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A BAYESIAN PROCEDURE FOR PROCESS CAPABILITY ASSESSMENT

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SUMMARY

The usual practice of judging process capability by evaluating point estimates of some process capability indices has a flaw that there is no assessment on the error distributions of these estimates. However, the distributions of these estimates are usually so complicated that it is very difficult to obtain good interval estimates. In this paper we adopt a Bayesian approach to obtain an interval estimation, particularly for the index C_{pm}. The posterior probability p that the process under investigation is capable is derived; then the credible interval, a Bayesian analogue of the classical confidence interval, can be obtained. We claim that the process is capable if all the points in the credible interval are greater than the pre-specified capability level ω, say 1.33. To make this Bayesian procedure very easy for practitioners to implement on manufacturing floors, we tabulate the minimum values of $\hat{C}_{pm}/\omega$, for which the posterior probability p reaches the desirable level, say 95%. For the special cases where the process mean equals the target value for C_{pm} and equals the midpoint of the two specification limits for C_{pk}, the procedure is even simpler; only chi-square tables are needed. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: process capability indices; quality; Bayesian approach; confidence interval; credible interval; prior; posterior

1. INTRODUCTION

Process capability indices (PCIs) are unitless measures for the capability of a process in meeting specification limits. These indices have been widely used in assessing the capability of manufacturing processes by many companies during the last decade. More and more efforts have been devoted to studies and applications of PCIs. For example, Rado [1] presented how Imprimis Technology, Inc. used the PCIs to enhance product development, and the C_p and C_{pk} indices have been used in Japan and in the US automotive industry such as Ford Motor Company [2,3]. To incorporate the departure of the process mean μ from the target value T, the index C_{pm} was proposed [4]. This index has been getting more and more recognition in industries in recent years.

A capable process is usually defined as a process with a certain process capability index greater than a pre-specified value ω. The usual practice is to estimate the PCI from process data. If the estimate is greater than the pre-specified value ω, say 1 or 1.33, then it is claimed that the process is capable. Of course, the estimate is not the index itself, so when the estimate is greater than ω, it does not guarantee that the index is greater than ω, and *vice versa*. Thus it is usually preferable to obtain an interval estimate, for which we can assert with a reasonable degree of certainty that it contains the true PCI value. However, the construction of such an interval estimate is not an easy task, since the distributions of the commonly used PCI estimators are usually quite complicated [4–8].

Therefore it is very natural to consider a Bayesian approach. By a Bayesian approach, it means that we first specify a prior distribution for the parameter of interest, obtain the posterior distribution of the parameter and then infer about the parameter by only using its posterior distribution given the observations. The reason why it is natural to consider a Bayesian approach is that for Bayesian estimation it is always very easy to obtain the posterior distribution when a prior distribution is given; and even when the form of the posterior distribution is complicated, it is still easy

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to use numerical methods or Monte Carlo methods [9] to obtain an approximate point estimate or interval estimate. This is a great advantage of the Bayesian approach over the classical frequentist approach.

More specifically, to assess the process capability, it is natural to consider the posterior probability $\Pr\{\text{process is capable}|\mathbf{x}\}$. Compared with the usual practice of just obtaining point estimates of PCIs, this Bayesian approach has the advantage of providing a statement on the posterior probability that the process is capable given the observed process data.

A nice Bayesian procedure for assessing process capability was proposed in Reference [5] for the index C_{p}, also in Reference [4] for the index C_{pm} under the assumption that the process mean μ is equal to the target value T. In general, C_{pm} is a better PCI than C_{p} [4]. However, the restriction that $\mu = T$ is a notable shortcoming, since the process mean may be quite deviated from the target value T in many industrial applications.

The main objective of this paper is to provide a Bayesian procedure for the general situation—no restriction on the process mean μ. In addition, for the restricted case in which $\mu = T$, we provide a simple procedure for computing the posterior probability of the process being capable. Instead of using approximation or numerical integration as in Reference [4], this posterior probability can be obtained by simply looking up the commonly available chi-square tables. A similar Bayesian procedure was given in Reference [10] for the restricted case.

Throughout this paper it is assumed that the process measurements are independent and identically distributed from a normal distribution. In other words, the process is under statistical control. We remark that estimation of PCIs is meaningful only when the process is under statistical control.

This paper is organized as follows. We give a brief review on four popular PCIs—C_{p}, C_{pk}, C_{pm}, and C_{pmk}—in Section 2. In Section 3 we present a Bayesian procedure for assessing the process capability based on C_{pm}. All the derivations are given in the Appendix. In Section 4 we describe a Bayesian procedure based on C_{pk}, but only for the special case in which the process mean is equal to the midpoint of the two specification limits. In Section 5 we present some examples to illustrate the Bayesian procedure, and compare the results with those obtained from the procedure given in Reference [4]. Finally we conclude the paper in Section 6.

2. A REVIEW ON SOME POPULAR PCIs

The index C_{p} is defined as

$$C_{\mathrm{p}} = \frac{\mathrm{USL} - \mathrm{LSL}}{6\sigma}$$

where USL and LSL denote the upper and lower specification limits respectively and σ is the process standard deviation of the quality characteristic of interest. The process standard deviation is usually unknown and can be estimated from a sample of n measurements $x_1, x_2, \ldots, x_n$. The most common estimate of σ is the sample standard deviation

$$s = \left(\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^{1/2}$$

where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

is the sample mean. This gives an estimate of C_{p},

$$\hat{C}_{\mathrm{p}} = \frac{\mathrm{USL} - \mathrm{LSL}}{6s}$$

We remark that other estimates of σ can be used. For example, it is very common to use subgroup ranges to obtain an estimate of σ to guard against shifting of the mean in practice, since many processes in the industry may be just semi-stable.

