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Theory and Methodology

A global optimization method for nonconvex separable programming problems

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Abstract

Conventional methods of solving nonconvex separable programming (NSP) problems by mixed integer programming methods requires adding numerous 0–1 variables. In this work, we present a new method of deriving the global optimum of a NSP program using less number of 0–1 variables. A separable function is initially expressed by a piecewise linear function with summation of absolute terms. Linearizing these absolute terms allows us to convert a NSP problem into a linearly mixed 0–1 program solvable for reaching a solution which is extremely close to the global optimum.

Keywords: Goal programming; Piecewise linear function; Separable programming

1. Introduction

Separable programs are nonlinear programs in which the objective functions and constraints can be expressed as the sum of all functions, each nonlinear term involving only one variable. The nonconvex separable programming (NSP) problem discussed herein, denoted as Problem P1, is expressed as follows:

Problem P1 (NSP Problem)

$$\text{minimize} \quad \sum_{i=1}^{n} f_i(x_i)$$

$$\text{subject to} \quad \sum_{i=1}^{n} h_{ij}(x_i) \geqslant 0 \ \text{ for all } j,$$

$$x_i \geqslant 0 \ \text{ for } i = 1, 2, \ldots, n,$$

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where $f_i(x_i)$ could be nonconvex functions and $h_{ij}(x_i)$ are linear functions.

By assuming that all of $f_i(x_i)$ in Problem P1 are convex, Problem P1 can be solved by the simplex method to obtain the global optimum. The conventional means of solving Problem P1 with nonconvex $f_i(x_i)$ [3,5,11] is discussed below.

Assume that $f_i(x_i)$ is approximately linearized over the interval $[a, b]$. Define $a_{i,k},\ k = 1, 2, \ldots, m_i$, as the kth break point on the x_i-axis such that $a_{i,1} < a_{i,2} < \cdots < a_{i,m_i}$ with $a_{i,1} = a$ and $a_{i,m_i} = b$. Then $f_i(x_i)$ can be approximated as

$$f_i(x_i) = \sum_{k=1}^{m_i} f_i(a_{i,k}) t_{i,k}, \tag{1}$$

where $x_i = \sum_{k=1}^{m_i} a_{i,k} t_{i,k},\ \sum_{k=1}^{m_i} t_{i,k} = 1,\ t_{i,k} \geqslant 0$, in which only two adjacent $t_{i,k}$, e.g. $(t_{i,k-1}, t_{i,k})$ and $(t_{i,k}, t_{i,k+1})$, are allowed to be nonzero. In reference to Eq. (1), conventional methods [3,5,11] treat the NSP Problem as the following program.

Program 1 (Conventional NSP methods [3,5,11])

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} \sum_{k=1}^{m_i} f_i(a_{i,k}) t_{i,k} \\
\text{subject to} \quad & \sum_{i=1}^{n} h_{ij}(x_i) \geqslant 0 \ \text{ for all } j, \\
& x_i = \sum_{k=1}^{m_i} a_{i,k} t_{i,k} \ \text{ for } i = 1, 2, \ldots, n, \\
& t_{i,1} \leqslant y_{i,1} \ \text{ for } i = 1, 2, \ldots, n, \\
& t_{i,k} \leqslant y_{i,k-1} + y_{i,k} \ \text{ for } i = 1, 2, \ldots, n,\ k = 2, 3, \ldots, m_i, \\
& \sum_{k=1}^{m_i} y_{i,k} = 1 \ \sum_{k=1}^{m_i} t_{t_{k_i}} = 1 \ \text{ for } i = 1, 2, \ldots, n,
\end{aligned}$$

where $y_{i,k} = 0$ or $1,\ t_{i,k} \geqslant 0,\ k = 1, 2, \ldots, m_i,$ and $i = 1, 2, \ldots, n.$

Obviously, there exists a unique k where $y_{i,k} = 1$ and $t_{i,k} + t_{i,k+1} = 1$, in which Problem 1 becomes

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} [f_i(a_{i,k}) t_{i,k} + f_i(a_{i,k+1})(1 - t_{i,k})] \\
\text{subject to} \quad & \sum_{i=1}^{n} h_{ij}(x_i) \geqslant 0 \ \text{ for all } j, \\
& x_i = a_{i,k+1} + (a_{i,k} - a_{i,k+1}) t_{i,k} \ \text{ for } i = 1, 2, \ldots, n, \\
& x_i \geqslant 0,\ \ t_{i,k} \geqslant 0.
\end{aligned}$$

Program 1 is a linearly mixed integer problem which can obtain the global optimum. Program 1 is seriously limited in that it contains a large number of 0–1 variables which incur heavy computational burden. The number of newly added 0–1 variables required to approximately linearize a function $f_i(x_i)$ equals the number of breaking intervals. For instance, Program 1 requires using $\sum_{i=1}^{n}(m_i - 1)$ zero–one variables (i.e., $y_{i,1},\ y_{i,2}, \ldots,\ y_{i,m_i-1}$).

An alternative means of solving Problem P1 is the restricted-basis simplex method [3,11]. This method specifies that no more than two positive $t_{i,k}$ can appear in the basis. Moreover, two $t_{i,k}$ can be positive only if they are adjacent. In this case, the additional constraints involving $y_{i,k}$ are disregarded. The restricted basis method, although computationally efficient in terms of solving Problem P1, can only guarantee to attain a local optimum [3,11].

In light of above discussion, this work presents a novel means of solving Problem P1. The proposed method is advantageous over conventional NSP methods in that it can find approximately global optimum of a NSP problem by using less number of 0–1 variables. The solution derived herein can be improved by adequately adding the break points with the searching intervals.

2. Preliminaries

Some propositions on how to linearize a nonconvex separable function $f(x)$ are described as follows.

Proposition 1. *Let $f(x)$ be the piecewise linear function of x, as depicted in Fig. 1, where $a_k,\ k = 1,2,\ldots,m$, are the break points of $f(x)$, $s_k,\ k = 1,2,\ldots,m-1$, are the slopes of line segments between s_k* and a_{k+1}, and $s_k = [f(a_{k+1}) - f(a_k)]/[a_{k+1} - a_k]$.

In addition, $f(x)$ can be expressed as follows:

$$f(x) = f(a_1) + s_1(x - a_1) + \sum_{k=2}^{m-1} \frac{s_k - s_{k-1}}{2}(|x - a_k| + x - a_k), \tag{2}$$

where $|o|$ is the absolute value of o.

This proposition can be examined as follows:

(i) If $x \leqslant a_2$, then

$$f(x) = f(a_1) + \frac{f(a_2) - f(a_1)}{a_2 - a_1}(x - a_1) = f(a_1) + s_1(x - a_1).$$

(ii) If $x \leq a_3$, then

$$f(x) = f(a_1) + s_1(a_2 - a_1) + s_2(x - a_2) = f(a_1) + s_1(x - a_1) + \frac{s_2 - s_1}{2}(|x - a_2| + x - a_2).$$

(iii) If $x \leqslant a_{k'}$, then $\sum_{k \geqslant k'}^{m-1}(|x - a_k| + x - a_k) = 0$, and $f(x)$ becomes

