

SYMMETRY-BREAKINGS FOR SEMILINEAR ELLIPTIC EQUATIONS ON FINITE CYLINDRICAL DOMAINS

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ABSTRACT. We study the existence and multiplicity of asymmetric positive solutions of a semilinear elliptic equation on finite cylinders with mixed type boundary conditions. By using a Nehari-type variational method, we prove that the numbers of asymmetric positive solutions are increasing without bound when the lengths of cylinders are increasing. On the contrary, by using the blow up technique, we obtain an a priori bound for positive solutions and then prove that all positive solutions must be symmetric when the cylinders are short enough.

1. INTRODUCTION

Let ω be a bounded smooth domain in \mathbb{R}^n , $n \geq 1$, $a > 0$, and $\Omega_a = (-a, a) \times \omega$ be a finite cylindrical domain in $\mathbb{R}^{n+1} = \{(x, y) : x \in \mathbb{R}^1 \text{ and } y \in \mathbb{R}^n\}$. In this paper, we shall study the existence and multiplicity of positive solutions of the following semilinear elliptic equation with mixed type boundary conditions

$$(1.1) \quad \Delta u + f(u) = 0 \quad \text{in } \Omega_a,$$

$$(1.2) \quad u = 0 \quad \text{on } [-a, a] \times \partial\omega,$$

$$(1.3) \quad \frac{\partial u}{\partial x} = 0 \quad \text{on } \{-a, a\} \times \omega,$$

where f satisfies the following conditions:

(H-0) $f \in C^1(\mathbb{R}^1)$, $f(u) > 0$ for $u > 0$,

(H-1) $f(0) = 0$ and $f'(0) = 0$,

(H-2) there exists $\sigma > 0$ such that $uf'(u) \geq (1 + \sigma)f(u)$ for all $u \geq 0$,

(H-3) $f(u) \leq C(u^p + 1)$ for some $C > 0$ and all $u \geq 0$, where $1 < p < (n + 3)/(n - 1)$ if $n \geq 2$ and $1 < p < \infty$ if $n = 1$.

A solution $u \in C^2(\overline{\Omega}_a)$ of (1.1)–(1.3) is said to be symmetric (with respect to the x -axis) if u is constant along the x -axis, i.e., $u = u(y)$ and satisfies

$$(1.4) \quad \Delta u + f(u) = 0 \quad \text{in } \omega,$$

$$(1.5) \quad u = 0 \quad \text{on } \partial\omega.$$

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Otherwise, u is called an asymmetric solution. The following problem was posed by Berestycki:

- (P) For $f(u) = u^p$, taking the length a as a bifurcation parameter, is there symmetry-breaking for (1.1)–(1.3)?

Previously, it has been known that there is an asymmetric solution when a is large enough. In this paper, we answer the problem (P) partially, and hope it will lead to more complete results of the problem. The main results are as follows.

Theorem 1.1. *Assume conditions (H-0)–(H-3) are satisfied. Then there exists an increasing sequence $a_k = a_k(\omega, \sigma) \rightarrow \infty$ as $k \rightarrow \infty$, such that for each $k \geq 1$ and $a \in (a_k, \infty)$, (1.1)–(1.3) have at least k -many asymmetric positive solutions.*

On the contrary, for short cylinders we have

Theorem 1.2. *Assume f satisfies (H-0)–(H-2) and (H-3)' $\lim_{u \rightarrow \infty} f(u)/u^p = C > 0$ for $1 < p < (n+3)/(n-1)$ if $n \geq 2$ and $1 < p < \infty$ if $n = 1$. Then there exists $a_* = a_*(\omega, p) > 0$ such that for any $a \in (0, a_*)$, every positive solution of (1.1)–(1.3) is symmetric.*

The last two theorems indicate that there are asymmetric bifurcations at certain critical numbers a_k^* , $a_k^* \rightarrow \infty$ as $k \rightarrow \infty$, and the bifurcation branches will go to the direction of increasing a . However, we still need more rigorous justifications.

To prove Theorem 1.1, we need to study the problems of asymmetric instability of symmetric positive solutions and then use a Nehari-type variational method to prove that there exist asymmetric positive solutions when all symmetric positive solutions are unstable with respect to certain asymmetric mode.

To prove Theorem 1.2, we need to obtain some a priori bounds for positive solutions of (1.1)–(1.3) and then prove that asymmetric positive solutions cannot exist when the cylinders are short enough.

