Operations Research Letters 25 (1999) 101–107

operations research letters

www.elsevier.com/locate/orms

Structure importance of consecutive-k-out-of-n systems

Fen-Hui Lin[a], Way Kuo[b,*], Frank Hwang[c]

[a] *Department of Information Management, National Sun Yat-sen University, Kaohsiung, Taiwan*
[b] *Department of Industrial Engineering, Texas A&M University, 241 Zachry Engineering Center, College Station, TX 77840-3131, USA*
[c] *Department of Applied Mathematics, National Chiao-Tung University, Hsin Chu, Taiwan*

Received 1 November 1996; received in revised form 1 February 1999

Abstract

A consecutive-k-out-of-n : F system is an n-component system that fails when k consecutive components fail. The structure importance is a measure, which indicates the importance of a component relative to its positioning in the system. Through the relationship to the Fibonacci sequence with order k, a closed-form solution of structure importance for each component is obtained. We obtain a complete ordering of the components with respect to their structure importance for some consecutive-k-out-of-n : F systems, and a partial ordering for other systems.

Keywords: Consecutive-k-out-of-n system; Structure importance; Fibonacci sequence with order k

1. Introduction

A consecutive-k-out-of-n : F system consists of an ordered sequence of n components such that the system fails if and only if k consecutive components fail. Similarly, a consecutive-k-out-of-n : G system exists where if the consecutive k components work, the system works.

Reliability importance as defined by Birnbaum [1] is a partial derivative of system reliability with respect to component reliability. Let $\boldsymbol{p}=(p_1,\ldots,p_n)$ be a vector of the component reliabilities of a given system. The reliability importance of component i is defined as

$$I_i(\boldsymbol{p})=\frac{\partial R(\boldsymbol{p})}{\partial p_i}=R(1_i;\boldsymbol{p})-R(0_i;\boldsymbol{p}), \tag{1}$$

where $R(1_i;\boldsymbol{p})$ and $R(0_i;\boldsymbol{p})$ are the conditional system reliabilities given that component i works and fails, respectively. Reliability importance of a component is interpreted as the probability that the component is critical to the system reliability. It means the system works when this component works and the system fails when this component fails. Reliability importance provides a quantitative measure of the importance of components so that system designers can decide which components deserve extra attention when the system is under going preventive maintenance or breaks down.

The magnitude of the reliability importance of a component in a given system depends on two factors. The first is the reliabilities of the rest of the components, and the other is the system structure. The effect of the first factor is neutralized when all components are i.i.d. with reliability $\frac{1}{2}$. In that case the reliability importance is called the structure importance which indicates the effect of a particular component's position

* Corresponding author. Fax: +1-409-847-9005.
E-mail address: way@acs.tamu.edu (W. Kuo)

0167-6377/99/$ - see front matter
PII: S0167-6377(99)00038-3

in the system. Although many researchers have discussed the consecutive-k-out-of-n systems during the last decade, but very few have considered the characteristics of structure importance. This paper is to discuss characteristics of structure importance associated with components of consecutive-k-out-of-n systems.

Papastavridis [6] derived the reliability importance for the consecutive-k-out-of-n : F system:

$$I_i(\boldsymbol{p}) = \frac{\partial R_k(n;\boldsymbol{p})}{\partial p_i} = \frac{R_k(i-1;\boldsymbol{p})R_k(n-i;\boldsymbol{p}) - R_k(n;\boldsymbol{p})}{1-p_i}, \tag{2}$$

where p_i is the reliability of component i and $R_k(n;\boldsymbol{p})$ is the reliability of a consecutive-k-out-of-n : F system with the component reliability $\boldsymbol{p} = (p_1,\dots,p_n)$.

Zuo and Kuo [7] provided the structure importance ordering for the consecutive-2-out-of-n system. They [7] also proved that the matching components of both consecutive-k-out-of-n : F and the G systems have the same reliability importance. Therefore throughout this paper, structure importance is only discussed for consecutive-k-out-of-n : F systems. Note that the structure importance ordering places the system components from the greatest structure importance to the smallest one.

Notation.