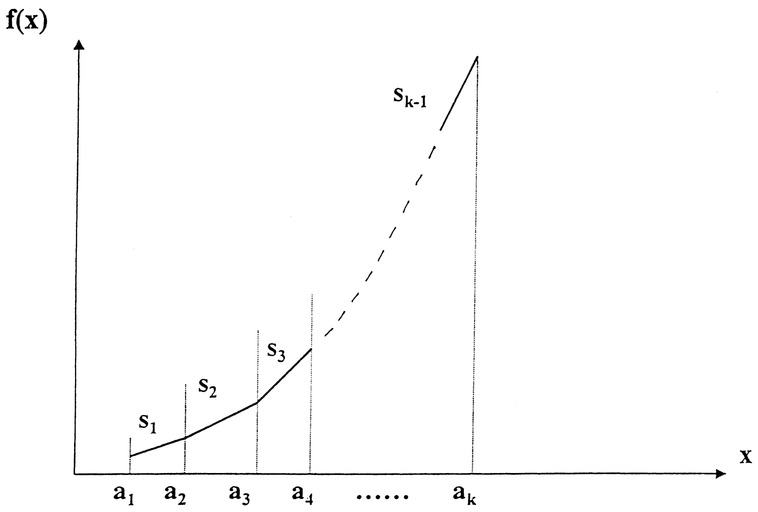


Fig. 1. A piecewise linear function.

$$f(x) = f(a_1) + s_1(x - a_1) + \sum_{k=2}^{k'-1} \frac{s_k - s_{k-1}}{2}(|x - a_k| + x - a_k).$$

Example 1. Consider a separable function $f(x_1) = x_1^3 - 4x_1^2 + 2x_1$ depicted in Fig. 2(a), where $0 \leqslant x_1 \leqslant 5$. Assume that the break points of $f(x_1)$ are 0, 0.5, 1, 1.5,..., 4.5, 5. In reference to Proposition 1, $f(x_1)$ can be approximately linearized as follows (Fig. 2(b)):

$$\begin{aligned} f(x_1) &= x_1^3 - 4x_1^2 + 2x_1 \\ &\cong 0.25x_1 - \frac{2.5}{2}(|x_1 - 0.5| + x_1 - 0.5) - \frac{1}{2}(|x_1 - 1| + x_1 - 1) + \frac{0.5}{2}(|x_1 - 1.5| + x_1 - 1.5) \\ &\quad + \frac{2}{2}(|x_1 - 2| + x_1 - 2) + \frac{3.5}{2}(|x_1 - 2.5| + x_1 - 2.5) + \frac{5}{2}(|x_1 - 3| + x_1 - 3) + \frac{6.5}{2}(|x_1 - 3.5| \\ &\quad + x_1 - 3.5) + \frac{8}{2}(|x_1 - 4| + x_1 - 4) + \frac{9.5}{2}(|x_1 - 4.5| + x_1 - 4.5). \end{aligned} \tag{3}$$

Expressing a separable linear function by Eq. (2) is advantageous in that the intervals of convexity and concavity for $f(x)$ can be easily known, as described by the following proposition.

Proposition 2. *Consider $f(x)$ in Eq. (2) where x is within the interval $[p, q]$, $a_p \leqslant x \leqslant a_q$. If $s_k > s_{k-1}$ then $f(x)$ is a convex for $a_{k-1} \leqslant x \leqslant a_{k+1}$, as depicted in Fig. 3(a). If $s_k < s_{k-1}$ then $f(x)$ is a concave for $a_{k-1} \leqslant x \leqslant a_{k+1}$, as depicted in Fig. 3(b).*

Consider Expression (3) and Fig. 2(b) as an example, in which $f(x)$ is concave when $0 \leqslant x \leqslant 1$ and $f(x)$ is convex when $1 \leqslant x \leqslant 5$.

Proposition 3. *Consider a goal programming PP1, as expressed below:*

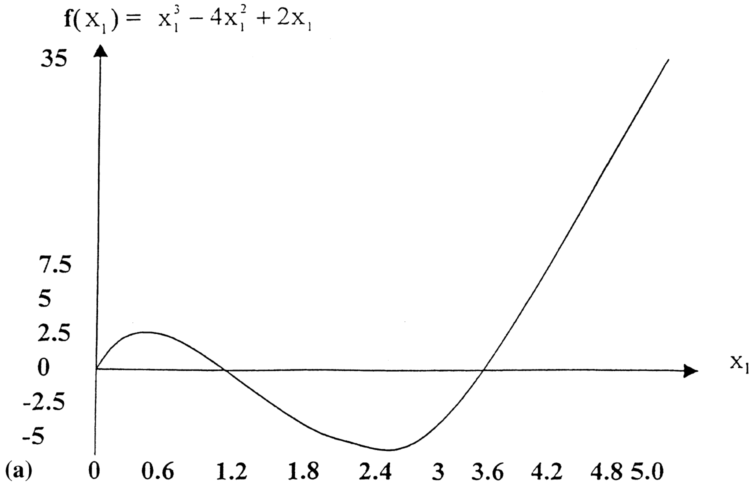
$$\begin{aligned} &PP1 \quad \textit{minimize} \quad w = \sum_{k=2}^{m-1} c_k(|x - a_k| + x - a_k) \\ &\qquad \textit{subject to} \quad x \in F(\textit{a feasible set}),\ x \geqslant 0,\ c_k \geqslant 0, \end{aligned} \tag{4}$$

where c_k are coefficients, $k = 2, 3, \ldots, m-1$, and $0 < a_1 < a_2 < \cdots < a_m$, can be linearized as PP2 below:

$$\begin{aligned} &PP2 \quad \textit{minimize} \quad w = 2\sum_{k=2}^{m-1} c_k\left(x - a_k + \sum_{l=1}^{k-1} d_l\right) \\ &\qquad \textit{subject to} \quad x + \sum_{l=1}^{m-2} d_l \geqslant a_{m-1}, \\ &\qquad\qquad 0 \leqslant d_l \leqslant a_{l+1} - a_l \ \ \textit{for}\ \ell = 1, 2, \ldots, m-1, \\ &\qquad\qquad x \in F(\textit{a feasible set}),\ x \geqslant 0,\ c_k \geqslant 0. \end{aligned} \tag{5}$$

Proof. According to Li [7], PP1 is equivalent to the following program:

$$\begin{aligned} &\text{PP3} \quad \text{minimize} \quad w = 2\sum_{k=2}^{m-1} c_k(x - a_k + r_k) \\ &\qquad \text{subject to} \quad x - a_k + r_k \geqslant 0,\ r_k \geqslant 0,\ \text{for } k = 2, 3, \ldots, m-1, \\ &\qquad\qquad x \in F(\text{a feasible set}),\ x \geqslant 0,\ c_k \geqslant 0, \end{aligned} \tag{6}$$