As for the related problems, the problems of symmetry-breaking of positive radial solutions on a ball were studied in [4, 6, 15–17], on annular domains in [3, 5, 8–11, 18, 19], and on sectorial domains in [2, 12].

The paper is organized as follows. In §2 we study the linear eigenvalue problems and obtain some asymmetric instability results for symmetric positive solutions for long cylinders. In §3 we prove the existence of asymmetric positive solutions for long cylinders. In §4 we prove the nonexistence of asymmetric positive solutions for short cylinders.

2. LINEARIZED EIGENVALUE PROBLEMS

Let $g \in C(\bar{\omega})$. Consider the following linear eigenvalue problem:

$$(2.1) \quad \Delta v + g(y)v = -\nu v \quad \text{in } \omega,$$

$$(2.2) \quad v = 0 \quad \text{on } \partial\omega.$$

Then there is the sequence $\{\nu_1, \nu_2, \dots\} = \{\nu_1(g), \nu_2(g), \dots\}$ of eigenvalues of (2.1) and (2.2) satisfying $\nu_1 < \nu_2 \leq \nu_3 \leq \dots$, where each distinct eigenvalue

is repeated according to its multiplicity. By separation of variables, the linear eigenvalue problem

$$(2.3) \quad \Delta w + g(y)w = -\lambda w \quad \text{in } \Omega_a,$$

$$(2.4) \quad w = 0 \quad \text{on } [-a, a] \times \partial\omega,$$

$$(2.5) \quad \frac{\partial w}{\partial x} = 0 \quad \text{on } \{-a, a\} \times \omega,$$

is equivalent to the problems

$$(2.6) \quad \Delta v + g(y)v = -(\lambda + \mu)v \quad \text{in } \omega,$$

$$(2.7) \quad v = 0 \quad \text{on } \partial\omega,$$

and

$$(2.8) \quad \varphi''(x) = \mu\varphi(x), \quad x \in (-a, a),$$

$$(2.9) \quad \varphi'(-a) = 0 = \varphi'(a),$$

with $w(x, y) = \varphi(x)v(y)$. It is clear that the eigenvalues of (2.8) and (2.9) are given by

$$(2.10) \quad \mu_k(a) = -(k\pi/2a)^2,$$

$k = 0, 1, 2, \dots$, and the associated eigenfunctions are

$$(2.11) \quad \varphi_k(x) = \begin{cases} \sin \frac{k\pi}{2a}x & \text{if } k \text{ is odd,} \\ \cos \frac{k\pi}{2a}x & \text{if } k \text{ is even.} \end{cases}$$

Therefore, λ is an eigenvalue of (2.3)–(2.5) if and only if $\lambda + \mu_k = \nu_l$ for some k and l . Hence, we denote

$$(2.12) \quad \lambda_{k,l} = \lambda_{k,l}(g, a) = \nu_l - \mu_k = \nu_l(g) + \left(\frac{k\pi}{2a}\right)^2$$

where $k = 0, 1, 2, \dots$ and $l = 1, 2, \dots$. Note that when $k = 0$ it corresponds to symmetric modes and $k \geq 1$ to asymmetric modes.

Let u_0 be any symmetric positive solution and $\nu_1 = \nu_1(u_0)$ be the least eigenvalue of (2.1) and (2.2) with $g(y) = f'(u_0(y))$. Then we have the following estimate for $\nu_1(u_0)$.

Lemma 2.1. *Assume condition (H-2) is satisfied. Then for any symmetric positive solution u_0 we have*

$$(2.13) \quad \nu_1(u_0) \leq -\sigma\bar{\nu}_1,$$

where $\bar{\nu}_1 = \bar{\nu}_1(\omega) > 0$ is the least eigenvalue of the Laplacian $-\Delta$ on ω with Dirichlet boundary conditions.

Proof. It is well known that $\nu_1(u_0)$ can be characterized by

$$\nu_1 = \inf \left\{ Q(v) / \int_{\omega} v^2 : v \in H_0^1(\omega) \setminus \{0\} \right\},$$

where

$$Q(v) = \int_{\omega} |\nabla v|^2 - f'(u_0)v^2.$$

Since u_0 is a solution of (1.4) and (1.5), we have

$$\int_{\omega} |\nabla u_0|^2 = \int_{\omega} u_0 f(u_0).$$

Hence, by (H-2), we have

$$\begin{aligned} Q(u_0) &= \int_{\omega} |\nabla u_0|^2 - f'(u_0)u_0^2 = \int_{\omega} u_0 f(u_0) - f'(u_0)u_0^2 \\ &\leq -\sigma \int_{\omega} u_0 f(u_0) = -\sigma \int_{\omega} |\nabla u_0|^2. \end{aligned}$$

Therefore, by using the Poincaré inequality, (2.13) follows. The proof is complete.