In order to reflect the impact of the deviation of the process mean μ from the midpoint m of the specification limits on the process capability, several indices have been proposed, including

$$\mathrm{CPU} = \frac{\mathrm{USL} - \mu}{3\sigma}$$

$$\mathrm{CPL} = \frac{\mu - \mathrm{LSL}}{3\sigma}$$

and

$$C_{\mathrm{pk}} = \min(\mathrm{CPL}, \mathrm{CPU}) \tag{1}$$

C_{pk} is sometimes defined as

$$C_{\mathrm{pk}} = (1-k)C_{\mathrm{p}} \tag{2}$$

where $k = 2|m - \mu|/(\mathrm{USL}-\mathrm{LSL})$. The above two definitions of C_{pk}, (1) and (2), are algebraically equivalent [2].

These indices are usually estimated respectively by

$$\widehat{\mathrm{CPU}} = \frac{\mathrm{USL} - \bar{x}}{3s}$$

$$\widehat{\mathrm{CPL}} = \frac{\bar{x} - \mathrm{LSL}}{3s}$$

and

$$\hat{C}_{pk} = \min(\widehat{CPL}, \widehat{CPU})$$

For the C_{pk} defined in (2), it can be estimated by

$$\hat{C}_{pk} = (1 - \hat{k})\hat{C}_p$$

where

$$\hat{k} = \frac{2\,|m - \bar{x}|}{USL - LSL}$$

Both C_p and C_{pk} are independent of the target value T. To account for the impact of the deviation of the process mean from the target value, another PCI called C_{pm} is defined [4] as

$$C_{pm} = \frac{USL - LSL}{6\sigma'} \tag{3}$$

where

$$\sigma' = [E(X - T)^2]^{1/2} = [\sigma^2 + (\mu - T)^2]^{1/2} \tag{4}$$

Chan *et al.* [4] estimated σ' by

$$\left(\frac{1}{n-1}\sum_{i=1}^{n}(x_i - T)^2\right)^{1/2}$$

In this paper, instead of using their estimator, we use

$$\hat{\sigma}' = \left(\frac{1}{n}\sum_{i=1}^{n}(x_i - T)^2\right)^{1/2}$$

to estimate σ'. The reason we use this estimator is that

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - T)^2$$

is both an unbiased estimator and the maximum likelihood estimator for σ'^2. The resulting estimator of C_{pm} is

$$\hat{C}_{pm} = \frac{USL - LSL}{6\hat{\sigma}'}$$

From (3) and (4), it is easy to see that C_{pm} and C_p have the relationship

$$C_{pm} = \frac{C_p}{\sqrt{1 + \left|\frac{T-\mu}{\sigma}\right|^2}} \tag{5}$$

and the relationship between $\hat{C}_{pm}$ and $\hat{C}_p$ is

$$\hat{C}_{pm} = \frac{\hat{C}_p}{\sqrt{\frac{n-1}{n} + \left|\frac{T-\bar{x}}{s}\right|^2}}$$

Thus by (5) it is clear that $C_{pm} = C_p$ when $\mu = T$.

Combining the ideas of C_{pk} and C_{pm}, Pearn *et al.* [11] proposed another index called C_{pmk} defined as

$$C_{pmk} = \min\left\{\frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\}$$

The estimator $\widehat{C}_{pmk}$ can be obtained by plugging in $\bar{x}$ for μ and s for σ. The study of this index is beyond the scope of this study.

There have been some studies on the distributions of these PCIs. When the process measurements follow a normal distribution, both $\widehat{CPL}$ and $\widehat{CPU}$ have a probability density function proportional to a non-central t distribution [4,6]. Chou and Owen [7] gave the exact distribution of $\hat{C}_{pk}$, distribution mean, variance, and mean-squared error. Another interpretation for the distribution of $\hat{C}_{pk}$ was given in Reference [8], where it was shown that the distribution of $\hat{C}_{pk}$ is related to the folded normal distribution. Many properties of C_{pm} and $\hat{C}_{pm}$ were given in Reference [4]. More distributional and estimation properties for the above PCIs were given in Reference [11]. These studies indicated that the statistical distributions associated with these PCI estimators are quite complicated.

In the next section we derive a Bayesian interval estimate for C_{pm} and propose accordingly a Bayesian procedure for process capability assessment. Other approaches to obtaining interval estimates for PCIs have been suggested in the literature. For example, Bittanti *et al.* [12] suggested a curve-fitting approach based on the Pearson system of curves for PCI estimation, followed by application of the bootstrap to obtain an interval estimate. Their method is applicable to non-normal processes, but with fairly high computational cost.

The following two sections are more mathematically/statistically involved. The proposed Bayesian procedure is illustrated by examples in Section 5. Readers who are not interested in the derivation of the procedure may skip to Section 5.

3. A BAYESIAN PROCEDURE BASED ON C_{pm}

Cheng and Spiring [5] proposed a Bayesian approach for assessing process capability by finding a credible interval for the index C_p. A $100p\%$ credible interval is the Bayesian analogue of the classical $100p\%$ confidence interval, where p is a number between

0 and 1, say 0.95 for 95% confidence interval. It covers $100p\%$ of the posterior distribution of the parameter [13]. Chan *et al.* [4] used the same approach to find an exact and an approximate credible interval for the index C_{pm} when $\mu = T$. Without assuming $\mu = T$, we present a Bayesian procedure based on C_{pm} in this section.