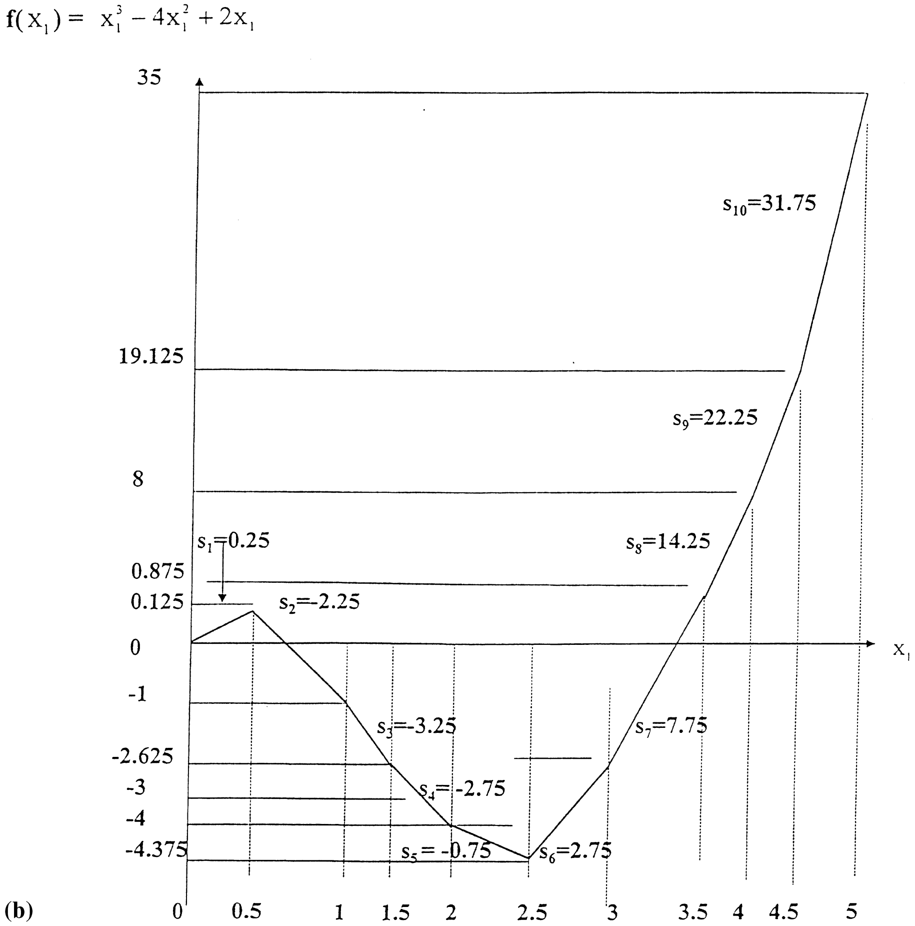


Fig. 2. (a) A nonconvex function $f(x_1)$. (b) A approximate piecewise linear function for $f(x_1)$.

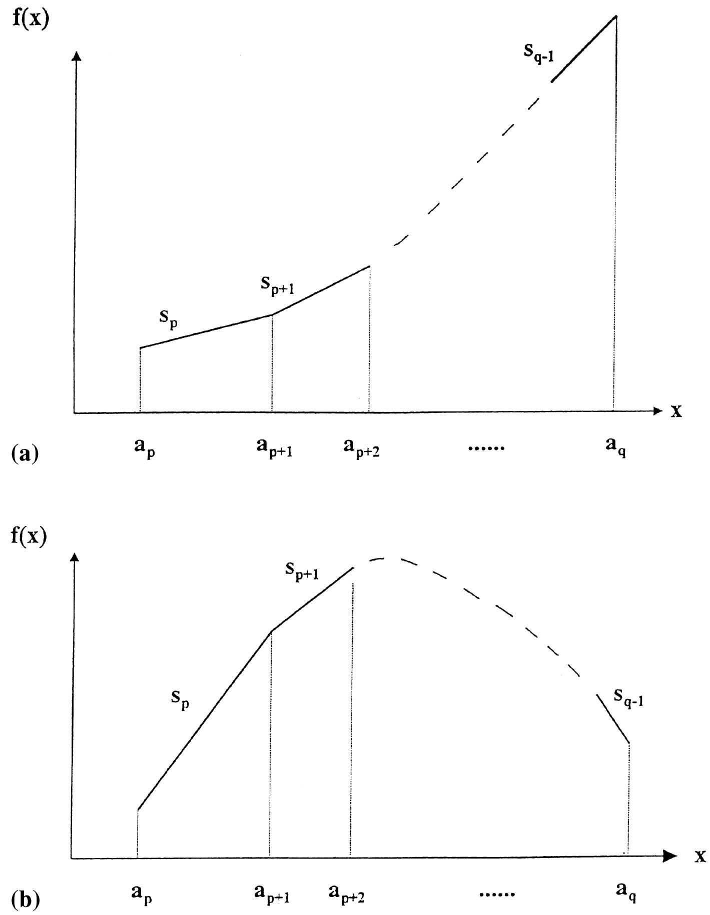


Fig. 3. (a) A convex function for $f(x)$. (b) A concave function for $f(x)$.

where r_k is a deviation variable. PP3 implies that: if $x < a_k$ then at optimal solution $r_k = a_k - x$; if $x \geqslant a_k$ then at optimal solution $r_k = 0$.

Substitute r_k by $\sum_{\ell=1}^{k-1} d_\ell,\ 0 \leqslant d_\ell \leqslant a_{\ell+1} - a_\ell$, PP3 then becomes

$$\begin{aligned}
\text{PP4}\qquad \text{minimize}\quad & w = 2\sum_{k=2}^{m-1} c_k\left(x - a_k + \sum_{l=1}^{k-1} d_l\right)\\
\text{subject to}\quad & x + d_1 \geqslant a_2,\\
& x + d_1 + d_2 \geqslant a_3,\\
& \vdots\\
& x + d_1 + d_2 + \cdots + d_{m-2} \geqslant a_{m-1},\\
& 0 \leqslant d_l \leqslant a_{l+1} - a_l,\\
& x \in F(\text{a feasible set}),\ x \geqslant 0,\ c_k \geqslant 0.
\end{aligned}$$

Since $a_{\ell+1} - a_\ell \geqslant d_\ell$, it is obvious that

$$x \geqslant a_{m-1} - \sum_{\ell=1}^{m-2} d_\ell \geqslant a_{m-2} - \sum_{\ell=1}^{m-3} d_\ell \geqslant \cdots \geqslant a_3 - d_1 - d_2 \geqslant a_2 - d_1 \geqslant 0.$$

Therefore, the first $(m-2)$ constraints in PP4 are covered by the first constraint in PP2. By doing so, Proposition 3 is proven. □

Many conventional goal programming methods (such as Charnes and Cooper method in [3,5]) can be utilized to solve (4). Comparing with conventional goal programming methods, linearizing (4) by (5) is more computationally efficient owing to the following two reasons.

(i) All constraints in (5) are simple upper or lower bounded constraints except for the first constraint in (5).

(ii) By utilizing Li's method [7] for linearizing an absolute term with positive coefficient, only (6) contains $m-2$ deviation variables (i.e., $r_2, r_3, \ldots, r_{m-1}$). In contrast, conventional goal programming techniques [3,5,11] require using $2(m-2)$ deviation variables.

Example 2. Consider the following goal programming:

$$\begin{aligned} \text{minimize} \quad & w = 2(x-1) + 2(|x-2| + x - 2) + 1(|x-3| + x - 3) \\ \text{subject to} \quad & x \geqslant 1.5. \end{aligned}$$

This program, as depicted in Fig. 4(a), can be transformed into the following linear program:

$$\begin{aligned} \text{minimize} \quad & w = 2(x-1) + 4(x - 2 + d_1) + 2(x - 3 + d_1 + d_2) \\ \text{subject to} \quad & x + d_1 + d_2 \geqslant 3, \ x \geqslant 1.5, \\ & 0 \leqslant d_1 \leqslant 1, \ 0 \leqslant d_2 \leqslant 1, \ x \in F. \end{aligned}$$

LINDO [9] is used to solve the above program, hereby obtaining $d_1 = 0.5$, $d_2 = 1$, $w = 1$, and $x = 1.5$.

Notably, Proposition 3 can be applied only if all coefficients in (4) are nonnegative. The technique of linearizing an absolute term with negative coefficient is introduced below.