$R_k(n)$ Reliability of a consecutive-k-out-of-n : F system containing i.i.d. components with reliability $\frac{1}{2}$. It can be expressed as the following:

$$R_k(n) = (\tfrac{1}{2})R_k(n-1) + (\tfrac{1}{2})^2R_k(n-2) + \cdots + (\tfrac{1}{2})^kR_k(n-k) \tag{3}$$

$$R_k(n) = R_k(n-1) - (\tfrac{1}{2})^{k+1}R_k(n-k-1). \tag{4}$$

$I_{k,n}(i)$ Structure importance of component i in a consecutive-k-out-of-n : F system as stated in Eq. (2) when $p_i = 1/2$, $i = 1,2,\dots,n$.

2. Closed-form solutions of the structure importance for consecutive-k-out-of-n systems

According to Eq. (2), we obtain Theorem 1 which states that the structure importance of components in the consecutive-k-out-of-n system are symmetric to the central component.

Theorem 1. $I_{k,n}(i) = I_{k,n}(n-i+1)$.

Proof.

$$\begin{aligned} I_{k,n}(n-i+1) &= 2[R_k(n-i)R_k(n-(n-i+1)) - R_k(n)] \\ &= 2[R_k(n-i)R_k(i-1) - R_k(n)] \\ &= I_{k,n}(i). \quad \square \end{aligned}$$

Miles [5] defined the Fibonacci sequence with order k, $f_{k,n}$, to be

$$f_{k,n} = \begin{cases} 0, & 0 \leqslant n \leqslant k-1, \\ 1, & n = k, \\ \sum_{j=n-k}^{n-1} f_{k,j}, & n \geqslant k+1. \end{cases}$$

Ferguson [3] gave a closed-form solution of $f_{k,n} - f_{k-1,n}$ (his W function). After some simplification, we obtain:

Lemma 1. *For $n \geqslant 2$, define $m = \lfloor (n-2)/(k+1) \rfloor$, where $\lfloor x \rfloor$ is the largest integer not exceeding x. Then,*

$$\begin{aligned} f_{k,n} &= \sum_{j=0}^{m} (-1)^j 2^{n-jk-k-j-1} \times \frac{(n-jk-k-1)!(n-jk-k+j)}{j!(n-jk-k-j)!} \\ &= 2^{n-k-1} + \sum_{j=1}^{m} (-1)^j 2^{n-jk-k-j-1} \times \frac{(n-jk-k-1)!(n-jk-k+j)}{j!(n-jk-k-j)!} \end{aligned}$$

given that $(-1)!/(-1)! \equiv 1$.

Corollary 1. $f_{k,n} = 2^{n-k-1}$ *for* $k+1 \leqslant n \leqslant 2k$.

Corollary 2. $f_{k,n} = 2^{n-k-1} - (n-2k+1)2^{n-2k-2}$ *for* $2k+1 \leqslant n \leqslant 3k+1$.

Lemma 2. $f_{k,n} = 2f_{k,n-1} - f_{k,n-k-1}$ *for* $n \geqslant k+2$.

Proof.

$$\begin{aligned} f_{k,n} &= f_{k,n-1} + (f_{k,n-2} + \cdots + f_{k,n-k}) \\ &= f_{k,n-1} + (f_{k,n-1} - f_{k,n-k-1}). \quad \square \end{aligned}$$

We show next that the structure importance is closely related to $f_{k,n}$. Call an n-system working if it does not contain k consecutive failed components. For consecutive-k-out-of-n : F systems containing i.i.d. components with reliability $\frac{1}{2}$, the system reliability is shown in Theorem 2.

Theorem 2. $R_k(n) = (\frac{1}{2})^n f_{k,n+k+1}$.

Proof. For $0 \leqslant n \leqslant k-1$, $R_k(n) = 1 = (\frac{1}{2})^n f_{k,n+k+1}$ by Corollary 1.