Assume that the measurements $\{X_i, i = 1, \ldots, n\}$ of the quality characteristic obtained from the process are independent and identically distributed (i.i.d.) from $N(\mu, \sigma^2)$. Denote $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, where x_i is the observed value of $X_i, i = 1, \ldots, n$. Then the likelihood function for μ and σ is

$$L(\mu, \sigma|\mathbf{x}) = (2\pi\sigma^2)^{-n/2} \times \exp\left(-\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^2}\right) \tag{6}$$

For the Bayesian approach the first step is to find an appropriate prior. Usually, when there is little or no prior information, we use non-informative priors. When there is only one parameter, one of the most widely used non-informative priors is the so-called reference prior, which is a non-informative prior that maximizes the difference between information (entropy) on the parameter provided by the prior and by the posterior. In other words, the reference prior allows the prior to provide information about the parameter as little as possible. See Reference [14] for more details. Also, with the reference prior the $100p\%$ credible interval has the coverage probability close to p up to the second order—in contrast to the first order for any other priors—in the frequentist sense [15]. More specifically, the credible interval obtained from a non-informative prior has a more precise coverage probability than that obtained from any other priors.

However, when there is more than one parameter, it is not always possible to find the reference prior by maximizing the information difference. For this reason, Berger and Bernardo [16] suggested a step-by-step procedure for finding a multiparameter prior. In this paper we adopt this step-by-step procedure and the resulting prior is

$$\pi(\mu, \sigma) = 1/\sigma \quad 0 < \sigma < \infty, \quad -\infty < \mu < \infty \tag{7}$$

As derived in the Appendix, the posterior probability density function (PDF) of (μ, σ) is

$$f(\mu, \sigma|\mathbf{x}) = \left(\frac{\sqrt{2n}}{\sqrt{\pi}\,\Gamma(\alpha)\beta^{\alpha}}\right)\sigma^{-(n+1)} \times \exp\left(-\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^2}\right) \tag{8}$$

where $\alpha = (n-1)/2$ and $\beta = [\sum_{i=1}^{n}(x_i - \bar{x})^2/2]^{-1} = [(n-1)s^2/2]^{-1}$. A reparametrized version of (7) and (8) with σ replaced by σ^2 can be found in Problem 16 of Chap. 4 in Reference [13].

As mentioned before, it is natural to consider the quantity Pr{process is capable$|\mathbf{x}$} in the Bayesian approach. Since the index C_{pm} is our major concern in this paper, we are interested in finding the posterior probability $p = \Pr\{C_{pm} > \omega|\mathbf{x}\}$ for some fixed positive number ω. Denote $\delta = |T - \bar{x}|/s$. It is derived in the Appendix that

$$p = \int_0^t \left(\frac{1}{\Gamma(\alpha)\gamma^{\alpha} y^{\alpha+1}}\right)\exp\left(-\frac{1}{\gamma y}\right) \times [\Phi(b_1(y) + b_2(y)) - \Phi(b_1(y) - b_2(y))]\,\mathrm{d}y \tag{9}$$

where

$$t = \frac{2}{n}\left(\frac{\hat{C}_{pm}}{\omega}\right)^2$$

$$\gamma = 1 + \frac{n}{n-1}\delta^2$$

$$b_1(y) = \sqrt{\frac{2}{y}}\left[\delta^2 \Big/ \left(\delta^2 + \frac{n-1}{n}\right)\right]^{1/2}$$

$$b_2(y) = \sqrt{n}\left(\frac{t}{y} - 1\right)^{1/2}$$

Note that the posterior probability p depends on n, δ, ω and $\hat{C}_{pm}$ only through n, δ and $\hat{C}_{pm}/\omega$. Denote $C^* = \hat{C}_{pm}/\omega$.

From expression (9) we can see that it is very difficult to compute p for any process either on-line or off-line in practice without serious computer programming. However, by noticing that there is a one-to-one correspondence between p and C^* when n and δ are given, and by the fact that $\hat{C}_{pm}$ can be easily calculated from the process data, we find that the minimum value of C^* required to ensure the posterior probability p reaching a certain desirable level can be useful in practice to assess the process capability. Denote this minimum value by $C^*(p)$.

For users' convenience in applying our Bayesian procedure in practice, we tabulate $C^*(p)$ (for various values of n and $\delta = |T - \bar{x}|/s$) in Tables 1(a)–1(c) for $p = 0.90$, 0.95, and 0.99 respectively. More specifically, the entries in these tables are values of $C^*(p)$ such that

$$P\left(C_{pm} > \frac{\hat{C}_{pm}}{C^*(p)}\,\middle|\,\mathbf{x}\right) = p \tag{10}$$