Proposition 4. *Consider the following program:*

$$\begin{aligned} \textit{minimize} \quad & w = c|x-a| \\ \textit{subject to} \quad & x \in F \ (\textit{a feasible set}), \ c \textit{ is negative coefficient } (\textit{i.e. } c < 0), \\ & 0 \leqslant x \leqslant \bar{x} \ (\bar{x} \textit{ is the upper bound of } x). \end{aligned}$$

This program can be replaced by the mixed 0–1 *program:*

$$\begin{aligned} \textit{minimize} \quad & w = c(x - 2z + 2au - a) \\ \textit{subject to} \quad & x + \bar{x}(u-1) \leqslant z, \\ & x \in F \ (\textit{a feasible set}), \ x \geqslant 0, \ z \geqslant 0, \ u \textit{ is a } 0\text{–}1 \textit{ variable and } c \textit{ is a negative constant}, \\ & 0 \leqslant x \leqslant \bar{x} \ (\bar{x} \textit{ is the upper bound of } x). \end{aligned}$$

Proof. By introducing a 0–1 variable u, where $u = 0$ if $x \geqslant a$, and otherwise $u = 1$. It is convenient to confirm whether if $u = 1$ then $z = x$ and if $u = 0$ then $z = 0$. Thus, w can be rewritten as $c|x-a| = c(1-2u)(x-a) = c(x - 2ux + 2au - a)$. Denote the polynomial term ux as z. By referring to Li and Chang [8], the relationship among x, z and u is expressed as $x + \bar{x}(u-1) \leqslant z$ and $z \geqslant 0$. By doing so, Proposition 4 is proven. □

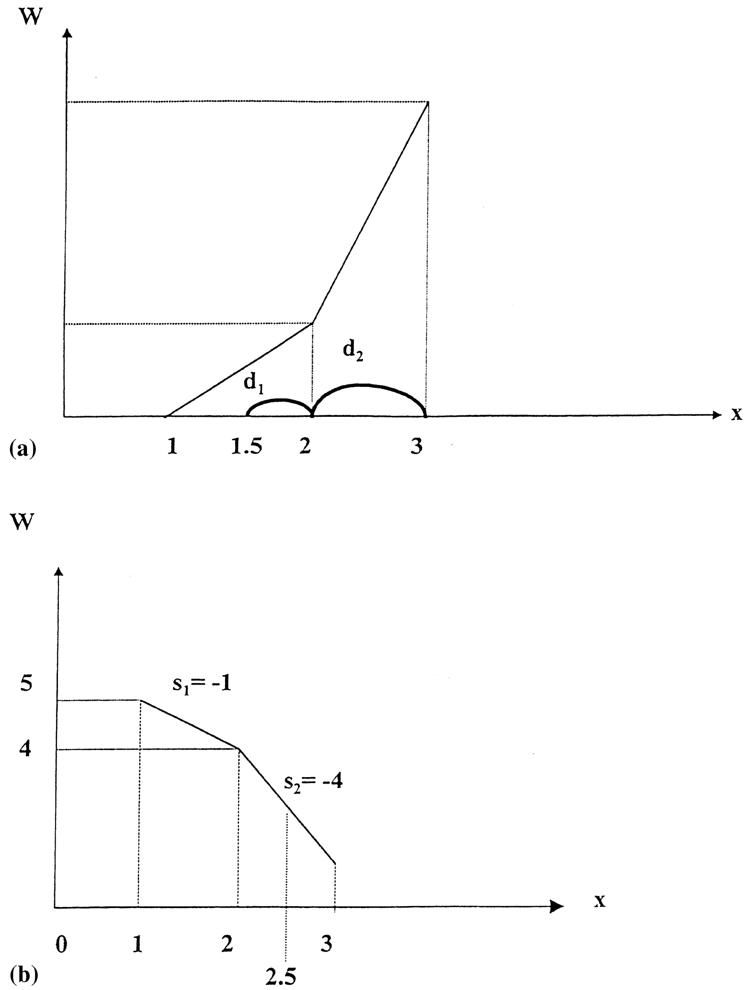


Fig. 4. (a) A goal programming problem with convex objective function (Example 2). (b) A goal programming problem with concave objective function (Example 3).

Example 3. Consider the following goal programming:

$$\begin{aligned} \text{minimize} \quad & w = 5 - (x-1) - 1.5(|x-2| + x - 2) \\ \text{subject to} \quad & x \leqslant 2.5. \end{aligned}$$

This program, as depicted in Fig. 4(b), can be transformed into the following linear program:

$$\begin{aligned} \text{minimize} \quad & w = 5 - (x-1) - 1.5[(x - 2z + 4u - 2) + (x-2)] = -4x + 3z - 6u + 12 \\ \text{subject to} \quad & x + 3(u-1) \leqslant z, \; x \leqslant 2.5, \\ & x \geqslant 0, \; z \geqslant 0, \; u \text{ is a 0–1 variable}, \; x \in F. \end{aligned}$$

Solve the above program by LINDO [9] to obtain $u=0$, $z=0$, $w=2$, and $x=2.5$.

Based on Propositions 1–3, Problem 1 can be approximated as the following program:

Program 2 (Proposed NSP method):

$$\text{minimize} \sum_{i=1}^{n}\Bigg[f(a_{i,1}) + s_{i,1}(x_i - a_{i,1}) + 2\sum_{\text{for } k,\text{ where } s_{i,k}>s_{i,k-1}}(s_{i,k} - s_{i,k-1})\left(x_i - a_{i,k} + \sum_{l=1}^{k-1} d_{i,l}\right)$$

$$+ \sum_{\text{for } k,\text{where } s_{i,k}<s_{i,k-1}}(s_{i,k} - s_{i,k-1})(x_i - 2z_{i,k} + 2a_{i,k}u_{i,k} - a_{i,k})\Bigg]$$

$$\text{subject to} \sum_{i=1}^{n} h_{ij}(x_i) \geqslant 0, \text{ for all } j,$$

$$\left.\begin{array}{l} x_i + \sum_{l=1}^{m_{j-2}} d_{i,l} \geqslant a_{m_i-1} \\ 0 \leqslant d_{i,l} \leqslant a_{i,l+1} - a_{i,l} \end{array}\right\} \text{ for } i = 1,\ldots,n \text{ and } k \text{ where } s_{i,k} > s_{i,k-1},$$

$$\left.\begin{array}{l} x_i + \bar{x}(u_{i,k} - 1) \leqslant z_{i,k} \\ z_{i,k} \geqslant 0 \end{array}\right\} \text{ for } i = 1,\ldots,n \text{ and } k \text{ where } s_{i,k} < s_{i,k-1},$$

where $x_i \geqslant 0, d_{i,l} \geqslant 0, z_{i,l} \geqslant 0$, and $u_{i,k}$ are 0–1 variables.

Table 1 lists the extra 0–1 and continuous variables used in Programs 1 and 2. Table 1 indicates that for solving a NSP problem the proposed method uses less number of 0–1 variables than used in Program 2.

3. Selection of break points

Accuracy of the piecewise linear estimate heavily depends on the selection of proper break points. With an increasing number of break points, the number of additional deviation variables for approximating a convex function (or zero–one variables for approximating a concave function) also increases. Consequently, inappropriate selecting of break points causes a computational burden when piecewise linearizing non-linear functions.

Bazarra et al. [3] and Meyer [10] presented a means of selecting adequate break points. Their method initially utilizes a coarse grid and then generates finer break points around the obtained optimal solution computed by the coarse grid. If necessary, break points around the optimal solution computed by the finer break are generated again until the precision is satisfied. Their method, although applicable to linearize a convex function, is difficult for use in linearizing a nonconvex function.

Therefore, in this work, we present an efficient means of selecting break points. For instance, consider a convex function $f(x_1) = 5x_1^3$ (Fig. 5(a)) where $a_1 \leqslant x_1 \leqslant a_5$. Assume that three break points a_2, a_3, and a_4 within $a_1 \leqslant x_1 \leqslant a_5$ are selected. The error of piecewisely linearizing $f(x_1)$ is computed as

Table 1
Comparison of Programs 1 and 2

	Extra 0–1 variables	Number of extra 0–1 variables	Extra continuous variables	Number of extra continuous variables
Program 1 (Conventional NSP Method)	$y_{i,k}$	Number of all piecewise segments for all f_i (x_i)	$t_{i,k}$	Number of all piecewise segments for all $f_i(x_i)$
Program 2 (Proposed NSP Method)	$u_{i,k}$	Number of concave piecewise segments only	$d_{i,\ell}$	Number of convex piecewise segments only
			$z_{i,k}$	Number of concave piecewise segments only

$$\text{Error} = f(a_1) + s(x_1 - a_1) - f(x_1) = 125x_1 - 5x_1^3.$$

By taking partial $\partial\text{Error}/\partial x_1 = 0$, the maximal error occurs at $x_1 = 2.89$ where $\partial\text{Error}/\partial x_1 = s - (\partial f(x_1)/\partial x_1) = 125 - 15x_1^2 = 0$. By doing so, we obtain the first break point $a_3 = 2.89$.