For $n \geqslant k$, let w denote the last working component. Then the n-system is working if and only if $n-k+1 \leqslant w \leqslant n$ and the first $w-1$ components form a working $(w-1)$-system. Hence, for $n = k$,

$$\begin{aligned} R_k(k) &= \sum_{w=1}^{k} (\tfrac{1}{2})^{k-w+1} R_k(w-1) \\ &= \sum_{w=1}^{k} (\tfrac{1}{2})^{k-w+1} (\tfrac{1}{2})^{w-1} f_{k,w+k} \\ &= (\tfrac{1}{2})^k f_{k,2k+1} = (\tfrac{1}{2})^n f_{k,n+k+1}. \end{aligned}$$

For $n > k$, it can be obtained that $R_k(n) = (\frac{1}{2})^n f_{k,n+k+1}$ by induction. □

Thus, $f_{k,n+k+1}$ can be interpreted as the number of working n-systems. We are now ready to give a closed-form solution of the structure importance.

Theorem 3. $I_{k,n}(i) = (\frac{1}{2})^{n-1}(2 f_{k,i+k} f_{k,n-i+k+1} - f_{k,n+k+1})$.

Proof.

$$\begin{aligned} I_{k,n}(i) &= 2[R_k(i-1)R_k(n-i) - R_k(n)] \\ &= 2[(\tfrac{1}{2})^{i-1} f_{k,i+k} (\tfrac{1}{2})^{n-i} f_{k,n-i+k+1} \\ &\quad -(\tfrac{1}{2})^n f_{k,n+k+1}] \\ &= (\tfrac{1}{2})^{n-1}[2 f_{k,i+k} f_{k,n-i+k+1} - f_{k,n+k+1}]. \end{aligned}$$ □

Corollary 3. $I_{k,n}(1) = (\frac{1}{2})^{n-1} f_{k,n}$

Proof.

$$\begin{aligned} I_{k,n}(1) &= (\tfrac{1}{2})^{n-1}[2 f_{k,n+k} - f_{k,n+k+1}] \\ &= (\tfrac{1}{2})^{n-1} f_{k,n} \quad \text{by Lemma 2.} \end{aligned}$$ □

We now give physical meaning to the structure importance. Let S_i denote the set of n-systems where its first $i-1$ components form a working $(i-1)$-system and its last $n-i$ components form a working $(n-i)$-system. Since component i can be either working or failed, $|S_i| = 2 f_{k,i+k} f_{k,n-i+k+1}$. Let W denote the set of working n-systems. Then $|W| = f_{k,n+k+1}$. Note that $W \subseteq S_i$. Furthermore, any system in S_i with a working component i is also in W. Therefore, $S_i \setminus W$ is the set of failing n-systems with a failing component i such that if component i works, then the system would also work.

Next we give a solution of $I_{k,n}(i)$ in the closed form, including an explicit solution for $n \leqslant 2k$.

Lemma 3. *For* $1 \leqslant i \leqslant n-k$, $I_{k,n}(i) = \sum_{j=n-k}^{n-1} (\frac{1}{2})^{n-j} I_{k,j}(i)$.

Proof. For $i \leqslant n-k$, by Theorem 3 and Lemma 2,

$$\begin{aligned} 2^{n-1} I_{k,n}(i) &= 2 f_{k,i+k} f_{k,n-i+k+1} - f_{k,n+k+1} \\ &= 2 f_{k,i+k} \sum_{j=n-i+1}^{n-i+k} f_{k,j} - \sum_{j=n+1}^{n+k} f_{k,j} \\ &= \sum_{j=n+1}^{n+k} (2 f_{k,i+k} f_{k,j-i} - f_{k,j}) \\ &= \sum_{j=n-k}^{n-1} (2 f_{k,i+k} f_{k,j-i+k+1} - f_{k,j+k+1}) \\ &= \sum_{j=n-k}^{n-1} 2^{j-1} I_{k,j}(i). \end{aligned}$$ □

Theorem 4. *For* $k+1 \leqslant n \leqslant 2k$,

$$I_{k,n}(i) = i/2^k \quad \textit{if } 1 \leqslant i \leqslant n-k$$

and

$$I_{k,n}(i) = (n-k+2)/2^k \quad \textit{if } n-k < i \leqslant \lceil n/2 \rceil.$$