Table 1(a). Values of $C^*(p)$ for $p = 0.90$

	$\|T - \bar{x}\|/s$				
n	0	0.5	1	1.5	2
5	2.3863	2.1643	1.8222	1.5857	1.4394
10	1.6326	1.5466	1.4068	1.3036	1.2360
15	1.4386	1.3856	1.2954	1.2255	1.1777
20	1.3464	1.3082	1.2406	1.1859	1.1477
25	1.2915	1.2617	1.2070	1.1613	1.1287
30	1.2546	1.2302	1.1838	1.1440	1.1154
35	1.2279	1.2072	1.1667	1.1312	1.1055
40	1.2075	1.1895	1.1534	1.1212	1.0976
45	1.1913	1.1754	1.1427	1.1131	1.0913
50	1.1781	1.1639	1.1339	1.1064	1.0860
55	1.1672	1.1542	1.1265	1.1008	1.0815
60	1.1578	1.1460	1.1201	1.0958	1.0776
65	1.1498	1.1389	1.1145	1.0915	1.0743
70	1.1428	1.1326	1.1097	1.0878	1.0713
75	1.1366	1.1271	1.1054	1.0845	1.0686
80	1.1312	1.1222	1.1015	1.0815	1.0663
85	1.1262	1.1178	1.0981	1.0788	1.0641
90	1.1217	1.1138	1.0949	1.0763	1.0621
95	1.1178	1.1102	1.0920	1.0741	1.0603
100	1.1141	1.1068	1.0894	1.0720	1.0587
110	1.1075	1.1010	1.0846	1.0684	1.0557
120	1.1020	1.0960	1.0806	1.0651	1.0532
130	1.0972	1.0916	1.0771	1.0624	1.0509
140	1.0929	1.0877	1.0739	1.0599	1.0490
150	1.0892	1.0843	1.0712	1.0577	1.0472
160	1.0859	1.0812	1.0687	1.0557	1.0456
170	1.0828	1.0785	1.0664	1.0539	1.0442
180	1.0801	1.0759	1.0644	1.0523	1.0429
190	1.0776	1.0736	1.0625	1.0509	1.0416
200	1.0753	1.0715	1.0608	1.0494	1.0405
210	1.0733	1.0695	1.0592	1.0482	1.0395
220	1.0712	1.0678	1.0577	1.0470	1.0386
230	1.0694	1.0661	1.0563	1.0459	1.0377
240	1.0678	1.0645	1.0550	1.0449	1.0369
250	1.0662	1.0631	1.0538	1.0439	1.0361
260	1.0647	1.0617	1.0527	1.0430	1.0353
270	1.0634	1.0605	1.0516	1.0421	1.0346
280	1.0621	1.0592	1.0507	1.0414	1.0339
290	1.0608	1.0581	1.0497	1.0406	1.0334
300	1.0597	1.0571	1.0488	1.0398	1.0328

Table 1(b). Values of $C^*(p)$ for $p = 0.95$

	$\|T - \bar{x}\|/s$				
n	0	0.5	1	1.5	2
5	2.9272	2.6268	2.1584	1.8293	1.6234
10	1.8319	1.7209	1.5389	1.4033	1.3139
15	1.5687	1.5017	1.3862	1.2952	1.2330
20	1.4465	1.3989	1.3127	1.2420	1.1925
25	1.3746	1.3377	1.2682	1.2092	1.1672
30	1.3265	1.2965	1.2377	1.1865	1.1496
35	1.2919	1.2665	1.2153	1.1697	1.1365
40	1.2655	1.2435	1.1979	1.1565	1.1262
45	1.2447	1.2254	1.1840	1.1460	1.1179
50	1.2277	1.2105	1.1726	1.1372	1.1110
55	1.2136	1.1979	1.1629	1.1298	1.1051
60	1.2017	1.1873	1.1547	1.1235	1.1001
65	1.1914	1.1782	1.1475	1.1179	1.0958
70	1.1824	1.1702	1.1412	1.1131	1.0918
75	1.1745	1.1631	1.1356	1.1088	1.0884
80	1.1674	1.1568	1.1306	1.1049	1.0853
85	1.1612	1.1511	1.1261	1.1014	1.0825
90	1.1554	1.1460	1.1221	1.0982	1.0800
95	1.1503	1.1413	1.1183	1.0953	1.0777
100	1.1456	1.1370	1.1149	1.0926	1.0755
110	1.1373	1.1294	1.1089	1.0879	1.0717
120	1.1302	1.1230	1.1036	1.0838	1.0685
130	1.1240	1.1174	1.0990	1.0802	1.0655
140	1.1187	1.1125	1.0950	1.0771	1.0630
150	1.1139	1.1080	1.0914	1.0742	1.0607
160	1.1097	1.1041	1.0882	1.0717	1.0586
170	1.1058	1.1005	1.0853	1.0693	1.0568
180	1.1023	1.0974	1.0827	1.0673	1.0550
190	1.0991	1.0944	1.0802	1.0653	1.0536
200	1.0961	1.0916	1.0781	1.0636	1.0521
210	1.0935	1.0892	1.0760	1.0619	1.0508
220	1.0910	1.0869	1.0741	1.0604	1.0496
230	1.0887	1.0848	1.0723	1.0590	1.0484
240	1.0866	1.0828	1.0706	1.0576	1.0473
250	1.0846	1.0809	1.0691	1.0564	1.0463
260	1.0827	1.0791	1.0677	1.0552	1.0454
270	1.0809	1.0775	1.0663	1.0542	1.0445
280	1.0793	1.0759	1.0650	1.0531	1.0437
290	1.0778	1.0745	1.0638	1.0521	1.0429
300	1.0763	1.0731	1.0627	1.0513	1.0421