Similarly, finer break points a_2 and a_4 can also be generated at maximal errors occur at x_1 for $0 \leqslant x_1 \leqslant 2.89$ and $2.89 \leqslant x_1 \leqslant 5$, respectively, as depicted in Fig. 5(b). Therefore, the second break point is

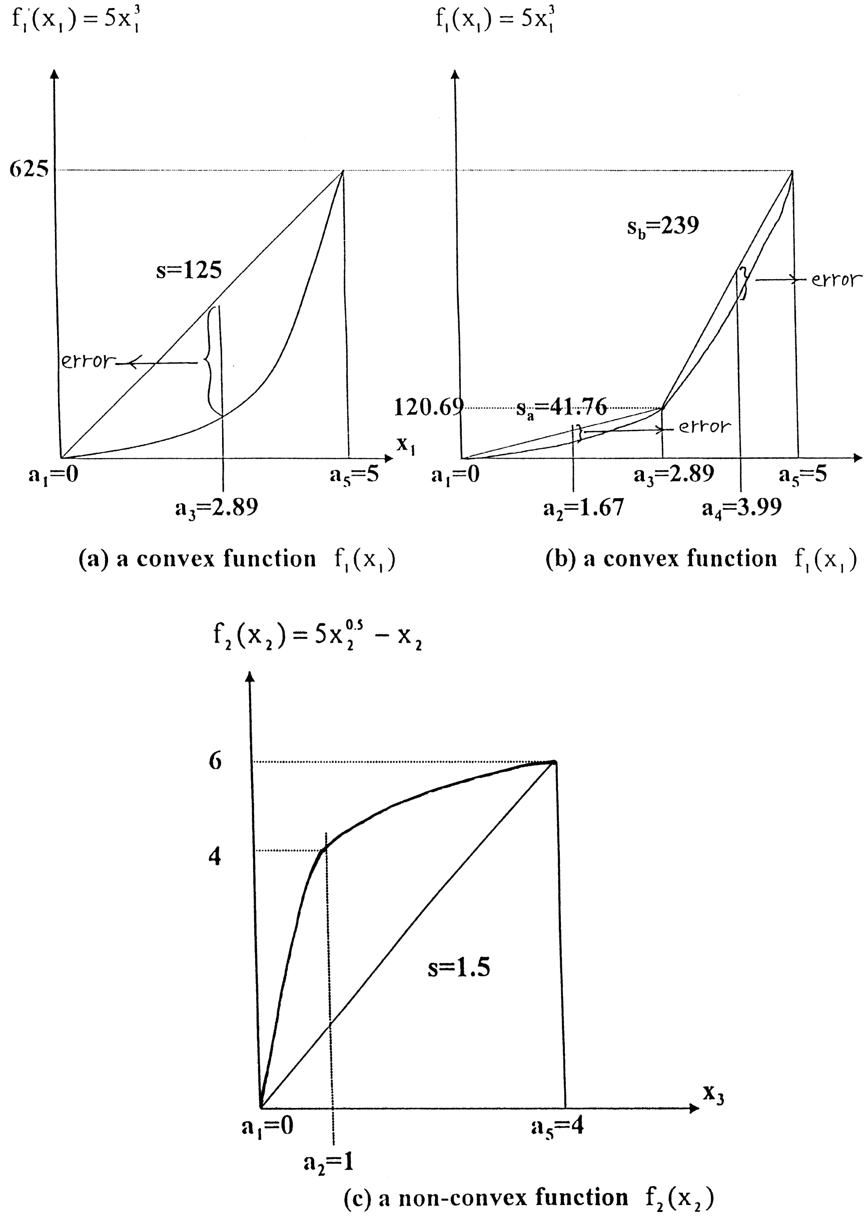


Fig. 5. (a) A convex function $f_1(x_1)$. (b) A convex function $f_1(x_1)$. (c) A nonconvex function $f_2(x_2)$.

$a_2 = 1.67$ (for $0 \leqslant x_1 \leqslant 2.89$) where $s_a - \partial f(x_1)/\partial x_1 = 41.76 - 15x_1^2 = 0$ and the third break point is $a_4 = 3.99$ (for $2.89 \leqslant x_1 \leqslant 5$) where $s_b - \partial f(x_1)/\partial x_1 = 239 - 15x_1^2 = 0$.

Similarly, for a concave function $f(x_2) = 5x_2^{0.5} - x_2$ (Fig. 5(c)) where $a_1 \leqslant x_2 \leqslant a_3$. Assume we want to choose a break point a_2 within $a_1 \leqslant x_1 \leqslant a_3$. The maximal error of piecewisely linearizing x_2 is computed as

$$\text{Error} = f(x_2) - (f(a_1) + s(x_2 - a_1)) = 5x_2^{0.5} - x_2 - 1.5x_2.$$

By taking $\partial\text{Error}/\partial x_2 = 0$, the maximal error occurs at x_2 where $(\partial f(x_2)/\partial x_2) - s = (2.5x_2^{-0.5} - 1) - 1.5 = 0$. After calculating, the obtained break point $a_2 = 1$.

Owing to that treating continuous variables is more computational efficient than treating zero–one variables, we recommend selecting three break points for linearizing a convex function and one break point for linearizing a concave function at each iteration.

4. Solution algorithm

The solution algorithm of solving Problem P1 is described in the following steps:

Step 1. Select initial break points.

(i) For each function $f_i(x_i)$ where $f_i(x_i)$ is convex for the interval $\underline{x}_i \leqslant x_i \leqslant \overline{x}_i$, three break points within this interval are selected by the method described in the Section 3.

(ii) For each function $f_i(x_i)$ where $f_i(x_i)$ is concave for the interval $\underline{x}_i \leqslant x_i \leqslant \overline{x}_i$, one point within this interval is selected by the method described in the Section 3.

Step 2. Formulate piecewise functions. Proposition 1 can be used to approximately linearize each function $f_i(x_i)$, expressed as

$$\hat{f}_i(x_i) = f_i(a_{i1}) + s_{i1}(x_i - a_{i1}) + \sum_{k=2}^{i_{m-1}} \frac{s_{i,k} - s_{i,k-1}}{2}(|x_i - a_{ik}| + x_i - a_{ik}),$$

where a_{ik} are break points selected in Step 1.

Step 3. Linearize the program. Using Proposition 3 linearizes the absolute terms where $s_{i,k} > s_{i,k-1}$, and using Proposition 4 linearizes the absolute terms where $s_{i,k} < s_{i,k-1}$.

Step 4. Solve the program and assess the tolerable error. Solve the linear mixed integer program to obtain the solution $x^\Delta = (x_1^\Delta, x_2^\Delta, \ldots, x_n^\Delta)$. If $|f_i(x_i^\Delta) - \hat{f}_i(x_i^\Delta)| \leqslant \varepsilon$ for all i, where $\hat{f}_i(x_i)$ is the approximate linear function expressed in Step 2, then terminate the solution process; and otherwise go to Step 5.

Step 5. Add finer break points. If $a_k \leqslant x_i^\Delta \leqslant a_{k'}$, then add new break points within the interval, reiterate Step 2.