Proof. Consider $k+1 \leqslant n \leqslant 2k$. Suppose $n-k < i \leqslant \lceil n/2 \rceil$. Now every k consecutive components in an n-system contains component i, which implies an n-system fails if its component i fails. In other words, $S_i \setminus W$ consists of all failed n-systems. Thus,

$$\begin{aligned} I_{k,n}(i) &= (\tfrac{1}{2})^{n-1} |S_i \setminus W| = 2[1 - R_k(n)] \\ &= 2[1 - (\tfrac{1}{2})^n f_{k,n+k+1}] \quad \text{by Theorem 2} \end{aligned}$$

$$= 2\left\{1 - (\tfrac{1}{2})^n[2^n - (n-k+2)2^{n-k-1}]\right\}$$
by Corollary 2
$$= (n-k+2)/2^k.$$

Next, suppose $1 \leqslant i \leqslant n-k$. Then $k \leqslant n-i \leqslant 2k-1$,

$$I_{k,n}(i) = 2[R_k(i-1)R_k(n-i) - R_k(n)]$$
$$= 2[(\tfrac{1}{2})^{n-i} f_{k,n-i+k+1} - (\tfrac{1}{2})^n f_{k,n+k+1}].$$

Because $2k+1 \leqslant n-i+k+1 \leqslant 3k$ and $2k+1 \leqslant n+k+1 \leqslant 3k+1$, by Corollary 2,

$$I_{k,n}(i) = (\tfrac{1}{2})^{n-i-1}[2^{n-i} - (n-i-k)2^{n-i-k-1}] - (\tfrac{1}{2})^{n-1}[2^n - (n-k)2^{n-k-1}] = i(\tfrac{1}{2})^k. \qquad \square$$

Note that Lemma 3 and Theorem 4 overlap but neither is comprehensive. For example, the case $k < n/2$ and $i > n-k$ is not covered by Lemma 3 but is covered with the application of Theorem 1 and the special case for $k+1 \leqslant n \leqslant 2k$ is given in Theorem 4.

Corollary 4. *For $k+1 \leqslant n \leqslant 2k$, $I_{k,n}(1) = 1/2^k$.*

Proof. This can be proven either by Theorem 4, or by Corollaries 1 and 3. □

The following are other significant results of the structure importance of consecutive-k-out-of-n systems.

Lemma 4. *For $n \geqslant 2k+1$,*

1. *$I_{k,n}(i) - I_{k,n}(i+1) = (\frac{1}{2})^{k+1}[I_{k,n-k-1}(i-k) - I_{k,n-k-1}(i)]$, given that $i \geqslant k+1$;*
2. *$I_{k,n}(i) - I_{k,n}(i+1) = \sum_{j=2}^{k} (\frac{1}{2})^j [I_{k,n-j}(i) - I_{k,n-j}(i-j+1)]$.*

Proof. (1)

$$\begin{aligned} I_{k,n}(i) - I_{k,n}(i+1) &= (\tfrac{1}{2})^{n-1}(2f_{k,i+k}f_{k,n-i+k+1} - 2f_{k,i+k+1}f_{k,n-i+k}) \\ &= (\tfrac{1}{2})^{n-1}[2f_{k,i+k}(2f_{k,n-i+k} - f_{k,n-i}) \\ &\quad -2(2f_{k,i+k} - f_{k,i})f_{k,n-i+k}] \\ &= (\tfrac{1}{2})^{n-1}(2f_{k,i}f_{k,n-i+k} - 2f_{k,i+k}f_{k,n-i}) \\ &= (\tfrac{1}{2})^{n-1}2^{n-k-2}[I_{k,n-k-1}(i-k) - I_{k,n-k-1}(i)]. \end{aligned} \tag{5}$$

(2) Eq. (5) is equal to $(\frac{1}{2})^{n-1}[2f_{k,i+k}(\sum_{j=1}^{k} f_{k,n-i+j}) - 2(\sum_{j=1}^{k} f_{k,i+j})f_{k,n-i+k}]$. Then, apply Theorem 3 and one can obtain the expression. □

Extended from Lemma 4, we get a more generalized form in Theorem 5. The proof which is the same as Lemma 4 is omitted.