Table 1(c). Values of $C^*(p)$ for $p = 0.99$

	$\lvert T - \bar{x}\rvert/s$				
n	0	0.5	1	1.5	2
5	4.5430	4.0165	3.1800	2.5761	2.1891
10	2.3203	2.1454	1.8567	1.6404	1.4974
15	1.8678	1.7660	1.5891	1.4496	1.3541
20	1.6689	1.5981	1.4685	1.3618	1.2872
25	1.5550	1.5011	1.3976	1.3096	1.2469
30	1.4804	1.4371	1.3501	1.2741	1.2194
35	1.4272	1.3910	1.3155	1.2482	1.1993
40	1.3871	1.3561	1.2890	1.2281	1.1836
45	1.3557	1.3284	1.2680	1.2122	1.1710
50	1.3303	1.3060	1.2508	1.1991	1.1607
55	1.3092	1.2874	1.2363	1.1879	1.1519
60	1.2914	1.2715	1.2239	1.1785	1.1445
65	1.2762	1.2579	1.2133	1.1703	1.1380
70	1.2629	1.2460	1.2039	1.1630	1.1322
75	1.2513	1.2356	1.1957	1.1567	1.1271
80	1.2409	1.2262	1.1884	1.1510	1.1226
85	1.2317	1.2179	1.1817	1.1459	1.1185
90	1.2233	1.2103	1.1756	1.1412	1.1148
95	1.2158	1.2035	1.1702	1.1369	1.1114
100	1.2089	1.1972	1.1652	1.1330	1.1083
110	1.1969	1.1861	1.1564	1.1260	1.1027
120	1.1865	1.1767	1.1487	1.1201	1.0979
130	1.1775	1.1684	1.1421	1.1148	1.0937
140	1.1697	1.1613	1.1363	1.1102	1.0900
150	1.1628	1.1549	1.1310	1.1061	1.0867
160	1.1566	1.1491	1.1263	1.1025	1.0837
170	1.1511	1.1440	1.1221	1.0990	1.0810
180	1.1460	1.1393	1.1183	1.0960	1.0786
190	1.1414	1.1350	1.1148	1.0933	1.0763
200	1.1372	1.1310	1.1116	1.0907	1.0743
210	1.1334	1.1275	1.1086	1.0883	1.0724
220	1.1298	1.1242	1.1058	1.0862	1.0706
230	1.1265	1.1210	1.1033	1.0841	1.0689
240	1.1234	1.1181	1.1009	1.0822	1.0674
250	1.1205	1.1154	1.0987	1.0804	1.0659
260	1.1178	1.1129	1.0966	1.0788	1.0646
270	1.1153	1.1106	1.0946	1.0772	1.0634
280	1.1129	1.1083	1.0928	1.0757	1.0621
290	1.1106	1.1062	1.0910	1.0743	1.0610
300	1.1085	1.1042	1.0893	1.0730	1.0599

We comment that the computations in creating these tables are rather involved and quite time-consuming.

According to Definition 3 on p. 102 of Reference [13], we can see from (10) that $[\hat{C}_{pm}/C^*(p), \infty)$ is a $100p\%$ credible interval for C_{pm}, which means that the posterior probability that the credible interval contains C_{pm} is p. In our Bayesian approach we say that the process is capable in a Bayesian sense if all the points in this credible interval are greater than a pre-specified value of ω, say 1 or 1.33. When this happens, we have Pr{process is capable|$\mathbf{x}$}$> p$. In other words, to see if a process is capable (with capability level ω and confidence level p), we only need to check if $\hat{C}_{pm} > \omega C^*(p)$.

From these tables we observe that for each fixed p and n the value of $C^*(p)$ decreases as δ increases. This phenomenon can be explained by the following argument. For a fixed $\hat{C}_{pm}$, since

$$\hat{C}_{pm} = (\text{USL} - \text{LSL}) \Big/ \left(6s\sqrt{\frac{n-1}{n} + \delta^2}\right)$$

s becomes smaller when δ becomes larger, and a smaller s means that it is plausible that the underlying process is tighter (i.e. with smaller σ). Since the estimation is usually more accurate with the data drawn from a tighter process, it is then plausible that the estimate $\hat{C}_{pm}$ is more accurate with a smaller s. In this case the required minimum value $C^*(p)$ is smaller, since we need only a smaller $C^*(p)$ to account for the smaller uncertainty in the estimation. Intuitively, if the estimation error in our estimate is potentially large, then it is reasonable that we need a large $\hat{C}_{pm}$ to be able to claim that the process is capable, and this means that the corresponding minimum value $C^*(p)$ should be large as well. Thus the value of $C^*(p)$ decreases as δ increases. Another observation from the tables is that the value of $C^*(p)$ decreases as n increases for fixed δ and p. This can also be explained by the same reasoning as above, since a larger n implies that $\hat{C}_{pm}$ is more accurate.

4. A BAYESIAN PROCEDURE FOR C_{pk} WHEN $\mu = m$ AND C_{pm} WHEN $\mu = T$

Owing to the complication of the distribution of $\hat{C}_{pk}$, we can only discuss the special case in which $\mu = m$, where m is the midpoint of the two specification limits. In this case, in fact, C_{pk} is reduced to C_p, since $C_{pk} = (d - |\mu - m|)/3\sigma = d/3\sigma = C_p$, where $d = (\text{USL} - \text{LSL})/2$. Then we can estimate C_{pk} by

$\hat{C}_{pk} = (\text{USL}-\text{LSL})/6\tilde{\sigma}$, where

$$\tilde{\sigma} = \left(\frac{1}{n}\sum_{i=1}^{n}(x_i - m)^2\right)^{1/2}$$

Suppose that the measurements are i.i.d. from $N(m, \sigma^2)$. Then the likelihood function for σ is

$$L(\sigma|\mathbf{x}) = (2\pi\sigma^2)^{-n/2} \times \exp\left(-\frac{\sum_{i=1}^{n}(x_i - m)^2}{2\sigma^2}\right)$$

Consider the non-informative reference prior

$$\pi(\sigma) = 1/\sigma \quad 0 < \sigma < \infty$$

Then the posterior distribution of σ^2 is an inverse Gamma distribution with the probability density function

$$f(\sigma^2|\mathbf{x}) = \frac{1}{\Gamma(n/2)}\left(\frac{n\tilde{\sigma}^2}{2}\right)^{n/2}(\sigma^2)^{-((n/2)+1)} \times \exp\left(-\frac{n\tilde{\sigma}^2}{2\sigma^2}\right) \quad 0 < \sigma^2 < \infty$$

We remark that this posterior PDF is exactly the same as that of C^2_{pm} when $\mu = T$, the case considered in Reference [4]. This is quite obvious, since the indices in both cases are reduced to the index C_p. Thus many results in Reference [4] for C_{pm} when $\mu = T$ are applicable to C_{pk} when $\mu = m$. Chan *et al.* [4] tabulated approximate $C^*(p)$ values for C_{pm} when $\mu = T$. These values can be used for C_{pk} when $\mu = m$ with some minor modification. However, there is a more straightforward Bayesian procedure to assess the process capability in these two cases.

