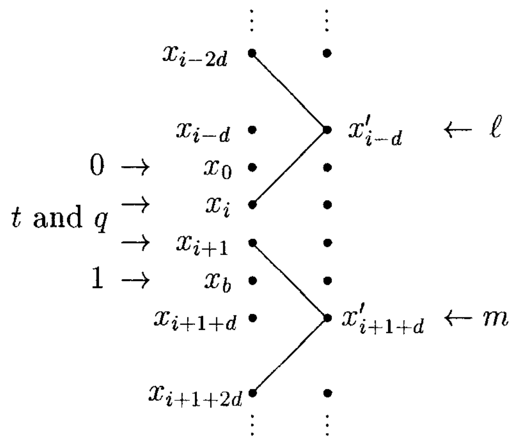

FIGURE 1. Vertex colors near $V_{0,b}$ (for Claim 3).

If $r \geq 4$, then $c(x'_{i+1+2d}) = m$ for some $m \notin \{0, 1, p, t, q\}$. Hence, the $dr+1-4d$ vertices of $V_{i+1+3d,i-d}$ are colored by $r-4$ colors, a contradiction (see Fig. 2). If $r = 3$, then $x_{i+1+2d} = x_{i-d}$. Hence, $c(x_{i-d}) = 1$ and $c(x'_{i-d}) = p$. Therefore, $c(x_i) = 0$ and $b > i+1+d$. Since $t, q \notin c(V_{i-d,i})$, we have that $x_{i-d}x_i$ is an edge with $\{c(x_{i-d}), c(x_i)\} = \{0, 1\}$ and $c(x'_{i-d}) = c(x'_i)$, but $|V_{i-d,i}| = d+1 \leq |V_{0,b}|$ and $|c(V_{i-d,i})| < |c(V_{0,b})|$, a contradiction to the choice of x_0x_b. Hence, $c(x_{i+1+d}) = 1$.

We now consider the set $c(V'_{i+1,i+1+d})$. If $q \in c(V'_{i+1,i+1+d})$ for some $q \notin \{0, 1, p, t\}$, then the $dr+1-3d$ vertices of $V_{i+1+2d,i-d}$ are colored by $r+2-|\{0, 1, p, t, q\}| = r-3$ colors, a contradiction. Hence, $c(x'_{i+1}) = c(x'_{i+1+d}) = t$ by Claim 2; and moreover, $c(V_{i+1,i+1+d}) \subseteq \{0, 1, t, p\}$.

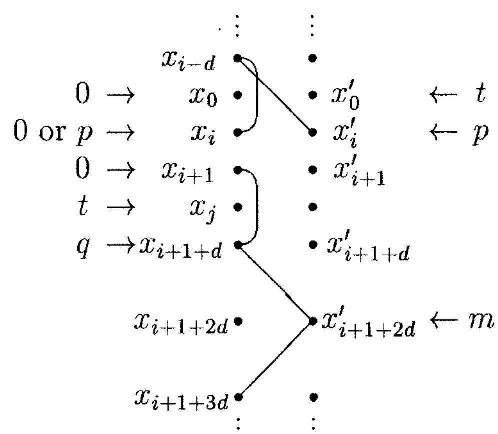

FIGURE 2. Vertex colors near $V_{0,b}$ (for the first paragraph of Case 1).

If $p \in c(V_{i+1,i+1+d})$, then $p \in c(V_{j+2,i+1+d})$, since i is maximal and since $c(x_{j+1}) \neq p$. By the assumption that $p \in V_{1,j-2}$, we have $c(x'_{j-d}), c(x'_{j+d}) \notin \{0,1,p,t\}$, say $c(x'_{j-d}) = \ell$ and $c(x'_{j+d}) = m$. If $r \geq 4$, then $\ell \neq m$ by Claim 2. Therefore, the $dr+1-(4d-1)$ vertices of $V_{j+2d,j-2d}$ must be colored by $r-4$ colors, a contradiction (see Fig. 3). If $r=3$, then $\ell = m$ by Claim 2 and $|V_{j+d,j-d}| = (dr+1)-(2d-1) = d+2$. Since all vertices of $V'_{j+d,j-d}$ are colored by $\ell = m$, we have that $c(x_{j+d}) = c(x_{j+d+1}) = 1, c(x_{j-d}) = c(x_{j-d-1}) = 0$, and $c(V_{j+d,j-d}) = \{0,1,l\}$. Hence, $x_{j+d}x_{j-d-1}$ is an edge with $\{c(x_{j+d}), c(x_{j-d-1})\} = \{0,1\}$ and $c(x'_{j+d}) = c(x'_{j-d-1})$, but $|V_{j+d,j-d-1}| = d+1 \leq |V_{0,b}|$ and $|c(V_{j+d,j-d-1})| < |c(V_{0,b})|$, a contradiction to the choice of x_0x_b. Therefore, $c(V_{i+1,i+1+d}) = \{0,1,t\}$, which also implies that $x_{i+1}x_{i+1+d}$ is an edge with $\{c(x_{i+1}), c(x_{i+1+d})\} = \{0,1\}$ and $c(x'_{i+1}) = c(x'_{i+1+d})$, but $|V_{i+1,i+1+d}| = d+1 \leq |V_{0,b}|$ and $|c(V_{i+1,i+1+d})| < |c(V_{0,b})|$, again a contradiction to the choice of x_0x_b.