Theorem 5. *For $s \geqslant 1$,*

1. *$I_{k,n}(i+s) - I_{k,n}(i) = (\frac{1}{2})^{k+1} \sum_{j=1}^{s} [I_{k,n-k-j}(i) - I_{k,n-k-j}(i+s-k-j)]$.*
2. *$I_{k,n}(i+s) - I_{k,n}(i) = \sum_{j=1}^{k} (\frac{1}{2})^j [I_{k,n-j}(i+s-j) - I_{k,n-j}(i)]$.*

To order the structure importance, Theorems 4 and 5 provide tools for direct comparison between the structure importance of two components. To compare the structure importance for two components next to each other, Theorem 6 below provides another perspective through examining the relationship between Fibonacci sequence with order k and the consecutive-k-out-of-n system.

Theorem 6. *Let $a_{k,i} = f_{k,i+1}/f_{k,i}$. Let $n \geqslant 3$ be an arbitrary integer. For $1 \leqslant i \leqslant \lfloor (n-1)/2 \rfloor$, $I_{k,n}(i+1) > I_{k,}(i)$ if and only if $a_{k,i+k} > a_{n-i+k}$.*

Proof. ($\Rightarrow$) Given $n \geqslant 3$, $I_{k,n}(i+1) > I_{k,n}(i)$ implies $f_{k,i+k+1}f_{k,n-i+k} > f_{k,i+k}f_{k,n-i+k+1}$. That is,

$$\frac{f_{k,i+k+1}}{f_{k,i+k}} > \frac{f_{k,n-i+k+1}}{f_{k,n-i+k}}. \tag{6}$$

($\Longleftarrow$) The proof is to reverse the previous proof from the end to the beginning. □

3. Structure importance ordering for consecutive-k-out-of-n systems

In this section, Propositions 1–5 summarize the structure importance ordering for different values of k. It is then concluded that the structure importance ordering of consecutive-k-out-of-n systems is affected mainly by the value of k. As stated in the introduction, structure importance serves as an important index for component positions in a given system. Knowing the structure importance ordering

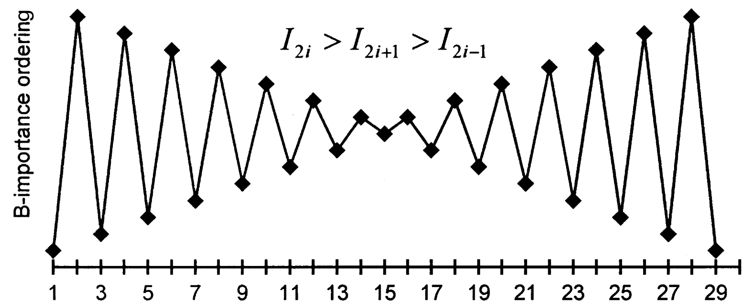

Fig. 1. Structure importance ordering of a consecutive-2-out-of-29 : F system to illustrate Proposition 2.

for a consecutive-k-out-of-n system helps understand which component positions are more crucial to system reliability. Through the discussion in this section, it can be generally concluded that components positioned at k and $n-k+1$ have the highest structure importance while components positioned at 1 and n have the lowest structure importance.

Proposition 1 states the simplest cases for consecutive-k-out-of-n : F systems, when $k=1$ and $k=n$. When $k=1$, the consecutive-k-out-of-n : F system becomes a series system. On the other hand, when $k=n$, a consecutive-k-out-of-n : F system becomes a parallel system. For either a series system or a parallel system, the structure importance of every component in the system is equal.

Proposition 1. *For both cases* $k=1$ and $k=n$, $I_{k,n}(i)=I_{k,n}(j)$, *for* $1 \leqslant i < j \leqslant n$.

For the case $k=2$, that is the consecutive-2-out-of-n : F system, Derman et al. [2] initiated the discussion about assigning higher component reliabilities to components in the system that are more important than others in order to raise the system's reliability. Zuo and Kuo [7] provided the structure importance ordering and offered a proof. Here in Proposition 2, we give another proof which is briefer compared than the one in [7].

Proposition 2. *For* $k=2$, *let* $1 \leqslant i \leqslant \lfloor (n+1)/2 \rfloor$. *Then* $I_{2,n}(2i-1) < I_{2,n}(2i+1) < I_{2,n}(2i)$, *and* $I_{2,n}(2i) > I_{2,n}(2i+2)$.

Proof. Because of the symmetric property of the structure importance stated in Theorem 1, $I_{2,n}(i)$'s will be discussed only for $1 \leqslant i \leqslant \lfloor (n+1)/2 \rfloor$.