Note that the upper bound $-\sigma\bar{\nu}_1$ of $\nu_1(u_0)$ is independent of symmetric positive solutions u_0 .

A consequence of Lemma 2.1 is the following asymmetric instability results for long cylinders.

Theorem 2.2. *Assume condition (H-2) is satisfied. For each $k \geq 1$, let*

$$(2.14) \quad a_k = \frac{k\pi}{2}(\sigma\bar{\nu}_1)^{-1/2}.$$

Then for any $a \in (a_k, \infty)$ and any symmetric positive solution u_0 we have $\lambda_{k,1}(u_0, a) < 0$.

Proof. By (2.12), $\lambda_{k,1}(u_0, a) = \nu_1(u_0) + \mu_k(a)$; the result follows from Lemma 2.1.

Remark 2.3. Assume conditions (H-0)–(H-3) are satisfied.

(i) If $u \in C^2(\Omega_a) \cap C^0(\bar{\Omega}_a)$ is a solution of (1.1)–(1.3), then u is positive in Ω . Indeed, by (H-0) and the maximum principle, the minimum of u in $\bar{\Omega}_a$ is achieved on $\partial\Omega_a$. On the other hand, by (1.3) and the strong maximum principle, the minimum cannot be achieved on $\{-a, a\} \times \omega$. Hence the minimum will be achieved on $[-a, a] \times \partial\omega$, which implies u is positive in Ω_a .

(ii) There exist positive solutions of (1.1)–(1.3) and (1.4) and (1.5), respectively, see, e.g., [1, 14].

3. LONG CYLINDERS

In this section, we shall prove that if the cylinders are long enough, $a > a_k$, then there exist at least k -many asymmetric positive solutions. We shall use a Nehari-type variational method.

For any function u defined on Ω_a , it can be extended to \tilde{u} on $\Omega_{\infty} = (-\infty, \infty) \times \omega$. In fact, we first define

$$\tilde{u}(x, y) = u(2a - x, y) \quad \text{for } x \in (a, 3a) \text{ and } y \in \omega,$$

a reflection with respect to $x = a$. Then \tilde{u} is extended to the whole of Ω_{∞} by $4a$ periodically along x -axis. Due to the Neumann boundary conditions on $\{-a, a\} \times \omega$, the eigenfunctions $w_{k,l}$ and solutions of (1.1)–(1.3) can be extended smoothly on Ω_{∞} in this way.

We need some notation. Let $V = \{u \in H^1(\Omega_a) : u = 0 \text{ on } [-a, a] \times \partial\omega\}$. Define the functionals

$$J(u) = \int_{\Omega_a} \frac{1}{2} |\nabla u|^2 - F(u) \quad \text{and} \quad M(u) = \int_{\Omega_a} |\nabla u|^2 - uf(u)$$

on V , where $F(u) = \int_0^u f(t) dt$.

For each $k \geq 1$, the submanifolds V_k and V_∞ of V are defined by

$$V_k = V_{k,a} = \left\{ u \in V : \tilde{u} \left(x + \frac{4a}{k}, y \right) = \tilde{u}(x, y) \right. \\ \left. \text{for } (x, y) \in \Omega_\infty \text{ and } M(u) = 0 \right\}$$

and

$$V_\infty = \{u \in H_0^1(\omega) : M(u) = 0\}.$$

Let

$$I_k = I_k(a) = \inf\{J(u) : u \in V_k\}, \quad I_\infty = I_\infty(a) = \inf\{J(u) : u \in V_\infty\}.$$

Let u_0 be a symmetric positive solution, and $w_0 = v_1(y) > 0$ and $w_k = \varphi_k(x)v_1(y)$ be eigenfunctions with respect to eigenvalues $\lambda_{0,1} = \lambda_{0,1}(u_0, a)$ and $\lambda_{k,1} = \lambda_{k,1}(u_0, a)$ with $\int_{\Omega_a} w_0^2 = \int_{\Omega_a} w_k^2 = 1$. Note that for $k \geq 1$, we have

$$(3.1) \quad \int_{\Omega_a} w_k = 0.$$

Then we have the following results concerning the change of $J(u)$ along the direction of asymmetric mode w_k .