Case 2:. $c(x_{i+1}) = 1$.

If $c(x_i) = p$, then $c(x'_{i-d}) \notin \{p,t\}$, say $c(x'_{i-d}) = \ell$. Therefore, the $dr+1-3d$ vertices of $V_{i+1+d,i-2d}$ are colored by $r+2-|\{0,1,p,t,\ell\}| = r-3$ colors, a contradiction.

If $c(x_i) = 0$, then since $c(x'_i) = p$, the $dr+1-2d$ vertices of $V_{i+1+d,i-d}$ are colored by $r-2$ colors, also a contradiction.

Case 3:. $c(x_{i+1}) \notin \{0,1,p\}$.

If $c(x_{i+1}) = t$, then $c(x_i) = 0$ and $c(x'_i) = p$. Since some vertex of $V_{1,j-2}$ is colored by p, we have $c(x'_{i+1+d}) \notin \{p,t\}$, say $c(x'_{i+1+d}) = m$. Therefore, the $dr+1-3d$ vertices of $V_{i+1+2d,i-d}$ are colored by $r-3$ colors, a contradiction.

If $c(x_{i+1}) = q$ for some $q \notin \{0,1,p,t\}$, then $c(x'_{j+d}) \notin \{0,1,p,q,t\}$, say $c(x'_{j+d}) = m$. Then we have $r \geq 4$. If $c(x_i) = 0$, then, since $c(x'_i) = p$, the vertices of $V_{j+2d,i-d}$ are colored by $r-4$ colors, a contradiction to $|V_{j+2d,i-d}| =$

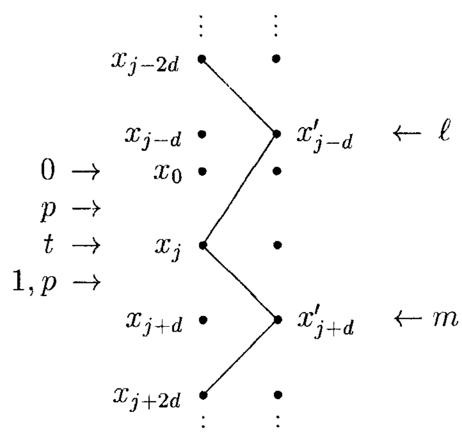

FIGURE 3. Vertex colors near $V_{0,b}$ (for the last paragraph of Case 1).

$dr + 1 - 3d - (j - i - 1) > dr + 1 - 4d$. If $c(x_i) = p$, then $c(x'_{i-d})$ cannot be colored by $0, 1, p, q, t$, say $c(x'_{i-d}) = \ell$. Note that $0 > j - d > i - d, b < j + d$, and $j < d$. If $r \geq 5$, then $|V_{j+2d,i-2d}| > dr + 1 - 5d$ and $\ell \neq m$, i.e., at least $dr + 1 - 5d$ vertices of V are colored by $r - 5$ colors, a contradiction (see Fig. 4). If $r = 4$, then $\ell = m$ and $|V_{j+d,i-d}| > dr + 1 - 3d = d + 1$. It is clear that $c(x_{i-d}) = 0$. Since $x_{i-2d} \in V_{j+d,i-d}$, we have that $c(x_{i-2d}) = 1$ and $c(V_{j+d,i-d}) = \{0, 1, \ell\}$. Therefore, $x_{i-2d}x_{i-d}$ is an edge with $\{c(x_{i-2d}), c(x_{i-d})\} = \{0, 1\}$ and $c(x'_{i-2d}) = c(x'_{i-d})$, but $|V_{i-2d,i-d}| = d+1 \leq |V_{0,b}|$ and $|c(V_{i-2d,i-d})| < |c(V_{0,b})|$, a contradiction to the choice of x_0x_b. ∎

By the three cases above, we conclude that no vertex of $V_{1,j-2}$ can be colored by p. A similar argument shows that no vertex of $V_{j+2,b-1}$ can be colored by p. Hence, $c(V_{0,b}) = \{0, 1, t\}$. This completes the proof of Claim 4.