Let $Y = n\tilde{\sigma}^2/2\sigma^2$. It can be derived easily that $2Y$ has a chi-square distribution with n degrees of freedom. Then the posterior probability of C_{pk} being greater than a value ω is

$$p = \Pr\{C_{pk} > \omega|\mathbf{x}\} = \int_0^a f(\sigma|\mathbf{x})\,d\sigma = \int_b^\infty \frac{1}{\Gamma(n/2)} y^{(n/2)-1}\,e^{-y}\,dy = \Pr\{2Y > 2b\}$$

where $a = (\text{USL}-\text{LSL})/6\omega$ and $b = (n/2)(\omega/\hat{C}_{pk})^2$. Thus, to compute p, we can use the commonly available chi-square tables. If p is greater than a desirable level, say 90% or 95%, then we may claim that the process is capable in a Bayesian sense with 90% or 95% confidence.

By the same nature, the Bayesian procedure based on C_{pm} under the assumption $\mu = T$ is similar. Thus we can summarize our Bayesian procedure for these two special cases as follows. Let $C^* = \widehat{\text{PCI}}/\omega$, where $\widehat{\text{PCI}}$ can be either $\hat{C}_{pm}$ or $\hat{C}_{pk}$. Then the process is capable in a Bayesian sense with $100p$% confidence if $\Pr\{\chi^2_n > n(1/C^*)^2\} > p$, where χ^2_n is a random variable following the chi-square distribution with n degrees of freedom.

Note that 'the degrees of freedom' of the posterior distribution for C_{pk} when $\mu = m$ (or for C_{pm} when $\mu = T$) are one more than those of the posterior distribution for C_p given in Reference [5]. The reason is that σ is estimated by s in $\hat{C}_p$, which uses an extra degree of freedom to estimate μ by $\bar{x}$.

5. EXAMPLES AND DISCUSSION

In Section 3, we have derived a Bayesian process capability assessment procedure based on the index C_{pm}. We have also provided tables (for various values of sample size n and off-target quantity $\delta = |T - \bar{x}|/s$) of the minimum values $C^*(p)$ of $C^* = \hat{C}_{pm}/\omega$ required to ensure that the posterior probability p of the process being capable (i.e. $p = P(C_{pm} > \omega|\hat{C}_{pm})$) reaches the desirable confidence levels, such as 0.90, 0.95 and 0.99. With these tables the procedure is as simple as comparing $\hat{C}_{pm}$ with ω times the tabulated value $C^*(p)$. If $\hat{C}_{pm} > \omega C^*(p)$, then we claim that the process is capable in a Bayesian sense.

For example, when $p = 0.9$, $n = 100$ and $\delta = 0.5$, we can find $C^*(p) = 1.1068$ from Table 1(a). Thus, when ω is given, say $\omega = 4/3$, the minimum $\hat{C}_{pm}$ required for the process to be capable is $1.1068 \times 4/3 = 1.4757$. That is, if $\hat{C}_{pm}$ is greater than 1.4757, we say that the process is capable in a Bayesian sense.

For the special case in which $\mu = T$, as described in Section 4, we do not even need to use the tables given in Section 3. We need only look up the commonly available chi-square tables for the posterior probability p of the process being capable (i.e. $p = \Pr\{\chi^2_n > n(1/C^*)^2\}$, with $C^* = \hat{C}_{pm}/\omega$) and then judge the process capability by comparing this posterior probability with the desirable confidence level, say 0.95. In this case, if $p > 0.95$, then we may claim that the process is capable in a Bayesian sense with 95% confidence.

We first illustrate our procedure via an example given in Reference [4], which was first given in Reference [2]. In this example the measurements were taken on the radial length of machined holes with upper and lower specification limits of 20 and −20 units respectively and target value $T = 0$.

The results of stages 1–3 of the example are used

Table 2. Results of machined holes example

Radial Length ($\times 10^3$ inches)							
Stage	n	$\bar{x}$	s	$\lvert T-\bar{x}\rvert/s$	$\hat{C}_{pm}$	P_T	P
1	201	4.7	8.7	0.5402	0.67	0.0000	0.0000
2	96	10.4	21.1	0.4929	0.28	0.0000	0.0000
3	316	5.0	5.4	0.9259	0.91	0.0067	0.0032

to illustrate the two Bayesian procedures—the one proposed in Reference [4] and the Bayesian procedure proposed in this paper. Take $\omega = 1$, which means that the process is capable if $C_{pm} > 1$. We summarize the results of this example in Table 2, where P denotes $\Pr\{C_{pm} > \omega|\hat{C}_{pm}\}$ given in (9) and P_T denotes the approximate posterior probability obtained in Reference [4].

From Table 2, we see that both P_T and P are very small for all three stages, indicating that the process is incapable. Both Bayesian procedures have the same conclusion as the traditional procedure, since the values of $\hat{C}_{pm}$ are smaller than 1 in all three stages. At first glance, it is a little bit surprising to see that at stage 3, P_T and P are so small for $\omega = 1$ even when $\hat{C}_{pm} = 0.91$. This can be explained by the high precision of the estimate $\hat{C}_{pm}$ resulting from the large sample size $n = 316$. Both posterior probabilities are too small in this example to show the effect of $\delta = |T - \bar{x}|/s$ on P_T and P.