5. Numerical examples

Example 4. Consider the following separable programming problem with nonconvex objective function, in which one of the constraints is nonconvex:

$$\begin{aligned} \text{minimize} \quad & w = x_1^3 - 4x_1^2 + 2x_1 + x_2^3 - 4x_2^2 + 3x_2 \\ \text{subject to} \quad & 3x_1 + 2x_2 \leqslant 11.75, \\ & 2x_1 + 5x_2^{0.5} - x_2 \geqslant 9, \\ & 0 \leqslant x_1 \leqslant 5,\ 0 \leqslant x_2 \leqslant 4, \end{aligned}$$

where $x_1^3 - 4x_1^2 + 2x_1,\ x_2^3 - 4x_2^2 + 3x_2$, and $5x_2^{0.5} - x_2$ are depicted in Figs. 2(a), 6(a) and 5(b), respectively.

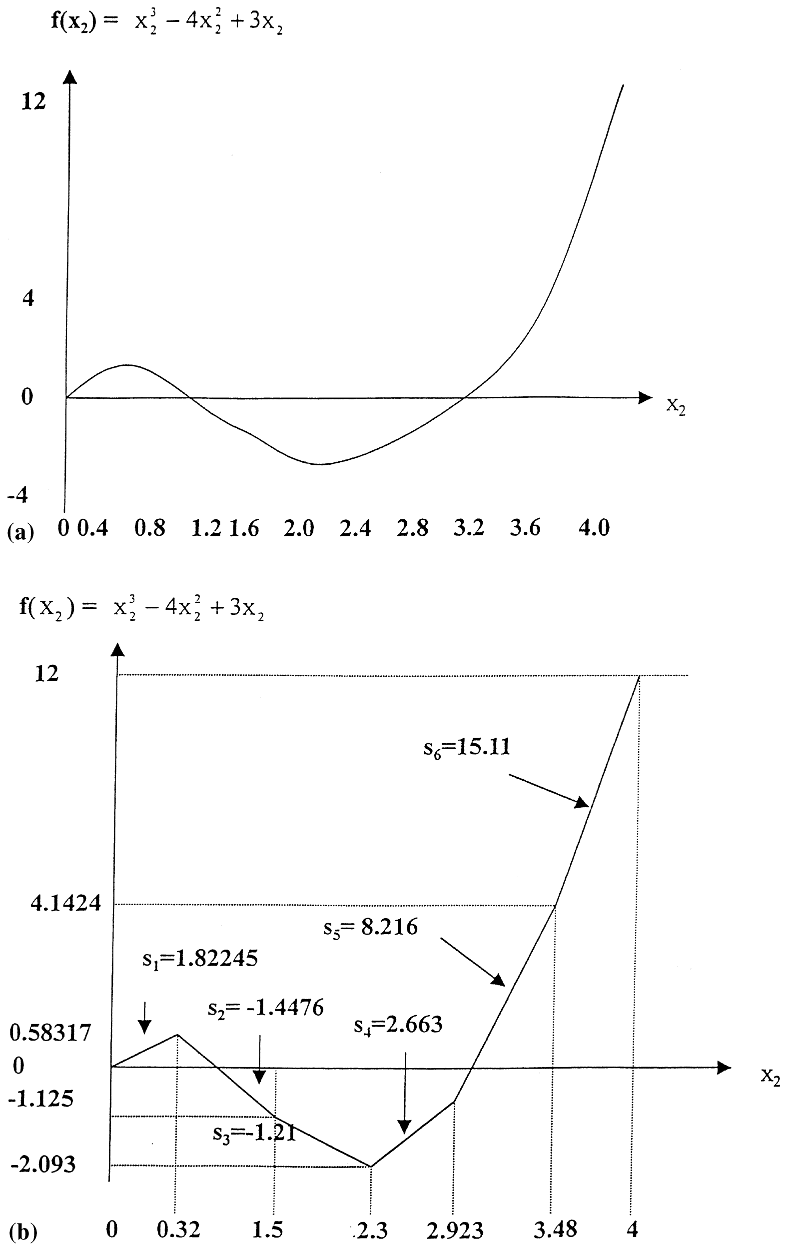


Fig. 6. (a) A nonconvex function $f(x_2)$. (b) A approximate piecewise linear function for $f(x_2)$.

Step 1. Select initial break points. From the basis of Section 3, one break point ($x_2 = 1$) is selected for the function $5x_2^{0.5} - x_2$ within $0 \leqslant x_2 \leqslant 4$ as depicted in Fig. 5(c). For the function $x_2^3 - 4x_2^2 + 3x_2$, one break point ($x_2 = 0.32$) is selected for the concave portion in which $0 \leqslant x_2 \leqslant 1.5$ and three break points ($x_2 = 2.3$, 2.923 and 3.48) are selected for the convex portion in which $1.5 \leqslant x_2 \leqslant 4$ (Fig. 6(b)).

Step 2. Formulate the piecewise functions. The original problem is expressed piecewisely as

$$\begin{aligned}\text{minimize}\quad & w = (\text{right-hand side of expression (3)}) + 1.8224\,x_2 \\ & -\frac{3.27}{2}(|x_2-0.32|+x_2-0.32)+\frac{0.2376}{2}(|x_2-1.5|+x_2-1.5) \\ & +\frac{3.873}{2}(|x_2-2.3|+x_2-2.3)+\frac{5.553}{2}(|x_2-2.923|+x_2-2.923) \\ & +\frac{6.894}{2}(|x_2-3.48|+x_2-3.48) \\ \text{subject to}\quad & 3x_1+2x_2\leqslant 11.75, \\ & 2x_1+4x_2-\frac{3.3334}{2}(|x_2-1|+x_2-1)\geqslant 9, \\ & 0\leqslant x_1\leqslant 5,\ 0\leqslant x_2\leqslant 4.\end{aligned}$$

Step 3. Linearize the program. The above problem is converted into following linearly mixed 0-1 program:

$$\begin{aligned}\text{minimize}\quad & w = 31.75x_1+2.5z_{11}+z_{12}-1.25u_{11}-u_{12}+35d_{11}+34.5d_{12} \\ & +32.5d_{13}+29d_{14}+24d_{15}+17.5d_{16}+9.5d_{17}+15.11x_2 \\ & +3.27z_{21}-1.046u_{21}+16.5576d_{21}+16.32d_{22}+12.447d_{23}-6.894d_{24}-172.19 \\ \text{subject to}\quad & x_1+d_{11}+d_{12}+d_{13}+d_{14}+d_{15}+d_{16}+d_{17}\geqslant 4.5, \\ & x_2+d_{21}+d_{22}+d_{23}+d_{24}\geqslant 3.84, \\ & x_1+5(u_{11}-1)\leqslant z_{11}, x_1+5(u_{12}-1)\leqslant z_{12},\ x_2+4(u_{21}-1)\leqslant z_{21}, \\ & 3x_1+2x_2\leqslant 11.75, \\ & 2x_1+0.666x_2-3.334z_{22}-3.334u_{22}\geqslant 5.666, \\ & x_2+4(u_{22}-1)\leqslant z_{22}, \\ & 0\leqslant x_1\leqslant 5,\ 0\leqslant x_2\leqslant 4. \\ & d_{1j}\leqslant 0.5,\ j=1,2,\ldots,7,\ d_{21}\leqslant 1.18,\ d_{22}\leqslant 0.8,\ d_{23}\leqslant 0.623,\ d_{24}\leqslant 0.557, \\ & u_{11},u_{12},u_{21},u_{22}\text{ are 0–1 variables.}\end{aligned}$$

Step 4. Solve the program and assess the tolerable error. By running on the LINDO [9], the optimal solution is $x_1=2.38333$, $x_2=2.3$, $w=-6.380064$ and the error of approximation is 0.129. Assume that the pre-specified tolerable error should be less than 0.01. Then, go to Step 5.