Having proved the claims, we are now ready to prove the theorem. Suppose $3 \leq t \leq r$. Since $t \notin c(V \setminus V_{0,b})$, there exists an integer i such that $c(x_i), c(x_{i-1}) \notin \{0, 1, t\}$ and $2 \leq |c(x_i) - c(x_{i-1})| \leq r$. Since $c(A_i) = r + 1$, there must be some vertex $x_j \in A_i \setminus \{x_i, x_{i-1}\}$ such that $c(x_j) \in \{0, 1\}$. Assume that $c(x_j) = 0$ (the case of $c(x_j) = 1$ is similar). Since the color of x'_j say q, cannot belong to $c(A_i)$, i.e., q is the only color not in $c(A_i)$ and $q \notin \{0, 1\}$. Hence, $1 \in c(A_i)$, some vertex $x_{j'} \in A_i \setminus \{x_i, x_{i-1}, x_j\}$ is colored by 1, and $c(x'_{j'}) = q$ is clear. It follows that either $2 \leq |c(x_j) - c(x'_j)| \leq r$ or $2 \leq |c(x_{j'}) - c(x'_{j'})| \leq r$. By Claim 1, there is a directed cycle in $D_c(\mu(G))$, a contradiction. Therefore, $t \in \{2, r + 1\}$.

Assume that $t = 2$ (the case of $t = r + 1$ is similar). Let i be the smallest integer such that $c(x_i) = 2$; and let j be the largest integer such that $c(x_j) = 1$ and $i \leq j \leq i + d$. Such a j exists. In fact, one may take $j = b$ if there is no larger value.

For the case of $j = i + d, c(x_{i+d}) = 1$ implies $c(x_{i-1}) \neq 1$. By the definition of $x_i, c(x_{i-1}) \neq 2$. Hence, by Claim 4, $c(x_{i-1}) = 0$. Also, $c(x_i) = 2$ implies

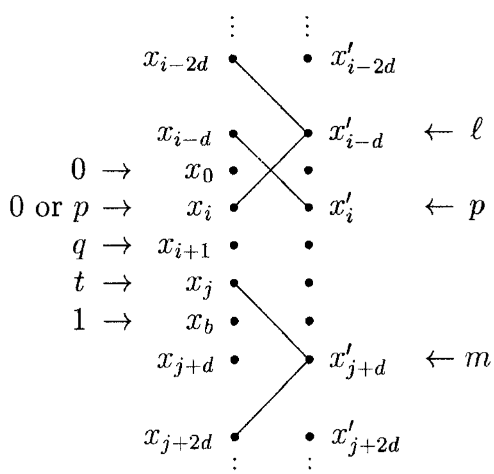

FIGURE 4. Vertex colors near $V_{0,b}$ (for Case 3).

$c(x'_{i+d}) \neq 2$. By Claim 1, $A_i \cup \{x'_{i+d}\}$ induces a directed cycle of $D_c(\mu(G))$, a contradiction.

For the case of $j < i+d$, $c(x_{j+1}) \neq 1$ by the choice of x_j; and $c(x_{j+1}) \notin \{0, 2\}$, since $j+1 > b$ and $c(x_0) = 0$ and $c(x'_0) = 2$. According to $0 \leq b-d \leq j-d < i$, we have $c(x_{j-d}) \neq 2$ by the choice of x_i, and $c(x_{j-d}) \neq 1$ as $c(x_j) = 1$. Hence, by Claim 4, $c(x_{j-d}) = 0$. We conclude that $c(x'_{j-d}) = 2$. By Claim 1, $A_{j+1} \cup \{x'_{j-d}\}$ induces a directed cycle of $D_c(\mu(G))$, a contradiction.

Therefore, $\chi_c(\mu(G^d_{dr+1})) = \chi(\mu(G^d_{dr+1})) = r + 2$. This completes the proof of the theorem. ∎

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