Next we describe some scenarios to illustrate the effect of δ. Table 3 gives two cases that are capable from the traditional point of view, i.e. $\hat{C}_{pm} > 1$. First we notice that as δ increases, P increases while P_T remains the same. This is because P_T neglects the deviation of the process mean from the target value. From Table 3, we observe that for both cases, if we require the posterior probability of the process being capable to be greater than 0.90, then the Bayesian procedure proposed in Reference [4] will claim that the process is incapable for all δ, while our procedure will claim that the process is capable when $\delta \geq 1.0$.

On the other hand, let us compare the minimum values of $\hat{C}_{pm}$ in Tables 1(a)–1(c) with the minimum values of Table 2(a) in Reference [4]. For $\delta = 0$ the Bayesian procedure we proposed has the minimum values of $\hat{C}_{pm}$ larger than those values given in Reference [4], since we do not assume the known information $\mu = T$. However, when δ gets larger, say $\delta \geq 1.0$, the minimum $\hat{C}_{pm}$ required for our procedure is less than that of Reference [4], which indicates that our procedure is more sensitive in claiming the process is capable.

Table 3. Results of examples in comparing P_T and P.

Case	n	δ	$\hat{C}_{pm}$	P_T	P
1	100	0.0	1.09	0.8858	0.8555
		0.5		0.8858	0.8730
		1.0		0.8858	0.9148
		1.5		0.8858	0.9550
		2.0		0.8858	0.9806
2	300	0.0	1.05	0.8826	0.8655
		0.5		0.8826	0.8773
		1.0		0.8826	0.9132
		1.5		0.8826	0.9519
		2.0		0.8826	0.9782

Finally we give a simple example to show how to use the tables in practice. Suppose a sample of size $n = 50$ is collected from a process and the data give that $\hat{C}_{pm} = 1.12$ and $|T - \bar{x}|/s = 1$. Consider $\omega = 1$ and $p = 0.95$. From Table 1(b), we find that $C^*(p) = 1.1726$, which implies that the minimum value of $\hat{C}_{pm}$—equal to $\omega C^*(p)$—is 1.1726. Since $1.12 < 1.1726$, we claim that this process is incapable in a Bayesian sense with 95% confidence. This shows that our procedure can differentiate processes with different δ values, which is definitely a desirable property for a process capability assessment procedure.

6. CONCLUSION

The index C_{pm} was proposed to take into account the departure of the process mean from the target value as well as the magnitude of the process variation [4]. However, the statistical distribution associated with its estimator $\hat{C}_{pm}$ is so complicated that it is very difficult to obtain an interval estimation of C_{pm}. Under a non-informative prior we obtain a simple Bayesian procedure for process capability assessment that provides a Bayesian credible interval estimation for C_{pm}. Thus this Bayesian procedure can serve as an alternative to the classical procedures in process capability assessment.

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APPENDIX

In this appendix we derive (8) and (9) given in Section 3.

Derivation of (8)

From (6) and (7), we have the posterior PDF of (μ, σ) as

$$f(\mu, \sigma|\mathbf{x}) \propto L(\mu, \sigma|\mathbf{x}) \times \pi(\mu, \sigma)$$
$$\propto \sigma^{-(n+1)} \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right). \quad (11)$$

Also

$$\int_0^\infty \int_{-\infty}^{\infty} \sigma^{-(n+1)} \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right) \mathrm{d}\mu\, \mathrm{d}\sigma$$
$$= \int_0^\infty \sigma^{-(n+1)} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\int_{-\infty}^{\infty} \exp\left(-\frac{n(\mu-\bar{x})^2}{2\sigma^2}\right) \mathrm{d}\mu\right] \mathrm{d}\sigma$$
$$= \sqrt{\frac{\pi}{2n}}\,\Gamma(\alpha)\,\beta^\alpha \quad (12)$$

where $\alpha = (n-1)/2$ and $\beta = [\sum_{i=1}^{n}(x_i-\bar{x})^2/2]^{-1} = [(n-1)s^2/2]^{-1}$.

Then from (11) and (12) the posterior PDF (8) is obtained.

Derivation of (9)

Recall that $\sigma'^2 = \sigma^2 + (\mu - T)^2$ and observe that

$$\hat{\sigma}'^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - T)^2$$
$$= \frac{1}{n}\left(\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-T)^2\right).$$

Denote $a = (\text{USL} - \text{LSL})/6\omega$ and $g(\sigma) = \sqrt{a^2-\sigma^2}$. Then

$$p = \Pr\{C_{\text{pm}} > \omega|\mathbf{x}\}$$
$$= \Pr\left\{\frac{\text{USL}-\text{LSL}}{6\sigma'} > \omega\,\middle|\,\mathbf{x}\right\}$$
$$= \Pr\{\sigma^2 + (\mu-T)^2 < a^2|\mathbf{x}\}$$
$$= \int_0^a \int_{T-\sqrt{a^2-\sigma^2}}^{T+\sqrt{a^2-\sigma^2}} f(\mu,\sigma|\mathbf{x})\,\mathrm{d}\mu\,\mathrm{d}\sigma$$
$$= \int_0^a \int_{T-g(\sigma)}^{T+g(\sigma)} \frac{2\sqrt{n}}{\sqrt{2\pi}\,\Gamma(\alpha)\beta^\alpha}\sigma^{-(n+1)} \times \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right)\mathrm{d}\mu\,\mathrm{d}\sigma$$
$$= \int_0^a \frac{2\sqrt{n}}{\sqrt{2\pi}\,\Gamma(\alpha)\beta^\alpha}\sigma^{-(n+1)}\exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\int_{T-g(\sigma)}^{T+g(\sigma)} \exp\left(-\frac{n(\mu-\bar{x})^2}{2\sigma^2}\right)\mathrm{d}\mu\right]\mathrm{d}\sigma$$
$$= \int_0^a \frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha}\exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(\frac{T-\bar{x}+g(\sigma)}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{T-\bar{x}-g(\sigma)}{\sigma/\sqrt{n}}\right)\right]\mathrm{d}\sigma \quad (13)$$

where Φ is the cumulative distribution function of the standard normal distribution.