Because $I_{2,n}(2) - I_{2,n}(1) = 2[R_2(n-2) - R_2(n-1)] > 0$ for $n \geqslant 3$, it is obtained that $I_{2,n}(3) - I_{2,n}(2) = \frac{1}{4}[I_{2,n-2}(1) - I_{2,n-2}(2)] < 0$ for $n \geqslant 5$ and $I_{2,n}(4) - I_{2,n}(3) = \frac{1}{4}[I_{2,n-2}(2) - I_{2,n-2}(3)] > 0$ for $n \geqslant 7$, by Lemma 3.

By induction, $I_{2,n}(2i) - I_{2,n}(2i-1) > 0$ for $n \geqslant 4i-1$ and $I_{2,n}(2i) - I_{2,n}(2i+1) > 0$ for $n \geqslant 4i+1$.

According to Theorem 5, $I_{2,n}(i+2) - I_{2,n}(i) = \frac{1}{2}[I_{2,n-1}(i+1) - i_{2,n-1}(i)]$. This implies, $I_{2,n}(2i+1) - I_{2,n}(2i-1) = \frac{1}{2}[I_{2,n-1}(2i) - I_{2,n-1}(2i-1)] > 0$ for $n \geqslant 4i+1$, and $I_{2,n}(2i) - I_{2,n}(2i+2) = \frac{1}{2}[I_{2,n-1}(2i) - I_{2,n-1}(2i+1)] > 0$ for $n \geqslant 4i+3$. □

Fig. 1 shows the structure importance ordering stated in Proposition 2. Theorem 6 indicates if the structure importance of component i is smaller (bigger) than component $(i+1)$, then $a_{2,i+k}$ is the upper (lower) bound for all $a_{2,i+k+j}$'s with j an arbitrary positive integer. Note that the sequence $(a_{2,n})$ is the ratio of one item to its previous item of the Fibonacci sequence. Therefore, it is obtained,

$$a_{2,2j-1} > a_{2,2j}, \quad a_{2,2j} < a_{2,2(j+1)}, \quad a_{2,2j-1} > a_{2,2j+1}.$$

Proposition 3 then states the case $k = n/2$ or $(n+1)/2$. It is found that components 1 to k and components n to $n-k+1$ have the structure importance placed in ascending order. Likewise, when $k > (n+1)/2$, components 1 to $n-k+1$ and components n to k have the structure importance placed in ascending order which is proven in Proposition 4. For illustrations of these two propositions, interested readers can refer to Fig. 4, in Kuo et al. [4].

Proposition 3. *For* $2k = n$ *or* $n+1$, $I_{k,n}(1) < I_{k,n}(2) < \cdots < I_{k,n}(k)$.

Proof. This can be proven by Theorem 4, or directly derive $I_{k,n}(i+1) - I_{k,n}(i) > 0$ for $1 \leqslant i \leqslant k-1$. □

Proposition 4. *For* $2k > n+1$, $I_{k,n}(1) < I_{k,n}(2) < \cdots < I_{k,n}(n-k+1) = I_{k,n}(n-k+2) = \cdots = I_{k,n}(k)$, *and* $I_{k,n}(n) < I_{k,n}(n-1) < \cdots < I_{k,n}(k)$.

Proof. The proof is similar to that in Proposition 3. □

Proposition 5 describes the cases for $3 \leqslant k \leqslant \lfloor (n-1)/2 \rfloor$. The structure importance ordering of components before $2k+1$ and after $n-2k$ have a general form. Although structure importance of components $2k+2$ to $n-2k-1$ are not completely ordered, they observe a strong partial order.

Proposition 5. *Given* $3 \leqslant k \leqslant \lfloor (n-1)/2 \rfloor$,

(1) $I_{k,n}(1) < I_{k,n}(2) < \cdots < I_{k,n}(k)$, $I_{k,n}(k+1) < I_{k,n}(k+2) < \cdots < I_{k,n}(2k)$, $I_{k,n}(2k+1) < I_{k,n}(2k+2) < \cdots < I_{k,n}(3k-1)$, $I_{k,n}(k) > I_{k,n}(2k) > I_{k,n}(3k)$, *and* $I_{k,n}(1) < I_{k,n}(k+1) < I_{k,n}(2k+1)$. *In addition,* $I_{k,n}(jk) > I_{k,n}(jk+1)$, *for* $j = 1, 2, 3$, *and* $I_{k,n}(k-1) < I_{k,n}(k+1)$.