Lemma 3.1. *Assume conditions (H-0)–(H-2) are satisfied. Then there exist $\varepsilon > 0$ and a smooth function $\delta : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^1$ with $\delta(0) = \delta'(0) = 0$ such that*

$$(3.2) \quad M(u_0 + \delta(t)w_0 + tw_k) = 0$$

for all $t \in (-\varepsilon, \varepsilon)$. Furthermore, we have

$$(3.3) \quad J(u_0 + \delta(t)w_0 + tw_k) = J(u_0) + \frac{1}{2}\lambda_{0,1}\delta(t)^2 + \frac{1}{2}\lambda_{k,1}t^2 + O(t^4),$$

as $t \sim 0$.

Proof. The proofs are similar to Lemmas 6.1 and 6.2 in [11]. (3.2) is proved by using the Implicit Function Theorem and (3.3) is proved by straightforward computations; the details are omitted. Note that (3.1) is used repeatedly in the proofs.

To prove Theorem 1.1, we need the following result, which indicates that asymmetric minimizers are different for different modes.

Lemma 3.2. *Assume conditions (H-0)–(H-3) are satisfied. If $I_{km} < I_\infty$ then $I_k < I_{km}$ for $k = 1, 2, \dots$ and $m = 2, 3, \dots$*

Proof. Under the assumptions of (H-0)–(H-3), it is rather standard to verify that the Palais-Smale condition holds, see, e.g., [1, 14]. Therefore, the minimizers of I_k and I_∞ are achieved by some functions $u_k \in V_k$ and $u_0 \in V_\infty$ for all k .

Let $u(x, y)$ be a minimizer of $J(u)$ on V_{km} . Since $I_{km} < I_\infty$,

$$(3.4) \quad \partial u / \partial x \neq 0 \quad \text{on } \Omega_a.$$

Let

$$v(x, y) = \begin{cases} u\left(\frac{x}{m}, y\right) & \text{for } x \in (-a, a) \text{ when } km \text{ is odd,} \\ u\left(\frac{x+a}{m}, y\right) & \text{for } x \in (-a, a) \text{ when } km \text{ is even.} \end{cases}$$

Then for km odd we have

$$(3.5) \quad v\left(x + \frac{4a}{k}, y\right) = u\left(\frac{x}{m} + \frac{4a}{mk}, y\right) = u\left(\frac{x}{m}, y\right) = v(x, y).$$

Furthermore, we have

$$(3.6) \quad \begin{aligned} \int_{\Omega_a} |\nabla v|^2 &= \int_{\omega} \int_{-a}^a \left\{ \left(\frac{\partial v}{\partial x} \right)^2 + (\nabla_y v)^2 \right\} dx dy \\ &= m \int_{\omega} \int_{-a/m}^{a/m} \left\{ \frac{1}{m^2} \left(\frac{\partial u}{\partial x} \right)^2 + (\nabla_y u)^2 \right\} dx dy < \int_{\Omega_a} |\nabla u|^2. \end{aligned}$$

Similarly, if km is even then (3.5) and (3.6) can also be proved. Since u and v are periodic in x of periods km and k , respectively, it can be proved that

$$(3.7) \quad \int_{\Omega_a} v f(v) = \int_{\Omega_a} u f(u) \quad \text{and} \quad \int_{\Omega_a} F(v) = \int_{\Omega_a} F(u).$$

Therefore, (3.6) and (3.7) imply $M(v) < 0$. Since $f'(0) = 0$, we have $M(tv) > 0$ for $t > 0$ and sufficiently small. This implies that there exists a $t_1 \in (0, 1)$ such that $M(t_1 v) = 0$.

Now, by (H-2), we have

$$\frac{d}{dt} \left\{ \frac{1}{2} t u f(tu) - F(tu) \right\} = \frac{u}{2} \{ t u f'(tu) - f(tu) \} > 0$$

for $u > 0$. Therefore, we have

$$\begin{aligned} I_k &\leq J(t_1 v) = \int_{\Omega_a} \frac{1}{2} t_1^2 |\nabla v|^2 - F(t_1 v) \\ &= \int_{\Omega_a} \frac{1}{2} t_1 v f(t_1 v) - F(t_1 v) < \int_{\Omega_a} \frac{1}{2} v f(v) - F(v) \\ &= \int_{\Omega_a} \frac{1}{2} u f(u) - F(u) = I_{km}. \end{aligned}$$

The proof is complete.