Step 5. Add finer break points. To derive a solution closer to the global optimum and satisfy the pre-specified approximated error $\leqslant 0.01$, three break points (2.285, 2.386, 2.48) can be further added for the function $x_1^3-4x_1^2+2x_1$ within $2.18\leqslant x_1\leqslant 2.58$. In addition, three break points (2.205, 2.307, 2.405) can be added for the function $x_2^3-4x_2^2+3x_2$ within $2.1\leqslant x_2\leqslant 2.5$. Similarly, one break point ($x_2=2.296$) is added for the function $5x_2^{0.5}-x_2$ within $2.1\leqslant x_2\leqslant 2.5$.

The problem then becomes

$$\begin{aligned}\text{minimize}\quad & w=-0.90507(x_1-2.18)+\frac{0.5873}{2}(|x_1-2.285|+x_1-2.285) \\ & +\frac{0.6145}{2}(|x_1-2.386|+x_1-2.386|+x_1-2.386)+\frac{0.6685}{2}(|x_1-2.48|+x_1-2.48) \\ & -0.131626(x_2-2.1)+\frac{0.5404}{2}(|x_2-2.2055|+x_2-2.2055) \\ & +\frac{0.5817}{2}(|x_2-.20369|+x_2-2.3069)+\frac{0.6203}{2}(|x_2-2.4049|+x_2-2.4049) \\ \text{subject to:}\quad & 3x_1+2x_2\leqslant 11.75, \\ & 2x_1+0.6868(x_2-2.1)-\frac{0.0719}{2}(|x_2-2.29564|+x_2-2.29564)\geqslant 9, \\ & 2.18\leqslant x_1\leqslant 2.58,\ 2.1\leqslant x_2\leqslant 2.5.\end{aligned}$$

The problem is linearized as follows:

$$\begin{aligned}\text{minimize}\quad & w = 0.965204x_1 + 1.870274d_{11} + 1.282974d_{12} + 0.668524d_{13} \\ & \quad + 1.42614x_2 + 1.7424d_{21} + 1.202d_{22} + 0.6203d_{23} - 5.4092 \\ \text{subject to}\quad & x_1 + d_{11} + d_{12} + d_{13} \geqslant 2.48,\ x_2 + d_{21} + d_{22} + d_{23} \geqslant 2.4049, \\ & 3x_1 + 2x_2 \leqslant 11.75, \\ & 2x_1 + 0.6149x_2 + 0.0719z - 0.16506u \geqslant 10.277223, \\ & x_2 + 2.5(u-1) \leqslant z,\ \text{u is an zero–one variable.} \\ & 2.18 \leqslant x_1 \leqslant 2.58,\ 2.1 \leqslant x_2 \leqslant 2.5, \\ & d_{11} \leqslant 0.105,\ d_{12} \leqslant 0.101,\ d_{13} \leqslant 0.094, \\ & d_{21} \leqslant 0.10548,\ d_{22} \leqslant 0.10139,\ d_{23} \leqslant 0.098.\end{aligned}$$

After running on the LINDO [9], the finer optimal values are $x_1 = 2.3875$, $x_2 = 2.2155$, the objective function's value is -6.5291 and the approximated error $= 0.00029 < 0.01$. The solution process is terminated since the approximated error is less than the pre-specified tolerable error.

Example 5. (*Taken from Klein et al.* [6]). The amount of electric power that can be produced from a multi-unit hydro-electric generating station depends on the amount of water discharged through each unit. A situation in which the discharge is not properly allocated among the generating units implies that the potential power output may not be fully achieved. More expensive sources such as nuclear, coal or oil (which are environmentally less attractive) would have to replace any loss. Thus, an electric utility should maximize hydro-electric generation which is the cheapest and cleanest source of energy. In addition, the quantity of electricity generated through each generating unit is a nonconvex function since the efficiency characteristics may not be the same for different units [6]. An illustrative example is provided in the following, which consists of two hydro-electric generating units, as depicted in Fig. 7(a) and (b), respectively:

$$\begin{aligned}\text{maximize}\quad & f_1(x_1) + f_2(x_2) \\ \text{subject to}\quad & x_1 \leqslant 241,\ x_2 \leqslant 250, \\ & x_1 + x_2 = Q,\ x_1, x_2 \geqslant 0,\end{aligned}$$

where Q are varying values of total discharge.

From the basis of Proposition 1, $f_1(x_1)$ and $f_2(x_2)$ can expressed as follows:

$$f_1(x_1) = 0.23256(x_1 - 11) + 0.00872(|x_1 - 54| + x_1 - 54) - 0.04924(|x_1 - 142| + x_1 - 142),$$
$$f_2(x_2) = 0.22727(x_2 - 11) + 0.040475(|x_1 - 55| + x_1 - 55) - 0.041865(|x_1 - 201| + x_1 - 201).$$

Based on Propositions 3 and 4, the problem can be reformulated as follows:

$$\begin{aligned}\text{minimize}\quad & -f_1(x_1) - f_2(x_2) = -0.15152x_1 + 0.01744z_1 - 0.94176u_{+}0.09848d_1 \\ & \quad - 0.22449x_2 + 0.08095z_2 - 4.45225u_2 + 0.08373d_2 - 20.36175 \\ \text{subject to}\quad & x_1 + d_1 \geqslant 142,\ d_1 \leqslant 88, x_1 + 241(u_1 - 1) \leqslant z_1, \\ & x_2 + d_2 \geqslant 201,\ d_2 \leqslant 146, x_2 + 250(u_2 - 1) \leqslant z_2, \\ & x_1 + x_2 = Q, \\ & u_1, u_2 \text{ are 0–1 variables},\ x_1, x_2 \geqslant 0.\end{aligned}$$

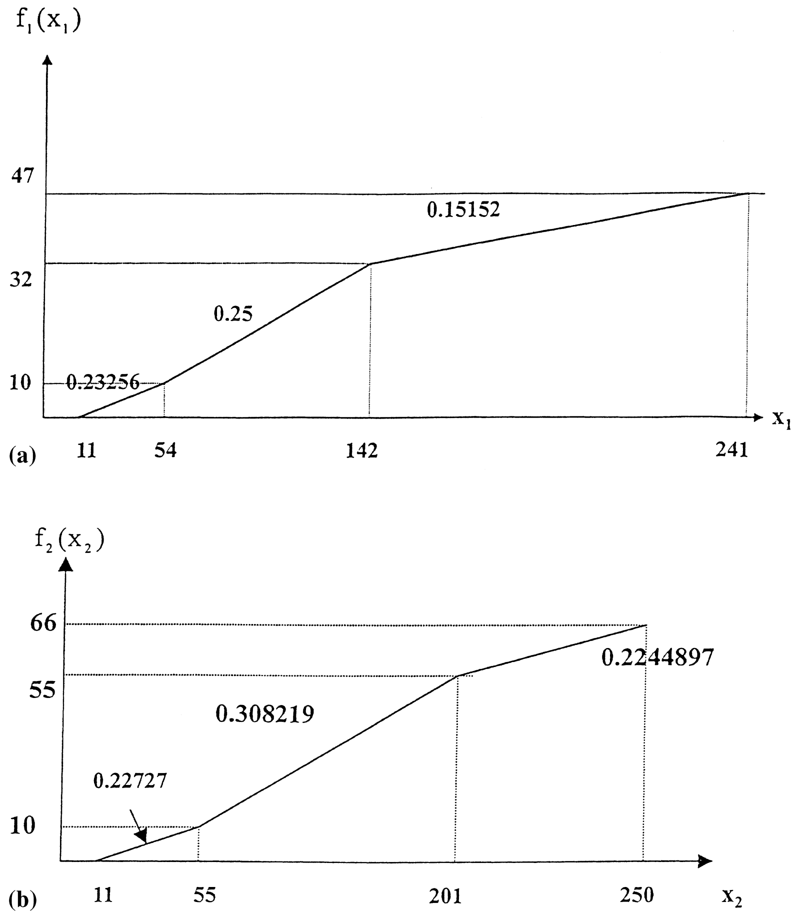


Fig. 7. (a) A hydro-electric generating function $f_1(x_1)$. (b) A hydro-electric generating function $f_2(x_2)$.