Let $\beta' = (\sum_{i=1}^{n}(x_i - T)^2/2)^{-1}$ and $y = \beta'\sigma^2$. Then

$$\frac{T-\bar{x}}{\sigma/\sqrt{n}} = \frac{1}{\sqrt{y}}(n(T-\bar{x})^2\beta')^{1/2}$$
$$= \frac{1}{\sqrt{y}}\left(\frac{2n(T-\bar{x})^2}{\sum_{i=1}^{n}(x_i-T)^2}\right)^{1/2}$$
$$= \sqrt{\frac{2}{y}}\left(\frac{n(T-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(T-\bar{x})^2}\right)^{1/2}$$
$$= \sqrt{\frac{2}{y}}\left[\delta^2 \Big/ \left(\delta^2 + \frac{n-1}{n}\right)\right]^{1/2}$$

and

$$\frac{g(\sigma)}{\sigma/\sqrt{n}} = \left(\frac{n(a^2-\sigma^2)}{\sigma^2}\right)^{1/2}$$
$$= \sqrt{n}\left(\frac{\beta' a^2}{y} - 1\right)^{1/2} = \sqrt{n}\left(\frac{t}{y} - 1\right)^{1/2}$$

where $t = \beta' a^2$.

Let

$$b_1(y) = \sqrt{\frac{2}{y}} \left[\delta^2 \Big/ \left(\delta^2 + \frac{n-1}{n} \right) \right]^{1/2}$$

$$b_2(y) = \sqrt{n} \left(\frac{t}{y} - 1 \right)^{1/2}$$

$$\gamma = \frac{\beta'}{\beta} = 1 + \frac{n}{n-1}\delta^2$$

Observe that

$$t = \beta' a^2 = \frac{2}{n\hat{\sigma}'^2} \left(\frac{\mathrm{USL} - \mathrm{LSL}}{6\omega} \right)^2$$

$$= 2 \left(\frac{\mathrm{USL} - \mathrm{LSL}}{6\hat{\sigma}'} \right)^2 \Big/ n\omega^2 = \frac{2\hat{C}^2_{\mathrm{pm}}}{n\omega^2}$$

Then (13) can be simplified to

$$p = \int_0^t \left(\frac{1}{\Gamma(\alpha)\gamma^\alpha y^{\alpha+1}} \right) \exp\left(\frac{-1}{\gamma y} \right) \times [\Phi(b_1(y) + b_2(y)) - \Phi(b_1(y) - b_2(y))]\, \mathrm{d}y$$

as given in (9).

REFERENCES

1. L. G. Rado, ‘Enhance product development by using capability indices’, *Qual. Prog.*, April, 38–41 (1989).
2. V. E. Kane, ‘Process capability indices’, *J. Qual. Technol.*, **18**, 41–52 (1986).
3. V. E. Kane, ‘Corrigenda’, *J. Qual. Technol.*, **18**, 265 (1986).
4. L. K. Chan, S. W. Cheng and F. A. Spiring, ‘A new measure of process capability: C_{pm}’, *J. Qual. Technol.*, **20**, 162–175 (1988).
5. S. W. Cheng and F. A. Spiring, ‘Assessing process capability: a Bayesian approach’, *IIE Trans.*, **21**, 97–98 (1989).
6. S. Kotz and N. L. Johnson, *Process Capability Indices*, Chapman & Hall, London. 1993.
7. Y.-M. Chou and D. B. Owen, ‘On the distributions of the estimated process capability indices’, *Commun. Statist.—Theory Meth.*, **18**, 4549–4560 (1989).
8. S. Kotz, W. L. Pearn and N. L. Johnson, ‘Some process capability indices are more reliable than one might think’, *Appl. Statist.*, **42**, 55–62 (1993).
9. M. H. Kalos and P. A. Whitlock, *Monte Carlo Methods*, Wiley New York. 1986.
10. J. H. Shiau, H.-N. Hung and C.-T. Chiang, ‘A note on Bayesian estimation of process capability indices’, *Statist. Probabil. Lett.*, **45**, (1999).
11. W. L. Pearn, S. Kotz and N. L. Johnson ‘Distributional and inferential properties of process capability indices’, *J. Qual. Technol.*, **24**, 216–231 (1992).
12. S. Bittanti, M. Lovera and L. Moiraghi, ‘Application of non-normal process capability indices to semiconductor quality control’, *IEEE Trans. Semicond. Manufact.*, **SM-11**, 296–303 (1988).
13. J. O. Berger, *Statistical Decision Theory*, Springer, New York, 1980, pp. 102, 255–267.
14. J. M. Bernardo and F. M. A. Smith, *Bayesian Theory*, Wiley, Chichester, 1993, pp. 222–226, 298–316, 371–373.
15. B. L. Welch and H. W. Peers, ‘On formulae for confidence points based on integrals of weighted likelihoods’, *J. R. Statist. Soc. B*, **25**, 318–329 (1963).
16. J. O. Berger and J. M. Bernardo, ‘Ordered group reference priors with application to the multinomial problem’, *Biometrika*, **79**, 25–37 (1992).

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