(2) $I_{k,n}(2k)$ *and* $I_{k,n}(2k+1)$ *are the upper and lower bounds, respectively, of the* $I_{k,n}(i)$'s *with* $2k+2 \leqslant i \leqslant \lfloor (n+1)/2 \rfloor$.

Proof. (1) When $1 \leqslant i \leqslant k-1$, $R_k(i) = 1$. Then $I_{k,n}(i+1) - I_{k,n}(i) = 2[R_k(n-i-1) - R_k(n-i)] > 0$.

Therefore, $I_{k,n}(1) < I_{k,n}(2) < \cdots < I_{k,n}(k-1) < I_{k,n}(k)$. With Lemma 3, $I_{k,n}(k+1) - I_{k,n}(k) = \sum_{j=2}^{k} (\frac{1}{2})^j [I_{k,n-j}(k+1-j) - I_{k,n-j}(k)] < 0$, and $I_{k,n}(k+1) - I_{k,n}(k-1) = (\frac{1}{2})^k [R_k(n-2k) + R_k(n-2k-1) - R_k(n-k-1)] > 0$.

Hence
$$I_{k,n}(1) < I_{k,n}(2) < \cdots < I_{k,n}(k-1) < I_{k,n}(k+1) < I_{k,n}(k). \tag{7}$$

When $k+1 \leqslant i \leqslant 2k-1$, $R_k(i) = 1 - (i-k+2)(\frac{1}{2})^{k+1}$.

$$I_{k,n}(i+1) - I_{k,n}(i) = (\tfrac{1}{2})^{2k+1} \left[\sum_{x=1}^{k} R_k(n-k-i-1-x) - (i-k+1) R_k(n-i-k-1) \right] > (\tfrac{1}{2})^{2k+1} \left[\sum_{x=1}^{k} R_k(n-k-i-1) - (i-k+1) R_k(n-i-k-1) \right] \geqslant 0.$$

Hence, $I_{k,n}(k+1) < I_{k,n}(k+2) < \cdots < I_{k,n}(2k-1) < I_{k,n}(2k)$. In addition, with Eq. (7), $I_{k,n}(k+2) - I_{k,n}(k) = \sum_{j=1}^{k} (\frac{1}{2})^j [I_{k,n-j}(k+2-j) - I_{k,n-j}(k)] < 0 \Rightarrow I_{k,n}(k) > I_{k,n}(k+2)$. By induction, $I_{k,n}(k) > I_{k,n}(k+j)$ for all $j \geqslant 1$. Therefore, $I_{k,n}(k) > I_{k,n}(2k)$, and Eq. (7) is extended as the following:

$$I_{k,n}(1) < I_{k,n}(2) < \cdots < I_{k,n}(k-1) < I_{k,n}(k+1) < \cdots < I_{k,n}(2k-1) < I_{k,n}(2k) < I_{k,n}(k). \tag{8}$$

When $2k+1 \leqslant i \leqslant 3k-2$, similar to the preceding proof, it can be derived that $I_{k,n}(i+1) - I_{k,n}(i) > 0$. Hence,

$$I_{k,n}(2k) < I_{k,n}(2k+1) < \cdots < I_{k,n}(3k-1). \tag{9}$$

Because $I_{k,n}(k) > I_{k,n}(2k)$ in Eq. (8),

$$I_{k,n}(2k+1) - I_{k,n}(2k) = (\tfrac{1}{2})^{k+1} [I_{k,n-k-1}(2k) - I_{k,n-k-1}(k)] < 0 \Rightarrow I_{k,n}(2k+1) < I_{k,n}(2k).$$

Therefore,

$$I_{k,n}(2k+2) - I_{k,n}(2k) = \sum_{j=1}^{k} (\tfrac{1}{2})^j [I_{k,n-j}(2k+2-j) - I_{k,n-j}(2k)] < 0 \Rightarrow I_{k,n}(2k+2) < I_{k,n}(2k).$$