By letting $Q = 450$, 400, 350, 300 and 250, the computed optimal discharge allocation $(x_1, x_2) = (200, 250)$, (150, 250), (142, 208), (142, 158), and (142, 108) respectively. The obtained solutions are the same as the ones found in Klein et al. [6].

Example 6 (*Modified from Hillier et al.* [5]). A farmer raises pigs for market, and he wishes to determine the quantities of the available types of feed that should be administered to each pig to fulfill certain nutritional requirements at a minimum cost. Table 2 provides the number of units of each type of basic nutritional

Table 2
Required nutritional ingredient

Nutritional ingredient	Kilogram of corn	Kilogram of tankage	Kilogram of alfalfa	Minimum daily requirement
Carbohydrates	90	20	40	2000
Protein	30	80	60	1800
Vitamins	10	20	60	1500
Costs	$f(x_1)$	$f(x_2)$	$f(x_3)$	

ingredient contained within a kilogram of each feed type, along with the daily nutritional requirements and feed cost:

By considering factors such as holding cost, order cost, and quantity discount, cost functions $f(x_1)$, $f(x_2)$, and $f(x_3)$ naturally become a non-convex shape [1,2,4,12], as depicted in Fig. 8(a)–(c), respectively.

Based on Proposition 1, the cost functions are formulated as follows:

$$f(x_1) = 40x_1 + 5(|x_1 - 10| + x_1 - 10) - 5(|x_1 - 12| + x_1 - 12),$$
$$f(x_2) = 20x_2 - 5(|x_2 - 10| + x_2 - 10) + 5(|x_2 - 12| + x_2 - 12),$$
$$f(x_3) = 30x_3 + 10(|x_3 - 10| + x_3 - 10) - 10(|x_3 - 20| + x_3 - 20).$$

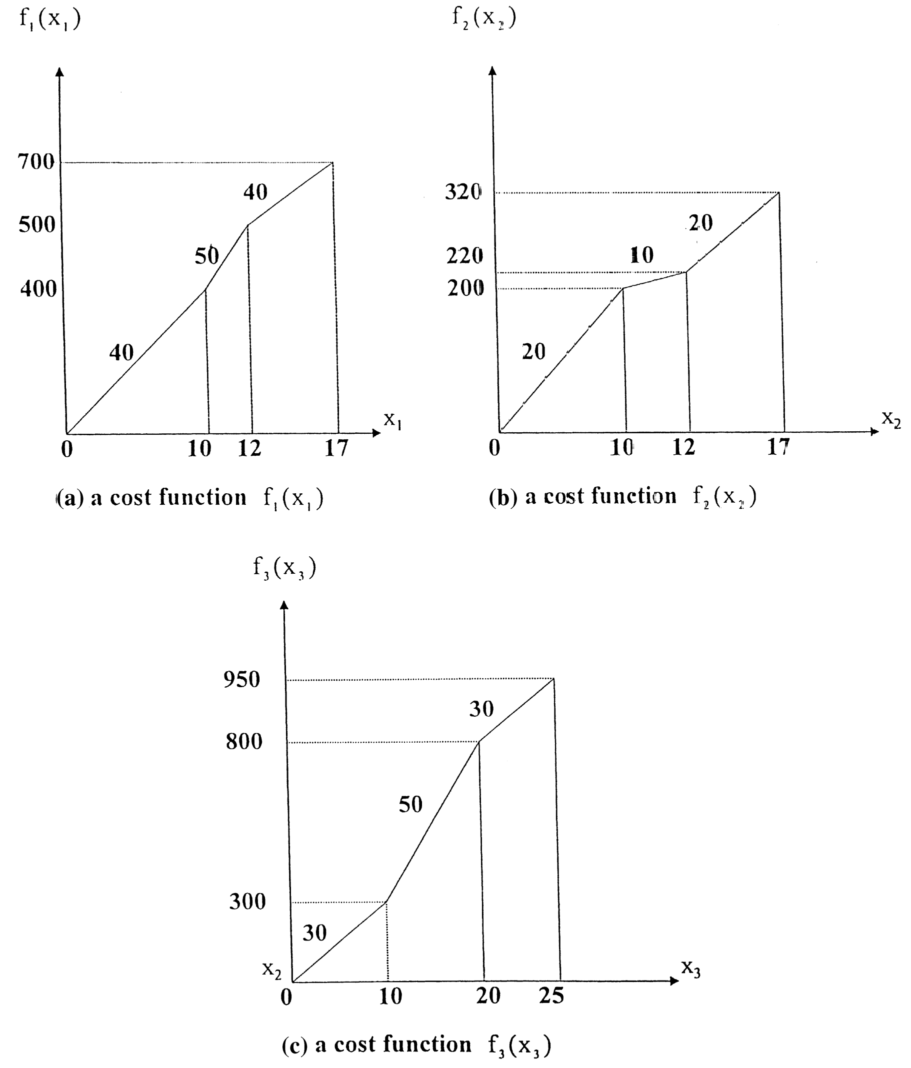


Fig. 8. (a) A cost function $f_1(x_1)$. (b) A cost function $f_2(x_2)$. (c) A cost function $f_3(x_3)$.

From the basis of Propositions 3 and 4, $f(x_1)$, $f(x_2)$ and $f(x_3)$ can be linearized as follows:

$$f(x_1) = 40x_1 + 10z_1 - 120u_1 + 10d_1 + 20,$$

where $x_1 + d_1 \geqslant 10,\ d_1 \leqslant 10,\ x_1 + 17(u_1 - 1) \leqslant z_1$, and u_1 is a 0–1 variable;

$$f(x_2) = 20x_2 + 10z_2 - 100u_2 + 10d_2 - 20,$$

where $x_2 + d_2 \geqslant 12,\ d_2 \leqslant 2,\ x_2 + 17(u_2 - 1) \leqslant z_2$, and u_2 is a 0–1 variable; and

$$f(x_3) = 30x_3 + 20z_3 - 400u_3 + 20d_3 + 200,$$

where $x_3 + d_3 \geqslant 10,\ d_3 \leqslant 10,\ x_3 + 25(u_3 - 1) \leqslant z_3$, and u_3 is a 0–1 variable.

Therefore, the problem is formulated as follows:

$$\begin{aligned}
\text{minimize}\quad & f(x_1) + f(x_2) + f(x_3) \\
\text{subject to}\quad & x_1 + d_1 \geqslant 10, d_1 \leqslant 10, x_1 + 17(u_1 - 1) \leqslant z_1, \\
& x_2 + d_2 \geqslant 12, d_2 \leqslant 2, x_2 + 17(u_2 - 1) \leqslant z_2, \\
& x_3 + d_3 \geqslant 10, d_3 \leqslant 10, x_3 + 25(u_3 - 1) \leqslant z_3, \\
& 90x_1 + 20x_2 + 40x_3 \geqslant 2000, 30x_1 + 80x_2 + 60x_3 \geqslant 1800, \\
& 10x_1 + 20x_2 + 60x_3 \geqslant 1500,\ x_1, x_2, x_3 \geqslant 0, \\
& u_1, u_2, \text{ and } u_3 \text{ are 0–1 variables.}
\end{aligned}$$

After running on the LINDO [9], the optimal values are $x_1 = 11.04$, $x_2 = 12$, and $x_3 = 19.16$.

6. Concluding remark

This paper treats nonconvex separable programming problems where the objective functions and the constraints might be nonconvex. Comparing the proposed method with conventional NSP methods reveals that the former can derive the approximately global optimum of a NSP problem by using less number of zero–one variables. The quality of derived solution can be improved by adequately adding the break points with the searching intervals.

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