Proof of Theorem 1.1. For each $k \geq 1$ and $a \in (a_k, \infty)$, a_k given by (2.14), the minimizers of $I_j(a)$ and $I_\infty(a)$ are achieved by some functions $u_j \in V_j$ and $u_0 \in V_\infty$, $j = 1, \dots, k$. By (3.3), $I_j(a) < I_\infty(a)$, $j = 1, \dots, k$.

Now let $1 \leq i < j \leq k$ and l be the least common multiple of i and j . If $j = l$ then, by Lemma 3.2, u_i and u_j are different. If $j < l$ then $V_i \cap V_j = V_l$. If $u_i = u_j = \hat{u} \in V_l$ then

$$(3.8) \quad I_l \leq J(\hat{u}) = I_i < I_\infty.$$

By (3.8) and Lemma 3.2, we obtain $I_i(a) < I_l(a)$, a contradiction to (3.8). Hence, u_i and u_j are different. The proof is complete.

4. SHORT CYLINDERS

In this section, we shall prove that there is no asymmetric positive solution of (1.1)–(1.3) when the cylinders are short enough.

We need the following a priori estimates.

Lemma 4.1. *Assume conditions (H-0)–(H-3)' are satisfied. Then there exists a constant $C = C(\omega) > 0$ such that for any $a > 0$ and any positive solution u_a of (1.1)–(1.3), we have*

$$(4.1) \quad \|u_a\|_\infty = \max\{|u_a(x)| : x \in \Omega_a\} \leq C.$$

Proof. Since the extension \tilde{u}_a of u_a is smooth in Ω_∞ , i.e., $\tilde{u}_a \in C^2(\overline{\Omega_\infty})$, the boundary $\{-a, a\} \times \omega$ is now an interior part of Ω_∞ . Since $\tilde{u}_a > 0$ in Ω_∞ and satisfies

$$\begin{aligned} \Delta \tilde{u}_a + f(\tilde{u}_a) &= 0 && \text{in } \Omega_\infty, \\ \tilde{u}_a &= 0 && \text{on } \partial\Omega_\infty, \end{aligned}$$

by using the technique of “blow up” as in [7], we can prove (4.1). The details are omitted.

Proof of Theorem 1.2. For $a > 0$, if u_a is an asymmetric positive solution of (1.1)–(1.3), then

$$(4.2) \quad w_a \equiv \partial u_a / \partial x \neq 0$$

and w_a satisfies

$$(4.3) \quad \Delta w_a + f'(u_a)w_a = 0 \quad \text{in } \Omega_a,$$

$$(4.4) \quad w_a = 0 \quad \text{on } \partial\Omega_a.$$

Let $\tilde{\lambda}_1(a) > 0$ be the least eigenvalue of the Laplacian $-\Delta$ on Ω_a with Dirichlet boundary condition. Then it is easy to verify that

$$(4.5) \quad \lim_{a \rightarrow 0^+} \tilde{\lambda}_1(a) = \infty.$$

On the other hand, the least eigenvalue $\eta_1(a)$ of the linearized eigenvalue problem

$$\begin{aligned} \Delta w + f'(u_a)w &= -\eta w && \text{in } \Omega_a, \\ w &= 0 && \text{on } \partial\Omega_a \end{aligned}$$

can be characterized by

$$\eta_1(a) = \inf \left\{ Q(v) / \int_{\Omega_a} v^2 : v \in H_0^1(\Omega_a) \setminus \{0\} \right\},$$

where

$$Q(v) = \int_{\Omega_a} |\nabla v|^2 - f'(u_a)v^2.$$

By Lemma 4.1, there exists a constant $C_1 > 0$ such that

$$(4.6) \quad \|f'(u_a)\|_\infty \leq C_1.$$

By the Poincaré inequality, we have

$$(4.7) \quad \int_{\Omega_a} |\nabla v|^2 \geq \tilde{\lambda}_1(a) \int_{\Omega_a} v^2$$

for all $v \in H_0^1(\Omega_a)$. Hence, by (4.5), (4.6), and (4.7), there exists $a_* > 0$ such that

$$(4.8) \quad \eta_1(a) > 0$$

for all $a \in (0, a_*)$. Therefore, (4.2), (4.3), (4.4), and (4.8) imply that there is no asymmetric positive solution for $a \in (0, a_*)$. The proof is complete.

Remark 4.2. The results are still valid if $f(u)$ is replaced by $f(y, u)$, which satisfies similar conditions as (H-0)–(H-3) and (H-3)'.

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