By induction it is obtained that

$$I_{k,n}(2k) > I_{k,n}(2k+j) \text{ for any positive integer } j \text{ and } 2k+j \leqslant \lfloor (n+1)/2 \rfloor. \tag{10}$$

Hence, $I_{k,n}(3k+1) - I_{k,n}(3k) = (\frac{1}{2})^{k+1} [I_{k,n-k-1}(3k) - I_{k,n-k-1}(2k)] < 0 \Rightarrow I_{k,n}(3k+1) < I_{k,n}(3k)$. □

(2) Eq. (10) shows that $I_{k,n}(2k)$ is the upper bound for those $I_{k,n}(i)$'s with $2k+1 \leqslant i \leqslant \lfloor (n+1)/2 \rfloor$. To show $I_{k,n}(2k+1)$ is the lower bound, we use Theorem 5 and Eq. (9):

$$I_{k,n}(3k) - I_{k,n}(2k+1) = \sum_{j=1}^{k} (\tfrac{1}{2})^j [I_{k,n-j}(3k-j) - I_{k,n-j}(2k+1)] > 0 \Rightarrow I_{k,n}(3k) > I_{k,n}(2k+1).$$

By induction, it can be derived that $I_{k,n}(2k+1)$ is the lower bound for those $I_{k,n}(i)$'s with $2k+2 \leqslant i \leqslant \lfloor (n+1)/2 \rfloor$. □

Remark. Because of the symmetrical structure importance stated in Theorem 1, the two pairs of $I_{k,n}(2k)$, $I_{k,n}(n-2k+1)$ and $I_{k,n}(2k+1)$, $I_{k,n}(n-2k)$ are the upper and lower bounds for $I_{k,n}(i)$'s with $2k+2 \leqslant i \leqslant n-2k-1$, respectively.

For a further illustration, let us consider the case $k=3$. That is the consecutive-3-out-of-$n:F$ system. The structure importance ordering is

$$\begin{aligned} I_{3,n}(1) &< I_{3,n}(2) < I_{3,n}(4) < I_{3,n}(5) \\ &< I_{3,n}(7) < I_{3,n}(10) < I_{3,n}(i) < I_{3,n}(9) < I_{3,n}(6) \\ &< I_{3,n}(3) \ \text{ for } 11 \leqslant i \leqslant \lfloor (n+1)/2 \rfloor \text{ or } i=8. \end{aligned} \quad (11)$$

Because the proof is similar to the proof of Proposition 5, it is omitted.

By applying the five Propositions, one can determine that when $k=1$, or 2, or $k \geqslant n/2$, the structure importance can be completely ordered. However, for other values of k, the structure importance ordering can only be partially determined rendering our results inconclusive.

Acknowledgements

This work is supported in part by an NSF project # DMI-9400051 and a Senior Fulbright Scholarship. We would also like to acknowledge Ms. K. Oh Kim and one referee for providing many useful suggestions.

References

[1] Z.W. Birnbaum, On the importance of different components in a multicomponent system, in: P.R. Krishnaiah (Ed.), Multivariate Analysis-II, Academic Press, New York, 1969, pp. 581–592.

[2] C. Derman, G.J. Lieberman, S.M. Ross, On the consecutive-k-out-of-$n:F$ systems, IEEE Trans. Reliab. 31 (1982) 57–63.

[3] D.E. Ferguson, An expression for generalized Fibonacci numbers, Fibonacci Quart. 4 (1966) 270–273.

[4] W. Kuo, W. Zhang, M.J. Zuo, A consecutive-k-out-of-$n:G$ system: the mirror image of a consecutive-k-out-of-$n:F$ system, IEEE Trans. Reliab. 39 (1990) 244–253.

[5] E.D. Miles, Generalized Fibonacci numbers and associated metrices, Ann. Math. Monthly 60 (1962) 745–752.

[6] S. Papastavridis, The most important component in a consecutive-k-out-of-$n:F$ system, IEEE Trans. Reliab. 36 (1987) 266–268.

[7] M.J. Zuo, W. Kuo, Design and performance analysis of consecutive-k-out-of-n structure, Naval Res. Logis. 37 (1990) 203–230.